下一节课．如何计算？

高斯积分；

动量空间；

2022.2.28．第2周第1节课．

上节课回顾：

Path Integral.（单粒子）.

Path Integral of field.（场）.

$$Z = \int D\phi \, e^{iS[\phi]}$$

$$S = \int \mathcal{L} \, dx. \quad (dx = d\vec{x}\, dt) \qquad x = (\vec{x}, t).$$

① 为什么这么复杂，但有用？

② 怎么去做计算？

☆ 注意：$D\phi$ 而非 $d\phi$.　　$Dx = A dx$

$$D\phi = \lim_{n\to\infty} A^n \int d\phi(x_1)\, d\phi(x_2) \cdots d\phi(x_n)$$

x_i 空间．

$$\prod_i \int D\phi(x_i) = J \prod_k \int \phi(k).$$

$\xRightarrow{FT}$ Gauss 积分 + Feynman 图

(Perturbation).

Path Integral.

① Bosen field.

② Fermion field. $\int D\psi D\bar{\psi}$.

③ Gauge field. $\int DA$. $\mathcal{L} \sim (\frac{\partial \vec{A}}{\partial t})^2 - (\nabla\times\vec{A})^2$. $A \simeq A + \nabla\varphi$.

④ Spin field.

以 Bosen field 为例.

Gauss 积分 (广泛易理解.易推广).

ref. A.zee.

$$\int dx\, e^{-Ax^2} = \sqrt{\frac{\pi}{A}}.$$

$$\int dx\, e^{-Ax^2+Jx} = \sqrt{\frac{\pi}{A}}\, e^{-\frac{J^2}{4A}}$$ (对 J 作展开)

$$\int dx\, e^{-Ax^2} \sum_n \frac{J^{2n}}{(2n)!} x^{2n} = \sqrt{\frac{\pi}{A}} \left(\frac{1}{4A}\right)^n \cdot J^{2n}$$

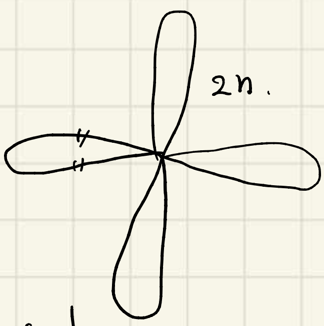
$$\Rightarrow \int dx\, e^{-Ax^2} x^{2n} = \sqrt{\frac{\pi}{A}} (2n)! \left(\frac{1}{4}\right)^n \cdot \frac{1}{A^n}$$

(2n阶) (n阶.)

$$\Rightarrow \langle x^{2n} \rangle = \frac{(2n)!}{n!} \left(\frac{1}{4A}\right)^n$$

图解!

图像:



$(\frac{1}{2A})$.

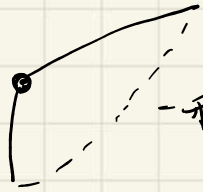
2n条线. 两两连通.

$$\frac{C_{2n}^2 C_{2n-2}^2 C_{2n-4}^2 \cdots}{n!\ 2^n}$$

$$= \frac{(2n)!}{n!\, 2^n}$$

$\times \left(\frac{1}{2A}\right)^n \leftarrow$ propagation. 归一化!

$$\frac{\int e^{-Ax^2} x^2\,dx}{\int e^{-Ax^2}\,dx} = \frac{1}{2A}.$$



一种连接方式 $\boxed{贡献\ \frac{1}{2A}}$

推广. $\mathbb{R}$.

$$\int dx_1 \cdots dx_n\, e^{-x^TAx} = ?$$

$$x = Oy.$$

$$= \int dy_1 dy_2 \cdots dy_n\, e^{-y^T \underbrace{O^TAO}\, y} \cdot J$$

$$O^TAO = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

$$= \int dy_1 dy_2 \cdots dy_n\, e^{-\sum_i \lambda_i y_i^2} \quad (J=1).$$

$$= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\lambda_1\lambda_2\cdots\lambda_n}}$$

$$= \prod_{i=1}^{n} \left(\frac{2\pi}{\lambda_i}\right)^{\frac{1}{2}} = \left(\frac{(2\pi)^n}{\det A}\right)^{\frac{1}{2}}$$

$$\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} \underbrace{\int dx_1 \cdots dx_n\, e^{-\frac{1}{2}x^TAx + J^Tx}}_{x \to x+c} = J_i x_i$$

Proof: $e^{-\frac{1}{2}x^TAx + J^Tx}$.

$$\xrightarrow{x \to x+c} e^{-\frac{1}{2}(x^T+c^T)A(x+c) + J^T(x+c)}.$$

$$= e^{-\frac{1}{2}x^TAx - \frac{1}{2}c^TAc \underbrace{- c^TAx + J^Tx}_{\to 0} + J^Tc}.$$

x^TAc?

$$J^T = c^TA \Leftrightarrow J = Ac \Leftrightarrow c = A^{-1}J$$

$$= e^{-\frac{1}{2}x^TAx - \frac{1}{2}J^T(A^{-1})^T J + J^TA^{-1}J}$$

$$= e^{-\frac{1}{2}x^TAx + J^T(\frac{1}{2A})J}$$

$$\Rightarrow \int dx_1 \cdots dx_n \, e^{-\frac{1}{2}x^TAx + Jx}$$

$$= \frac{(2\pi)^{n/2}}{\det A} e^{J^T \frac{1}{2A} J}$$

关联: $$\langle x_i x_j \rangle = \left. \frac{\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\int e^{-\frac{1}{2}x^TAx + Jx} Dx}{\int e^{-\frac{1}{2}x^TAx} Dx} \right|_{J=0}$$

$$= \left. \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} e^{J^T \frac{1}{2A} J} \right|_{J=0}$$

令 $\omega = J^T \frac{1}{2A} J$.

$$\left. \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} e^{\omega} \right|_{J=0} = \sim$$

eg: $\frac{\partial}{\partial J_i} e^{\omega} = \frac{\partial \omega}{\partial J_i} e^{\omega}$, ...

推广 $\mathbb{C}$.

$$\frac{\int dz\, d\bar{z}\, e^{-\frac{1}{2}A\bar{z}z}\, \bar{z}z}{\int dz\, d\bar{z}\, e^{-\frac{1}{2}A\bar{z}z}} = \langle \bar{z} z \rangle. \qquad z = x+iy.$$

$$= \frac{\int dx\, dy\, e^{-\frac{1}{2}A(x^2+y^2)}(x^2+y^2)}{\int dx\, dy\, e^{-\frac{1}{2}A(x^2+y^2)}}$$

推广.

$$e^{-\frac{1}{2}z^{+}Az} \quad e^{J^{+}z+Jz^{+}}$$

$$\begin{cases} z \to z+C \\ z^{+} \to z^{+}+C^{+} \end{cases}$$

然后仿照上述方式消去线性项.

$$\int Dz\, D\bar{z}\ e^{-\frac{1}{2}z^{+}Az}\, e^{J^{+}\frac{1}{2A}J} \quad , \quad U^{+}AU = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

相互作用与微扰.

$$\int dx\ e^{-\frac{1}{2}Ax^2 - \frac{\lambda}{4!}x^4} = e^{-F}$$

$$= \sum_{n=0}^{+\infty} \int dx\ e^{-\frac{1}{2}Ax^2} \underbrace{\left(-\frac{\lambda}{4!}\right)^n \left(\frac{1}{n!}\right)} x^{4n}$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!}\left(\frac{-\lambda}{4!}\right)^n \int dx\ e^{-\frac{1}{2}Ax^2} x^{4n}$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!}\left(\frac{-\lambda}{4!}\right)^n \frac{(4n)!}{(2n)!\,2^{2n}} \left(\frac{\sqrt{\lambda}}{A}\right)^{2n}$$

讨论: ① 收敛半径.

② 体会相互作用微扰. 分析它的收敛性.

下节课内容:

1. 发散问题: 1). 2d. $\delta(\vec{x})$ potential. (没有束缚态, 1d 有束缚态)

2) Casimir Force. $\propto \frac{\hbar c}{d^4}$

3) fine structure splitting. (g−2 ≠ 0). } 粒子物理.

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = -\frac{1}{12} \quad (\text{Zelta function}).$$

ϕ^4 理论. (ref. Peskin 的书).