path integral: ① Boson field ③ Gauge field $\int DA \quad \to A \sim A+\nabla\varphi$

② fermi field $\int D\psi D\bar{\psi}$ ④ spin field $\vec{\sigma}, \vec{S}$

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Gauss积分 + Feynman图 (Perturbation)

Gauss积分:(广泛,易理解,易推广)

ref: A.Zee $\quad \int dx\, e^{-Ax^2} = \sqrt{\frac{\pi}{A}} = B$

证明: Fubini公式 $\int f(x,y)\,dA = \int\left(\int (f(x,y)\,dx\right)dy = \int dy\left(\int dx\right)$

$$B^2 = \int dx\,dy\, e^{-A(x^2+y^2)} = 2\pi\int_0^\infty dr\, r\, e^{-Ar^2} = \pi\int_0^\infty dr\, e^{-Ar} = \frac{\pi}{A}$$

(一) $\quad B = \sqrt{\frac{\pi}{A}}$

$\int e^{-Ax^2}dx = \sqrt{\frac{\pi}{A}} \quad$ 问① $\int e^{-Ax^2+Jx}dx = \int e^{-A(x-\frac{J}{2A})^2+\frac{J^2}{4A}}dx = \sqrt{\frac{\pi}{A}}e^{\frac{J^2}{4A}}$

② $\int e^{-Ax^2}\cdot x^{2n+1} = 0 \quad$ 偶.奇

③ $\int e^{-Ax^2}\cdot x^{2n} = \quad \sum_{n=0}^{+\infty}\int e^{-Ax^2}\frac{J^{2n}}{(2n)!}x^{2n}dx$ 泰勒展开

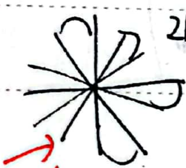
$$\int e^{-Ax^2}\cdot\frac{J^{2n}}{(2n)!}x^{2n} = \sqrt{\frac{\pi}{A}}\left(\frac{1}{4A}\right)^n\frac{J^{2n}}{n!} \quad (二)$$

2n阶 $\qquad$ n阶

$$\int e^{-Ax^2}\cdot x^{2n}dx = \sqrt{\frac{\pi}{A}}\cdot\frac{(2n)!}{n!}\left(\frac{1}{4A}\right)^n$$

$$\langle x^{2n}\rangle = \frac{\int e^{-Ax^2}\cdot x^{2n}dx}{\int e^{-Ax^2}dx} = \frac{(2n)!}{n!}\left(\frac{1}{4A}\right)^n = \frac{(2n)!}{n!\,2^{2n}}\left(\frac{1}{A}\right)^n$$

解释意义 $\quad \dfrac{\int e^{-Ax^2}\cdot x^{2n}dx}{\int e^{-Ax^2}dx}$



2n条 两两相连

共 $\dfrac{C_{2n}^2 C_{2n-2}^2 C_{2n-4}^2\cdots}{n!}$

Feynman diag $\quad = \dfrac{(2n)(2n-1)}{2}\dfrac{(2n-2)(2n-3)}{2}\cdots\dfrac{1}{n!}$

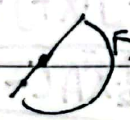
$$= \frac{(2n)!}{2^n\cdot n!} \times \left(\frac{1}{2A}\right)^n$$

权重 weight $\qquad$ 传播子 propagator

$+Jx$ ← 源 source

$$\int D\phi(k) \quad \frac{\int e^{-Ax^2}\cdot x^2dx}{\int e^{-Ax^2}dx} = \frac{-\frac{\partial}{\partial A}\sqrt{\frac{\pi}{A}}}{\sqrt{\frac{\pi}{A}}} = \sqrt{\frac{A}{\pi}}\left(-\frac{\partial}{\partial A}A^{-\frac{1}{2}}\right) = \frac{\frac{1}{2}A^{-\frac{3}{2}}}{\sqrt{A}} = \frac{1}{2A}$$

$$\frac{\int e^{-\frac{1}{2}Ax^2}\cdot x^2dx}{\int e^{-\frac{1}{2}Ax^2}dx} = \frac{1}{A}.$$



$\frac{1}{2A}$

扫描全能王 创建

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① $$\frac{\int dx_1\cdots dx_n e^{-\frac{1}{2}x^TAx}\cdot x_ix_j}{\int dx_1\cdots dx_n e^{-\frac{1}{2}x^TAx}} = \langle x_ix_j\rangle .$$

Gauss 积分 $\Leftrightarrow$ 多分量(实数)

② 复数场 $$\frac{\int dz\, d\bar{z}\, e^{-\frac{1}{2}A\bar{z}z}\ \bar{z}z}{\int dz\, d\bar{z}\, e^{-\frac{1}{2}A\bar{z}z}} = \langle \bar{z}z\rangle .$$

③ 多分量(复①) $$\frac{\int dz_1 dz_2\cdots d\bar{z}_1 d\bar{z}_2\, e^{-\frac{1}{2}A\bar{z}_1z_1}\cdots}{\cdots}$$

$$\int dx_1\cdots dx_n e^{-\frac{1}{2}x^TAx} = ? = \int dy_1\cdots dy_n e^{-\frac{1}{2}\sum_i \lambda_i y_i^2} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\lambda_1\cdots\lambda_n}} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(O^TAO)}} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(A)}}$$

1) 实数 $\Rightarrow$ $x = Oy,\ O\in O(N)$ Jacobi

$$\Leftrightarrow \int dx_1\cdots dx_n = \det\left(\frac{\partial x_i}{\partial y_i}\right) dy_1\cdots dy_n$$
$$= \det(O)\, dy_1\cdots dy_n$$
$$= dy_1\cdots dy_n$$

$$O^TAO = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$$

$$\int dx_1\cdots dx_n e^{-\frac{1}{2}x^TAx}\cdot x_ix_j =$$

补充: $\int d\phi(x_1)\cdots d\phi(x_n)\, e^{-\frac{1}{2}\phi^TA\phi}\,\phi(x_i)\phi(x_j)$

把 $x\to\phi$ 变到场.

$$\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\int dx_1\cdots dx_n e^{-\frac{1}{2}x^TAx + J^Tx}\quad J^Tx = J_ix_i$$

平移 $x\to x+c$ (此处 x 为向量) ...

$$e^{-\frac{1}{2}(x^T+c^T)A(x+c)+J^Tx+J^Tc}$$

$$= e^{-\frac{1}{2}x^TAx - \frac{1}{2}c^TAc\boxed{-\frac{1}{2}x^TAc - \frac{1}{2}c^TAx + J^Tx} + J^Tc}$$

$=0$,

$x^TAC = C^TAX$ $=0$

$= e^{-\frac{1}{2}x^TAx - \frac{1}{2}C^TAC - (C^TAx + J^Tx) + J^TC}$

$= e^{-\frac{1}{2}x^TAx - \frac{1}{2}J^T(A^{-1})^TA\cdot A^{-1}J + J^TA^{-1}J}$ $J^T = C^TA \Leftrightarrow J = AC$

$C = A^{-1}J$

$= e^{-\frac{1}{2}x^TAx + J^T(\frac{1}{2A})J}$

$= e^{-\frac{1}{2}x^TAx + J^Tx}$

则 $\int dx_1 \cdots dx_n\, e^{-\frac{1}{2}x^TAx + J^Tx}$

$= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(A)}} \cdot e^{J^T(\frac{1}{2A})J}$

$$\langle x_i x_j \rangle = \frac{\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} \int e^{-\frac{1}{2}x^TAx + J^Tx} Dx}{\int e^{-\frac{1}{2}x^TAx} Dx}\Bigg|_{J=0}$$

$\frac{(2\pi)^{\frac{n}{2}}}{\det(A)}$

$\left(\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} e^{\underbrace{J^T\frac{1}{2A}J}_{=w}}\right)\Big|_{J=0}$ $\frac{1}{2A}$要用余子式写出

$= \left(\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} e^{w}\right)\Big|_{J=0,\, w=0}$

$\frac{\partial}{\partial J_i}e^w = \frac{\partial w}{\partial J_i}e^w$

$= \frac{\partial}{\partial J_i}\left(\frac{\partial w}{\partial J_j}e^w\right)$

$= \left(\frac{\partial w}{\partial J_i}\right)\left(\frac{\partial w}{\partial J_j}\right)e^w + \left(\frac{\partial^2 w}{\partial J_i \partial J_j}\right)e^w$

$= \left(\frac{\partial^2 w}{\partial J_i \partial J_j} - \frac{\partial w}{\partial J_i}\cdot\frac{\partial w}{\partial J_j}\right)\Big|_{J=0}$

$= \frac{\partial^2 w}{\partial J_i \partial J_j}$

复变场中 $\langle \bar{z} z \rangle = \frac{\int dx\,dy\, e^{-\frac{1}{2}A(x^2+y^2)}(x^2+y^2)}{\int dx\,dy\, e^{-\frac{1}{2}A(x^2+y^2)}}$

$e^{-\frac{1}{2}z^+Az}$, $z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$, $z^+ = (z_1^+, z_2^+ \cdots z_n^+)$

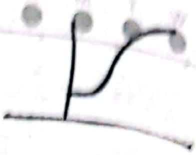
$\times e^{J^+z + z^+J}$

result : $\int DzD\bar{z}\, e^{-\frac{1}{2}z^+Az} + J^+\frac{1}{A}J$

$\begin{cases} z \to z + C \\ z^+ \to z^+ + C^+ \end{cases}$

$U^+AU = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ 对角化.

加入弱多体相互作用后积分形式为 与微扰



相互作用：$\int dx\, e^{-\frac{1}{2}Ax^2-\frac{\lambda}{4!}x^4} = e^{-F}$

$$=\sum_{n=0}^{+\infty}\int dx\, e^{-\frac{1}{2}Ax^2}\left(-\frac{\lambda}{4!}\right)^n\frac{1}{n!}x^{4n}$$

$$=\sum_{n=0}^{+\infty}\frac{1}{n!}\left(\frac{-\lambda}{4!}\right)^n\int dx\, e^{-\frac{1}{2}Ax^2}x^{4n}$$

（作业：体会相互作用微扰分析收敛性）.

☆

$$=\frac{1}{n!}\left(-\frac{\lambda}{4!}\right)^n\frac{(4n)!}{(2n)!\,2^{2n}}\left(\frac{1}{A}\right)^{2n}$$

收敛半径～0 $\left(\frac{\sqrt{\lambda}}{A}\right)^{2n}$

Renormalization Group 重整化群.

下节课：发散. (1) 2d $\delta(\vec{x})$ potential

(2) Casimir force $F\propto\frac{\hbar c}{d^4}$

Peskin.

(3) fine structure spliting 电子自旋 $(g-2)$.

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}=-\frac{1}{12}$

ϕ^4理论（发散，k空间 path integral）

$$\mathcal{L}=\left(\frac{\partial\phi}{\partial t}\right)^2-v^2\left(\frac{\partial\phi}{\partial x}\right)^2-m^2\phi^2-\frac{\lambda}{4!}\phi^4$$

☆ ϕ意义～坐标/振动大小 → 可能有Topo结构.

☆ $Z=\int D\phi\, e^{iS}$, $S=\int\mathcal{L}\,dt\,d\vec{x}$. Monte Carlo simulation.

path integral $D\phi=\lim_{N\to\infty}A^{N-1}d\phi(x_1)\cdots d\phi(x_N)$

☆ 求解：动量空间 $\prod_i d\phi(x_i)\to J\prod_k d\phi(k)$

一般为常数，例外：反常

原因：$\langle Q\rangle=\frac{\int P(x)Q\,dx}{\int P(x)\,dx}$ 与J无关

唯一可解N维积分 → Gauss积分.