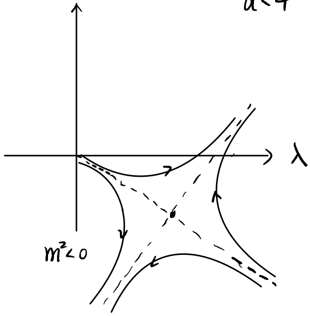
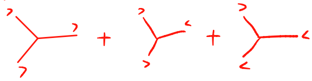
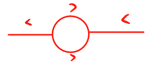


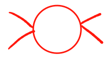
$d<4$

作业（2选1）.

1) 研究 ϕ^3 理论，$d=6+\varepsilon$ 维的 RG flow.

$\langle \delta S\rangle =$  无贡献. $\langle \delta S\rangle = 0$

$\langle \delta S^2\rangle_c =$  m^2 修正（负的）

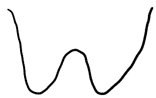
$\langle \delta S^3\rangle_c =$  λ 修正

$m^2<0$: $\mathcal{L} = (\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4$

$\langle\phi\rangle \neq 0$

SSB（自发磁化）

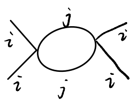
$m^2>0$: $\langle\phi\rangle = 0$

L:  SSB

2) $\phi^4 \Rightarrow$ 多分量

$\phi^2 = \sum_i \phi_i^2$

$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4 + V\sum_i \phi_i^4$



下节课：Bosonization. $\begin{cases} 1+1, & \partial_t, \partial_x \text{ (Sine-Gordon)} \\ 2+0. & \partial_x, \partial_y \text{ (BKT)} \end{cases}$ $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \lambda\cos(\beta\phi)$

(3.31)

3个R: ① Regularization : 正规化 $\longrightarrow 1+\frac{1}{2}+\frac{1}{3}+\cdots = -\frac{1}{12}$, $\int_0^{+\infty}\frac{d^dk}{k^2+m^2}$

② Renomalization : 重整化

③ Renomalization Group : 重整化群

1) cutoff

$\int_0^{\Lambda}\frac{d^dk}{k^2+m^2} \rightarrow$ 

2) 维度正规化

$\int_0^{+\infty}\frac{d^dk}{k^2+m^2} \rightarrow \int_0^{+\infty}\frac{k^{1-\varepsilon}dk}{k^2+m^2}$, $\varepsilon \to 0^+$

作业中的：$d=4-\varepsilon$ or $6-\varepsilon$

$\varepsilon \sim \frac{1}{\ln\Lambda}$

正规化：（局部）消除求和的发散.

② 重整化： $m^2 = m_0^2 + \delta m^2 \propto$ 〇 $\quad \lambda = \lambda_0 +$ ><

有限，观测值 $\quad$ 无穷大 $\quad$ 无穷大

$\lim_{\Lambda\to\infty} e(\Lambda)$

e^* 不是真实值，而是低能的有效值

③ 重整化群 $\quad$ Ising model： $k_{n+1} = R(k_n)$

半群. $\quad k_{n+1} = R(R(k_{n-1})) = R(k_{n-1})$

R是一个操作，没有逆.

群： $g_i g_j = g_k$

Abelian Group： $R_a R_b = R_{ab}$ $\quad k' = R_a(k)$

$e^{ia\theta} e^{ib\theta} = e^{i(a+b)\theta}$ $\quad k'' = R_b(k') = R_b(R_a(k)) = R_{ab}(k)$

* ε 与 Λ 的关系：

eg：两台仪器： $\Lambda_1 \ll \Lambda_2$

$\Downarrow \quad \Downarrow$

$\nu_1 \quad \nu_2 \qquad \nu_2 = \nu_1 + f(\Lambda_2)$

不可能出现： $\ln\Lambda_2$, Λ_2 $\cdots$

可能： $\frac{1}{\ln\Lambda_2}$, $\frac{1}{\Lambda_2}$ $\cdots$

$\Lambda\to\infty$, $f(\Lambda)$ 不发散.

如何确定 ε？ so far as I know，无法确定.

$m^2(b)$

不做微扰计算

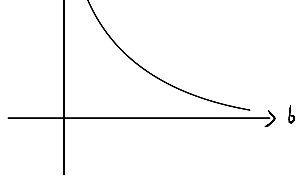
$$S = \int dk \frac{k^2+m^2}{2} \phi_k^* \phi_k + \int dk_1 dk_2 dk_3\, \phi\phi\phi\phi\,\lambda$$

令 $k = bk'$

$$\phi(k) = \phi(bk') = z\phi'(k')$$

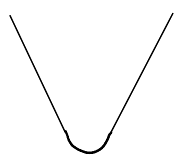
$$b^{d+2} z = 1 \longrightarrow z = b^{-d-2}$$

$$b^d m^2 z^2 = m^2(b) = m^2 b^{-2}$$

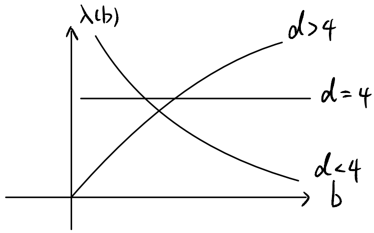


$\varepsilon_k = \sqrt{k^2+m^2}$

$k\to 0$, m 贡献大

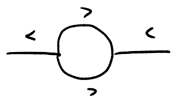


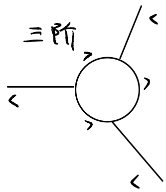
$$\lambda(b) = Z^4 b^{3d} \lambda = \lambda b^{d-4}$$



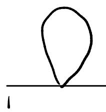
① ϕ^3 二阶 三阶

作业：





② ϕ^4 多分量：



Project： 50%

1人

2人

3人

重复论文推导.

DDL： 6月份.

Bosonization (玻色化).

几个 model 之间的关系

BKT 相变 (2D, 经典) ——— vortex excitation

$$H = \frac{1}{2}(\partial_x\phi)^2 + (\partial_y\phi)^2 + A\cos\phi$$

今天： 1D quantum： Sine-Gordon model

$$H = \frac{1}{2}\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right] + A\cos\phi$$ — 时间 vertex

不是振幅，而是相位.

$(x, y) \leftrightarrow (t, x)$

Quantum d-dim $\xleftrightarrow{\text{dual}}$ $d+1$-dim Classical

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

$\phi(x)$ 表示 x 处的振幅.

$\propto x^2 + g x^4$

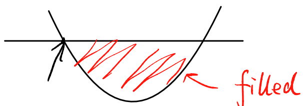
相位： $\phi \sim \phi + \text{const}$.

$\Rightarrow \cos\phi$.

(无法做 Taylor 展开)

Application of Sine-Gordon model:

- 物理上：$(1+1)=d=2$，复数中的 conformal transform / FT.
- 1d. for Fermion $\longrightarrow$ Dirac equation. $\Leftrightarrow$ Interacting Dirac eq.



附近展开：1d Dirac eq.

- Sidnet Coleman (1970s)

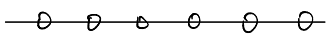
Sine-Gordon $\overset{\text{dual}}{\Longleftrightarrow}$ Thirring model

- 物理实验上存在（碳纳米管）
- QHE edge state

Nanotube, Nanowire, Ultracold Atom.

理论上证明：① $d\geq 3$，Fermion / Boson

② $d=1,2 \Rightarrow$ Anyon (Non-abelian)



$d=1$

位置无法交换，只会有 $\rho(x)$ 和 $\theta(x)$ 改变.

$\Rightarrow$ 不同的统计都对应 Bosonic 激发.

一种流体.

$$\frac{\partial \rho}{\partial t}+\frac{\partial j}{\partial x}=0$$

适用于任意粒子的物理

Refs: ① C. Kane. note

② Sachdev. PPT/讲义 or QFT书

③ Haldane. 1979–1981 论文

④ Cazalilla. Bosoniz liquid Cold atomic gases, 2004.

⑤ Senecild: An indroduction to Bosonization. 1999

⑥ Shanker 书 chap 17-18.

几个例子:

① C. Kane note.

$$\mathcal{L} = \frac{m}{2}\sum_i \dot{x}_i^2 - \frac{k}{2}(x_i - x_{i+1})^2$$

引入场论时提到.

$$x_i = \sqrt{a}\,\phi(x_i,t).$$

$$\mathcal{L} = \int dx\, \frac{m}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{k^2a^2}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 \longrightarrow \text{EOM: } m^2\frac{\partial^2\phi}{\partial t^2} = k^2a^2\frac{\partial^2\phi}{\partial x^2}$$

$$m\omega_q^2 = k^2a^2q^2,\quad \phi \sim e^{i(qx-\omega t)},\quad \omega_q = v|q|$$

$$N = \sum_q \frac{1}{e^{\beta v|q|}-1} \sim \int dq\, \frac{1}{e^{\beta vq}-1}$$ $\Longleftrightarrow$ 红外发散 Mermin-Wager Theorem

通常打开能隙: $m^2\phi^2$

若ϕ为相位,则不能通过此方式.

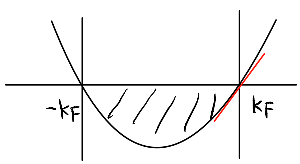
此时用 $\cos\phi$ (Sine-Gordon) 打开能隙.

Boson:

1 d chain: $\omega = v|q|$. $q = \frac{n\pi}{L}$, $\omega_q = \frac{\pi v}{L}n$

$$H = \frac{v\pi}{L}\sum_{n\geq 1} b_n^\dagger b_n$$

Fermion:



$$H = \int \psi^\dagger\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \mu\right)\psi \qquad \mu = \frac{\hbar^2k_F^2}{2m}$$

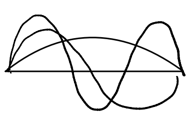
$$= \left(\frac{\hbar^2k^2}{2m} - \mu\right)\psi_k^\dagger\psi_k$$

定义: $\psi(x) = (e^{ik_Fx}\psi_R + e^{-ik_Fx}\psi_L)$

↑快 ↑慢(低能激发)

$$N = \sum_{|k|\leq k_F} \frac{L}{2\pi}\int_{-k_F}^{k_F} dk = \frac{Lk_F}{\pi} \Rightarrow k_F = \pi\rho,\quad \rho = N/L = 1/a$$

$$= \pi/a$$



缓变：一阶导：$\frac{\partial}{\partial x} e^{-ax^2} \sim -2ax\, e^{-ax^2}$

$\frac{\partial^2}{\partial x^2} e^{-ax^2} \sim a^2x^2 e^{-ax^2}$ $a \ll 1$，$\frac{\text{一阶导}}{\text{二阶导}} \sim \frac{1}{ax} \to \infty$

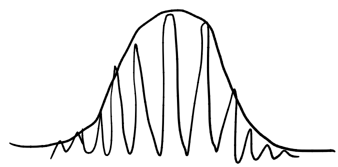
$$H = \int dx\, \psi^\dagger \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \mu\right)\psi$$

$$= \int dx\, [e^{-ik_F x}\psi_R + e^{ik_F x}\psi_L]\left(-\frac{\hbar^2}{2m} - \mu\right)[e^{ik_F x}\psi_R + e^{-ik_F x}\psi_L]$$

$$(fg)'' = f''g + 2f'g' + fg''$$

0，$\frac{\hbar^2 k_F^2}{2m} - \mu = 0$　　$\nwarrow 0$，g 为慢变.

$$\int e^{-2ik_F x}\psi_L^\dagger \psi_R\, dx \sim 0, \quad k_F = \pi/a.$$



只剩 $2f'g' \longrightarrow \psi_R^\dagger e^{-ik_F x}\left(-\frac{\hbar^2}{m}\right)\frac{d}{dx}(e^{ik_F x})\frac{d}{dx}\psi_R$.

$$= \psi_R^\dagger\left(-\frac{i\hbar^2 k_F}{m}\right)\frac{d}{dx}\psi_R$$

$$= v_F\, \psi_R^\dagger\left(-i\frac{d}{dx}\right)\psi_R$$

& Left: $v_F\, \psi_L^\dagger\left(i\frac{d}{dx}\right)\psi_L$.

$$H = \hbar(\psi_R^\dagger, \psi_L)\begin{pmatrix} -iv\frac{d}{dx} & 0 \\ 0 & iv\frac{d}{dx}\end{pmatrix}\begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}$$

$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi \Rightarrow E = \frac{\hbar^2 q^2}{2m}$

$v_F\left(-i\frac{d}{dx}\right)\psi = E\psi \Rightarrow E = vq$.

Fermion：$\varepsilon = vq = v\left(\frac{n\pi}{L}\right)$

Exact duality for Fermion / Boson in 1d.

i.e. $Z_F = Z_B$.

1个 Boson 和 $N_F = 1$ 对应

不是一个 Fermi 态.

$H = \omega + \omega b^\dagger b$

图示法：$H_F = \omega \sum_{n=1}^{+\infty} n f_n^\dagger f_n$

$H_B = \omega \sum_{n=1}^{+\infty} n\, b_n^\dagger b_n + E_0$

Fock态：$|1000\cdots\rangle \to \omega \longrightarrow n=0$

$|0100\cdots\rangle \to 2\omega \longrightarrow n=1$

$|001\cdots\rangle \to 3\omega \longrightarrow n=2$.