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$$\langle\phi(x)\phi(y)\rangle = \frac{\int D\phi\, e^{iS_0+iS_I}\phi(x)\phi(y)}{\int D\phi\, e^{iS_0+iS_I}} \longrightarrow \text{Cumulant expansion}$$

moment expansion

$\delta(k+k')dk'$

利 下连接图

Parseval公式：$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{+\infty} a_n\cos(nx) + b_n\sin(nx)$

$\langle f,f\rangle = \int f^2(x)dx = a_0^2 + \sum_{n=1}^{+\infty}(a_n^2+b_n^2)$

$\int f^2(x)dx = \int f^2(k)dk$ 类比 $\int\phi^2 dx \Rightarrow \int dk|\phi_k|^2 \propto \frac{1}{V}\sum_k|\phi_k|^2$

$\Longleftrightarrow \frac{1}{V}\sum_k = \int dk\left(\frac{1}{2\pi}\right)^d$

等式：① $\int e^{ikx}dx = 2\pi\delta(k)$

② $\left(\frac{1}{2\pi}\right)^d\int e^{i\vec{k}\cdot\vec{x}}\,d\vec{x} = \delta(\vec{x})$

③ $\frac{1}{V}\sum_k e^{i\vec{k}\cdot\vec{x}} = \frac{1}{V}\cdot\frac{V}{(2\pi)^d}\int e^{i\vec{k}\cdot\vec{x}}\cdot d\vec{k} = \delta(\vec{x})$

④ $\int dx\, e^{-\frac{A}{2}x^2} = \sqrt{\frac{2\pi}{A}} \Rightarrow \frac{\int dx\, e^{-\frac{A}{2}x^2}\cdot x^2}{\int dx\, e^{-\frac{A}{2}x^2}} = \frac{1}{A}$.

$\int dz d\bar{z}\, e^{-A\bar{z}z} = \frac{\pi(2i)}{A}$. $\qquad \frac{\int dz d\bar{z}\, e^{-A\bar{z}z}\cdot\bar{z}z}{\int dz d\bar{z}\, e^{-A\bar{z}z}} = \frac{1}{A}$.

目的：① Green function. propagator，图表示.

$D(x-y) = \frac{1}{V}\sum_k(\qquad)$ 成对出现.

② Introduction. 预备知识. $\begin{cases}\text{Cumulant 展开}\\ \text{moment 展开.}\end{cases}$

物理：linked cluster expansion

意义：

$\phi(x) = \frac{1}{V}\sum_k e^{ikx}\phi(k)$，$k=(k_0,\vec{k})$ 上一节课 ω_k，$x=(t,-\vec{x})$

$S_0 = \int\mathcal{L}_0 dx = \int\mathcal{L}_0 d\vec{x}dt$.

$\int\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2dx = \frac{1}{2V^2}\sum_{kk'}\phi(k)\phi(k')(ik_0)(ik_0')\int e^{i(k+k')x}dx$

$= \frac{1}{2V^2}\sum_{kk'}\phi(k)\phi(k')(ik_0)(ik_0')(2\pi)^d\delta(k+k')$

$= \frac{1}{2V_0}\sum_{kk'}$

$\frac{1}{V}\sum_k = \left(\frac{1}{2\pi}\right)^d\int dk$.

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$$= \frac{1}{2V^2}\sum_k \phi(k)\frac{V}{(2\pi)^d}\cdot(2\pi)^d\int dk'\,\phi(k')(ik_0)(ik_0')\delta(k+k')\phi(k')$$

要求 $k'=-k \Rightarrow k_0'=-k_0$ $\qquad\int\delta(k)dk=1$

$\phi(-k)=\phi^*(k)$

$$= \frac{1}{2V}\sum_k \phi(k)\phi(-k)k_0^2$$

对应关系：$\mathcal{L}_0=\frac{1}{2}(\partial\phi)^2-\frac{m^2}{2}\phi^2$

$$=\frac{1}{2}[(\partial_t\phi)^2-(\nabla\phi)^2]-\frac{m^2}{2}\phi^2$$

$$=\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2-\left(\frac{\partial\phi}{\partial x}\right)^2\right]-\frac{m^2}{2}\phi^2$$

$S_0=\int dx\,\mathcal{L}_0 \xrightarrow[\text{类比}]{\text{parseval公式}} \frac{1}{V}\sum_{k_0>0}(k_0^2-\vec{k}^2-m^2)\,\phi^*\phi$

$$\begin{cases}\left(\frac{\partial}{\partial t}\right)^2\leftrightarrow k_0^2\\ \left(\frac{\partial}{\partial x}\right)^2\leftrightarrow k^2\\ \phi^2\leftrightarrow 1\end{cases}$$

$$\int D\phi\, e^{iS_0}$$

$$=\int D\phi_k^*\phi_k\, e^{\frac{i}{V}\sum_k(k_0^2-\vec{k}^2-m^2)\phi_k^*\phi_k} \leftrightarrow \int dx\,p(x)$$

求 $\langle\phi(x)\phi(x)\rangle=\frac{1}{V^2}\sum_{qq'}\langle\phi(q)\phi(q')\rangle e^{i(q\cdot x+q'\cdot y)}$

$$\langle x\rangle=\frac{\int p(x)\cdot x\cdot dx}{\int p(x)dx}$$

$$\langle\phi(x)\phi(y)\rangle=\frac{1}{V^2}\sum_{qq'}e^{i(qx+q'y)}\frac{\int_{>0}D\phi_k^*D\phi_k\,e^{\frac{i}{V}\sum_k(k_0^2-\vec{k}^2-m^2)\phi_k^*\phi_k}\phi_q\phi_{q'}}{\int_{>0}D\phi_k^*D\phi_k\,e^{\frac{i}{V}\sum_k(\ )\phi_k^*\phi_k}}$$

特点：①表达式复杂 ②对角，Guass积分.

eg：$\dfrac{\int dxdy\,e^{-A(x^2+y^2)}\cdot x\cdot y}{\int dx\,dy\,e^{-(A(x^2+y^2))}}=\langle x,y\rangle$. 仅当 $x=y$ 时 $\neq 0$.

$\langle xy\rangle=0$ ， $\langle x^2\rangle=\frac{1}{A}$ ， $\langle y^2\rangle=\frac{1}{A}$.

$\Downarrow$ 可知.

唯一贡献只有 $q'=-q$.

$\langle\phi^4\rangle\sim\int D\phi_k^*D\phi_k\,e^{iS_0}\phi_{q_1}\phi_{q_2}\phi_{q_3}\phi_{q_4}$ 或

$q_1=-q_4$, $q_2=-q_3$；$q_2=-q_1$, $q_4=-q_3$

$\langle\phi^6\rangle$

$\langle\phi^8\rangle$ Feynman diag

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{V^2}\sum_{qq'} e^{i(qx+q'y)}\,\delta(q+q')\,\frac{\int D\phi_k^* D\phi_k\, e^{iS_0}\,\phi_q^*\phi_q}{\int D\phi_k^* D\phi_k\, e^{iS_0}}$$

$$= \underline{\qquad\qquad}\; \frac{\int D\phi_q^* D\phi_q\, e^{\frac{i}{V}(q_0^2-\vec{q}^2-m^2)\phi_q^*\phi_q}\,\phi_q^*\phi_q}{\int D\phi_q^* D\phi_q\, e^{\frac{i}{V}(q_0^2-\vec{q}^2-m^2)\phi_q^*\phi_q}}$$

$A = -i(q_0^2-\vec{q}^2-m^2)/V$

$$= \frac{1}{V^2}\sum_{qq'} e^{i(qx+q'y)}\,\delta(q+q') \times \frac{iV}{(q_0^2-q'^2-m^2)}$$

$$= \frac{1}{V^2}\sum_q \frac{V}{(2\pi)^d}\, e^{i(qx+q'y)}\,\delta(q+q')\,\frac{iV}{\Lambda_q}.$$

$$= \frac{1}{V(2\pi)^d}\sum_q e^{iq(x-y)}\cdot\left(\frac{iV}{\Lambda_q}\right)$$

$$= \frac{1}{V}\sum_q \frac{i e^{iq(x-y)}}{\Lambda_q} = D(x-y)$$ 传播子.

$G(x-y)$ propagator.

符号 $\underset{x\qquad y}{\sqcup} = x\underline{\qquad}y = D(x-y)$.

$$D(x-y) = \frac{1}{V}\sum_q \frac{e^{iq(x-y)}}{\Lambda_q} = G(x-y).$$

propagator $\boxed{L\phi=\rho}$ L: Linear operator.

通解 $\phi = \phi_0 + \int G(x-y)\rho(y)\,dy$

$$LG(x-y) = \delta(x-y)$$

$\delta(\vec{x}) = \frac{1}{V}\sum_k e^{i\vec{k}\cdot\vec{x}}$. $G(x-y) = \frac{1}{V}\sum_k e^{ikx}G(k)$

$$\frac{1}{V}\sum_k G(k)\cdot L e^{i\vec{k}\cdot\vec{x}} = \frac{1}{V}\sum_k e^{i\vec{k}\cdot\vec{x}}.$$

该 把L拿下来

$$= \frac{1}{V}\sum_k G(k)L(k)e^{ikx} = \frac{1}{V}\sum_k e^{i\vec{k}\cdot\vec{x}}$$

$G(k)L(k) = 1 \Leftrightarrow G(k) = \frac{1}{L(k)}$

则 $\Lambda_q = \mathcal{L}(q) \to$ Lagrange的 FT (傅利叶变换).

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传播子 $\underline{x\quad y} = D(x-y) = \frac{1}{V}\sum_q \frac{i e^{iq(x-y)}}{\mathcal{L}(q)}$ ← Fourier transformation

e^{iS_0} ↑

↖ 对 Lagrange

$\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4$，其中 λ 成对小量

$$\langle\phi(x)\phi(y)\rangle = \frac{\int D\phi\, e^{iS_0 + i\int \mathcal{L}_I dx}\phi(x)\phi(y)}{\int D\phi\, e^{iS_0 + i\int\mathcal{L}_I dx}}$$

$$= \frac{\int D\phi\, e^{i(S_0+S_I)}\phi(x)\phi(y)}{\int D\phi\, e^{i(S_0+S_I)}}$$

$S_I = \int \mathcal{L}_I\, dx$

$$\int D\phi\, e^{iS_0 + iS_I} = \sum_{n=0}^{\infty}\int D\phi\, e^{iS_0}\frac{(iS_I)^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int D\phi\, e^{iS_0}\int dx_1\cdots dx_n\, \mathcal{L}_I(x_1)\cdots\mathcal{L}_I(x_n)$$

$$= \sum_{n=0}^{\infty}\frac{i^n}{n!}\frac{\lambda^{4n}}{(4!)^n}\int\int D\phi\, e^{iS_0}\phi(x_1)\cdots\phi(x_1)\, dx_1\cdots dx_n$$

① 简化的一般式

② 图表示

方法一：Cumulant expansion 累积量展开公式

方法二：$\int D\phi\, e^{iS_0 + \int J\phi} \Leftrightarrow \int dx\, e^{-\frac{a}{2}x^2 + Jx}$ (形式理解)

source

矩；累积量 $\langle e^{tx}\rangle = \frac{\int p(x) e^{tx} dx}{\int p(x)\, dx} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\langle x^n\rangle$ ← moment 矩

$Z = \mathrm{Tr}(e^{-\beta H}) = e^{-\beta F}$

$= e^{\Omega}$ ($\Omega = \sum_{r=1}^{\infty}\mu_r\frac{t^r}{r!}$)

无一阶平移效应.

特点：1) 可加. 2) 关联，一阶，二阶全出现在 n 阶 $\langle x^n\rangle$ 中.

$\langle x^2\rangle = \langle x\rangle^2 + \sigma_2^2$

$\sigma_2^2 = \langle x^2\rangle - \langle x\rangle^2$ $x \to x + c$ 不变

$\langle x^3\rangle = \langle x\rangle^3 + 3\langle x^2\rangle\langle x\rangle + \sigma_3$ ← Cumulant 累积量

$\langle x^n\rangle = \int x^n p(x)\, dx = m_n$

$Z = \langle e^{tx}\rangle = \sum_{n=0}^{\infty}\frac{t^n}{n!}m_n = e^k \Leftrightarrow$ 求 $k = ?$

$k = \sum_{r=1}^{\infty}k_r\frac{t^r}{r!}$

比较左 $= 1 + \sum_{n=1}^{\infty}\frac{t^n}{n!}m_n = 1 + m_1 t + \sum_{n=2}^{\infty}\frac{t^n}{n!}m_n$

0阶

右 $= e^{\sum_{r=1}^{\infty}k_r\frac{t^r}{r!}} = e^{k_1 t}\cdot e^{k_2\frac{t^2}{2!}}\cdot e^{k_3\frac{t^3}{3!}}\cdots = \frac{1}{2!}k_1^2 t^2 + k_1 t\cdot k_2\frac{t^2}{2}$

则 $k_1 = m_1$，$k_2 = m_2 - m_1^2$，$k_3 = m_3 - 3m_2 m_1 + 2m_1^3$

讨论：$m_2 - m_1^2 = \langle x^2 \rangle - \langle x \rangle^2$

$$m_3 - 3m_2 m_1 - m_1^3 = \langle x^3 \rangle - 3\langle x^2 \rangle \langle x \rangle + 2\langle x \rangle^3$$

k_r 明确分类.

一个抽象的证明：methords of QFT in Cond'matt'，Abrikosov

$\langle e^{tx} \rangle = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \langle x^n \rangle$，$\langle x^n \rangle = \langle x^1 \rangle_c^{n_1} \langle x^2 \rangle_c^{n_2} \langle x^3 \rangle_c^{n_3} \cdots C_n^{n_1} C_{n-n_1}^{2n_2} C_{n-n_1-2n_2}^{3n_3}$

Connected. n_1个1阶关联

其中 $n = n_1 + 2n_2 + 3n_3 + \cdots$

$$\times \frac{(2n_2)!}{(2!)^{n_2} n_2!} \times \frac{(3n_3)!}{(3!)^{n_3} \cdot n_3!} \cdots$$

$$\langle e^{tx} \rangle = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \langle x \rangle^{n_1} \langle x^2 \rangle^{n_2} \langle x^3 \rangle^{n_3} \cdots C_n^{n_1} \cdot C_{n-n_1}^{2n_2} \cdot C_{n-n_1-2n_2}^{3n_3} \cdots \times \frac{(2n_2)!}{(2!)^{n_2} n_2!} \times \frac{(3n_3)!}{(3!)^{n_3} \cdot n_3!}$$

$$= \sum_{n_1 n_2} t^{n_1 + 2n_2 + 3n_3 + \cdots} \frac{\langle x \rangle^{n_1} \langle x^2 \rangle^{n_2} \langle x^3 \rangle^{n_3}}{(1! \, n_1!)(2!)^{n_2} n_2! \cdot (3!)^{n_3} n_3!} \cdots$$

$$= \left(\sum_{n=0}^{+\infty} \frac{\langle x \rangle_c^{n_1}}{n_1!} \right) \left(\sum_{n=0}^{+\infty} \frac{\langle x^2 \rangle_c^{n_2}}{n_2! (2!)^{n_2}} \right) \cdots$$

$\langle x^n \rangle_c = \langle x^n \rangle - 1$阶 $- \cdots - (n-1)$阶.

$$= e^{\sum_{n=1}^{+\infty} \frac{\langle x^n \rangle_c t^n}{n!}} = e^{\Omega}$$

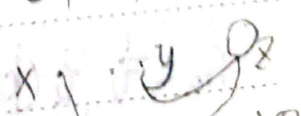
$$\frac{\int D\phi \, \phi(x) \phi(y) e^{i(S_0 + S_I)}}{\int D\phi \, e^{i(S_0 + S_I)}}$$

$$\langle O \rangle = \frac{\int D\phi \, O \, e^{-H_0 - u}}{\int D\phi \, e^{-H_0 - u}} = \frac{\int D\phi \, O e^{-H_0} \cdot e^{-u}}{\int D\phi \, e^{-H_0} \cdot e^{-u}} = \frac{\int D\phi \, O e^{-u} e^{-H_0}}{\int D\phi \, e^{-u} e^{-H_0}}$$

$$= \frac{\langle O e^{-u} \rangle_{H_0}}{\underbrace{\langle e^{-u} \rangle_{H_0}}_{e^{-\Omega H_0}}} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n} \langle O u^n \rangle_{H_0}^{c}$$

$c \leftarrow$ 连接图.

$\int \phi(x) \phi(y) \phi^4(z)$



除去极小数连接图

5 2d δ potential 一样发散.