3.10 parseval公式

$$f(x)=\frac{a_0}{\sqrt{2}}+\sum a_n\cos(nx)+b_n\sin\sin(nx)$$

$$\langle f,f\rangle=\int f^2(x)dx=a_0^2+\sum_{n=1}^{+\infty}(a_n^2+b_n^2)$$

or $\int f^2(x)dx=\int f^2(k)dk$ 可能差系数

类比: $\int\phi^2dx\Rightarrow\int dk|\phi_k|^2$

本次目的: ① Green function propagator

图表示, $D(x-y)=\frac{1}{V}\sum_k(\quad)$

② Interaction

预备知识 $\begin{cases}\text{Cumulant 展开}\\ \text{moment 展开}\end{cases}$

物理上对应: Linked cluster 展开

意义

$$\phi(x)=\underbrace{\frac{1}{V}\sum_k}_{\text{成对出现}}e^{ik\cdot x}\phi(k)$$ 其中 $k=(k_0,\vec{k})$, $x=(t,\vec{x})$

$$S_0=\int\mathcal{L}_0dx=\int\mathcal{L}_0d\vec{x}dt$$

$$\int\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2dx=\underbrace{\frac{1}{2V^2}\sum_{kk'}}_{\text{成对出现}}\phi(k)\phi(k')(ik_0)(ik_0')\int e^{i(k+k')x}dx$$

$$=\frac{1}{2V^2}\sum_{kk'}\phi(k)\phi(k')(ik_0)(ik_0')(2\pi)^d\delta(k+k')$$

$$=\frac{1}{2V^2}\sum_k\phi(k)\frac{V}{(2\pi)^d}(2\pi)^d\int dk'\phi(k')(ik_0)(ik_0')\,\delta(k+k')$$

$$= \frac{1}{2V} \sum_k \phi(k)\phi(-k) K_0^2 = \frac{1}{V} \sum_{k_0 \geq 0} \phi_k \phi_k^* K_0^2$$

对应关系： $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 = \frac{1}{2}[(\partial_t \phi)^2 - (\nabla\phi)^2] - \frac{m^2}{2}\phi^2$

$$= \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right] - \frac{m^2}{2}\phi^2$$

对角化

$$S_0 = \int dx\, \mathcal{L}_0 \xrightarrow{\text{Parseval}} \frac{1}{V} \sum (k_0^2 - k^2 - m^2)\phi_k^* \phi_k$$

$\phi(x)$

之后 $\int D\phi\, e^{iS_0} = \int_{k_0 \geq 0} D\phi_k^* D\phi_k\, e^{i \frac{1}{V} \sum_k (k_0^2 - k^2 - m^2)\phi_k^* \phi_k}$

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{V^2} \sum_{q q'} \langle \phi(q)\phi(q') \rangle e^{i(qx + q'y)}$$

$$\langle x \rangle = \frac{\int P(x)\, x\, dx}{\int P(x)\, dx}$$

其中 $\langle \phi(q)\phi(q') \rangle = \dfrac{\int_{>0} D\phi_k^* D\phi_k\, e^{\frac{i}{V}\sum_k (k_0^2 - k^2 - m^2)\phi_k^* \phi_k} \phi_q \phi_{q'}}{\int_{>0}}$

注意这是一个Gauss型复积分

$$\langle xy \rangle = \int dx dy\, e^{-A(x^2+y^2)} xy \Big/ \int dx dy\, e^{-A(x^2+y^2)} = 0$$

$$\langle x^2 \rangle = \frac{1}{A} \quad \langle y^2 \rangle = \frac{1}{A}$$

故上式积分中唯一有贡献的就是 $q = -q'$ 项

(同理 $\langle \phi^4 \rangle \sim \int D\phi_k^* D\phi_k\, e^{iS_0} \phi_{q_1}\phi_{q_2}\phi_{q_3}\phi_{q_4}$

唯一有贡献(非0)的情形就是 $q_{1\sim4}$ 成两对)

$$= \frac{1}{V^2}\sum_{qq'} e^{i(qx+q'y)}\,\delta(q+q')\int D\phi_k^* D\phi_k\, e^{iS_0}\,\phi_q^*\phi_q$$

移项 $= \dfrac{\int \cancel{D\phi_k^* D\phi_k} \int D\phi_q^* D\phi_q\, \phi_q^*\phi_q\, e^{iS_0}}{\int \cancel{D\phi_k^* D\phi_k} \int}$

$S_0 = \frac{1}{V}(q_0^2 - q^2 - m)\,\phi_q^*\phi_q$

$$= \frac{\int D\phi_q^* D\phi_q\, e^{iS_0}\,\phi_q^*\phi_q}{\int D\, D\, e^{iS_0}} = \frac{\int d\bar z\, dz\, e^{-\lambda \bar z z}\, \bar z z}{\int dz\, d\bar z\, e^{-\lambda \bar z z}}$$

$\phi_q^* = \bar z \quad \phi_q = z$

~~由已知公式~~

$$= \frac{1}{\lambda} \quad \underline{Pp} \quad \frac{iV}{q_0^2 - q^2 - m^2}$$

$$\therefore \langle \phi(x)\phi(y)\rangle = \frac{1}{V^2}\sum_{qq'} e^{i(qx+q'y)}\,\delta(q+q')\,\frac{iV}{(q_0^2-q^2-m^2)} \to \Lambda_q$$

$$= \frac{1}{V^2}\sum_q \int dq' \frac{V}{(2\pi)^d}\, e^{i(qx+q'y)}\,\delta(q+q')\,\frac{iV}{\Lambda_q}$$

$$= \frac{1}{V(2\pi)^d}\sum_q e^{iq(x-y)}\,\frac{iV}{\Lambda_q}$$

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{V}\sum_q \frac{i e^{iq(x-y)}}{\Lambda_q} = D(x-y)$$

物理意义: $D(x-y) = \frac{1}{V}\sum_q \frac{e^{iq(x-y)}}{\Lambda_q}$ Lagrangian 的 fourier Trans

传播子: $x \longrightarrow y$ $\quad D(x-y) = \underbrace{\frac{1}{V}\sum_q}_{\text{成对}} \frac{i e^{iq(x-y)}}{\mathcal{L}(q)}$

$\mathcal{L}_I$

考虑有相互作用的情形 $\mathcal{L} = \mathcal{L}_0 + \frac{\lambda}{4!}\phi^4 \quad \lambda \ll 1$

$$\langle\phi(x)\phi(y)\rangle = \frac{\int D\phi\, e^{iS_0 + i\int \mathcal{L}_I dx}\,\phi(x)\phi(y)}{\int D\phi\, e^{iS_0 + i\int \mathcal{L}_I dx}}$$

$$\int D\phi\, e^{iS_0 + iS_I} = \sum_{n=0}^{\infty} \int D\phi\, e^{iS_0} \frac{(iS_I)^n}{n!}$$

$$= \sum_{n=0}^{+\infty} \frac{i^n}{n!} \int D\phi\, e^{iS_0} \int dx_1 \cdots dx_n\, \mathcal{L}_I(x_1) \cdots \mathcal{L}_I(x_n)$$

$$= \sum_{n=0}^{+\infty} \frac{i^n}{n!} \frac{\lambda^{4n}}{(4!)^n} \iint D\phi\, e^{iS_0} \phi(x_1) \cdots \phi(x_n)\, dx_1 \cdots dx_n$$

moment cumulant

$$\langle e^{tX}\rangle = \frac{\int p(x) e^{tx} dx}{\int p(x) dx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle X^n\rangle$$

moment

$\langle X^n\rangle$（矩）的缺点：①复杂 ②关联性：一阶二阶…全现在 n阶 $\langle X^n\rangle$ 中

考虑将不同阶分开：

$$\langle X^2\rangle = \langle X\rangle^2 + \sigma_2^2$$

$$\Rightarrow \sigma_2^2 = \langle X^2\rangle - \langle X\rangle^2$$

$$\langle X^3\rangle = \langle X\rangle^3 + 3\langle X^2\rangle\langle X\rangle + \sigma_3$$

$$\Rightarrow \sigma_3 = \cdots$$

σ_n 称为累积量 $\quad \langle e^{tX}\rangle = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \langle X^n\rangle = e^{\Omega} \quad \Omega = \sum_{r=0}^{+\infty} \mu_r \frac{t^r}{r!}$

$$\langle X^n\rangle = \int x^n p(x) dx = m_n$$

$$Z = \langle e^{tX}\rangle = \sum_{n=0}^{+\infty} \frac{t^n}{n!} m_n = e^{K} \quad 求 K = ?$$

$$K = \sum_{r=1}^{+\infty} k_r \frac{t^r}{r!}$$ K_r 即是 累积量

比较 $$Z = \sum_{n=0}^{+\infty} \frac{t^n}{n!} m_n = 1 + \sum_{n=1}^{+\infty} \frac{t^n}{n!} m_n = 1 + m_1 t + \sum_{n=2}^{+\infty} \frac{t^n}{n!} m_n$$

$$Z = e^K = e^{k_1 t} e^{k_2 \frac{t^2}{2!}} e^{k_3 \frac{t^3}{3!}} \cdots e^{k_n \frac{t^n}{n!}}$$
$$= \left(1 + k_1 t + \frac{k_1^2 t^2}{2} + \cdots\right)\left(1 + k_2 \frac{t^2}{2!} + \frac{k_2^2 t^4}{(2!)^3}\right)(\cdots)$$

$$\therefore \begin{cases} m_1 = k_1 \\ \frac{m_2}{2} = \frac{k_2}{2} + \frac{k_1^2}{2} \\ \vdots \end{cases} \Rightarrow \begin{cases} k_1 = m_1 \\ k_2 = m_2 - m_1^2 \\ \vdots \end{cases}$$

一个抽象的证明 (Abrikosov)

$$\langle e^{tX} \rangle = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \langle x^n \rangle$$

$$\langle x^n \rangle = \underbrace{\langle x^1 \rangle_c^{n_1} \langle x^2 \rangle_c^{n_2} \langle x^3 \rangle^{n_3} \cdots}_{\text{connected}} C_n^{n_1} C_{n-n_1}^{2n_2} C_{n-n_1-2n_2}^{3n_3} \cdots \times \frac{(2n_2)!}{(2!)^{n_2} n_2!} \times \frac{(3n_3)!}{(3!)^{n_3} n_3!} \times \cdots$$

operator
↓
$$\langle O \rangle_H = \frac{\int D\phi\, O e^{-U} e^{-H_0}}{\int D\phi\, e^{-H_0} e^{-U}} = \frac{\int D\phi\, O e^{-U} e^{-H_0}}{\int D\phi\, e^{-U} e^{-H_0}} = \frac{\langle O e^{-U} \rangle_{H_0}}{\langle e^{-U} \rangle_{H_0}}$$

3.14

① Cumulant expansion

$$Z = \langle e^{tX} \rangle = e^{\Omega}$$

| Z: 生成函数/配分函数
| Ω: 自由能

$$X = \frac{\lambda}{4!} \int \phi^4 dx$$ X可以为变量，也可以是场

$$\Omega = \sum_{r=1}^{+\infty} \frac{t^r}{r!} k_r$$