因为场($\phi(x,t)$)同样是无穷多自由度，此考虑将 $S[x]$ 变化到场论中

$$\langle q_1 q_2 | e^{-iHT} | q_1' q_2' \rangle = \int Dq_1 Dq_2 \, e^{iS[q_1, q_2]}$$

$$\lim_{N\to\infty} \langle q_1 \cdots q_N | e^{-iHT} | q_1' \cdots q_N' \rangle = \int D\phi \, e^{iS[\phi]}$$

只有变换到动量空间，才有可能继续做下去

2.28

$$\prod_i \int D\phi(x_i) = J \prod_k \int D\phi(k) \qquad \phi(k) = \sum_i e^{ikx_i}\phi(x_i)$$

变为 Gauss integral Feynmann diagram

无穷维只有Gauss积分有解析形势

Path integral

① Boson Field ② Fermion Field ③ Gauge Field ④ Spin Field

$$\int D\psi D\psi^\dagger \qquad \int DA$$

Boson field: Gauss Integral (Ref. A. Zee

$$\int dx\, e^{-Ax^2} = \sqrt{\frac{\pi}{A}}$$

$$\int e^{-Ax^2+Jx} dx = \cdots = \sqrt{\frac{\pi}{A}}\, e^{\frac{J^2}{4A}}$$ → 左、右 Taylar series

$$\sum_{n=0}^{+\infty} \int e^{-Ax^2} \frac{J^{2n}}{(2n)!!} x^{2n} dx$$

$\int e^{-Ax^2} x^{2n} dx =$ 右边展开 $\frac{J^{2n}}{(2n)!!}$ 前的系数

$$\langle x^{2n} \rangle = \frac{(2n)!}{n!\, 2^n} \left(\frac{1}{2A}\right)^n$$

可用 $2n$ 个点两两配对，强度为 $\frac{1}{2A}$ 算 $\frac{C_{2n}^2 C_{2n-2}^2 \cdots C_2^2}{n!} \left(\frac{1}{2A}\right)^n$

source / perturbation

FUTURE PEOPLE

$$\frac{\int e^{-Ax^2+Jx} x^2 dx}{\int e^{-Ax^2} dx} = \cdots = \frac{1}{2A}$$

① $$\frac{\int dx_1 \cdots dx_n \, e^{-\frac{1}{2}x^T A x} x_i x_j}{\int dx_1 \cdots dx_n \, e^{-\frac{1}{2}x^T A x}} = \langle x_i x_j \rangle$$ Gauss integral $\Rightarrow$ multivariable

② $$\frac{\int dz d\bar{z} \, e^{-\frac{1}{2}A\bar{z}z} \bar{z} z}{\int dz d\bar{z} \, e^{-\frac{1}{2}A\bar{z}z}} = \langle \bar{z} z \rangle$$ $\Rightarrow$ Complex

③ $$\frac{\int dz_1 dz_2 \cdots d\bar{z}_1 d\bar{z}_2 \, e^{-\frac{1}{2}\bar{z}Az} \bar{z}_i z_j}{\int (\quad) e^{-\frac{1}{2}\bar{z}Az}} = \langle \bar{z}_i z_j \rangle \Rightarrow$$

计算：$\int dx_1 \cdots dx_n \, e^{-\frac{1}{2}x^T A x} = I$

变换 $x = Oy \qquad O \in O(N)$

$y = O^T x \qquad$ Jacobi det $= 1$

$$I = \int \left(\frac{\partial x_i}{\partial y_j}\right) dy_1 dy_2 \cdots dy_n \, e^{-\frac{1}{2}x^T A x}$$

把 A 对角化：$x^T A x = y O^T A O y \rightarrow$ diag : $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$= \int \left(\frac{\partial x_i}{\partial y_j}\right) dy_1 \cdots dy_n \, e^{-\frac{1}{2}\sum_i y_i^2 \lambda_i} = \sqrt{\frac{(2\pi)^n}{\prod_i \lambda_i}}$$

计算 $\int dx_1 \cdots dx_n \, e^{-\frac{1}{2}x^T A x + J^T x}$

$$e^{-\frac{1}{2}x^TAx+J^Tx} = e^{-\frac{1}{2}(x^T+C^T)A(x+C)+J^Tx+J^TC}$$

$$= e^{-\frac{1}{2}x^TAx - \frac{1}{2}C^TAC \underbrace{- \frac{1}{2}x^TAC - \frac{1}{2}C^TAx + J^Tx + J^TC}_{\text{让这几项相消}}}$$

$C = A^{-1}J$

$$= e^{-\frac{1}{2}x^TAx - \frac{1}{2}C^TAC + J^TC}$$

$$\langle x_i x_j \rangle = \frac{\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\int e^{-\frac{1}{2}x^TAx+J^Tx} Dx}{\int e^{-\frac{1}{2}x^TAx} Dx}$$

$$= \left(\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} e^{\underbrace{\frac{1}{2}J^T A^{-1} J}_{w}}\right)\Bigg|_{J=0}$$

$$= \left(\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} e^{w}\right)_{J=0}$$

$$\left(\frac{\partial}{\partial J} e^w = \frac{\partial w}{\partial J} e^w\right) \quad \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} e^w = \frac{\partial}{\partial J_i}\left(\frac{\partial w}{\partial J_j} e^w\right)$$

$$= \frac{\partial w}{\partial J_i}\frac{\partial w}{\partial J_j} e^w + \frac{\partial^2 w}{\partial J_i \partial J_j} e^w$$

推广到复数

$$e^{-\frac{1}{2}z^\dagger A z + J^\dagger z + z^\dagger J} \quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \quad z^\dagger = (z_1^\dagger \cdots z_n^\dagger)$$

$$\Rightarrow \int Dz D\bar{z} \, e^{-\frac{1}{2}z^\dagger A z + J^\dagger \frac{1}{2} A^{-1} J}$$

相互作用与微扰

$$\int dx\ e^{-\frac{1}{2}Ax^2-\frac{\lambda}{4!}x^4} = e^{-F}$$

$$=\sum_{n=0}^{\infty}\int dx\, e^{-\frac{1}{2}Ax^2}\left(-\frac{\lambda}{4!}\right)^n\frac{1}{n!}x^{4n}$$

3.3.

$$\mathcal{L} = \left(\frac{\partial\phi}{\partial T}\right)^2 - v^2\left(\frac{\partial\phi}{\partial x}\right)^2 - m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

☆ ϕ 含义～坐标/振动大小

☆ $Z = \int D\phi\, e^{iS}$ $\quad S = \int \mathcal{L}\, d\vec{x}\, dt$

path integral

一般为常数，偶尔有反常

☆ 求解：动量空间 $\prod_i d\phi(x_i) \longrightarrow J\prod_k d\phi(k)$

J 为常数时不重要，因为 $\langle Q\rangle = \dfrac{\int P(x)Q\,dx}{\int P(x)dx}$

唯一可解N维积分 ⟹ Gauss积分

恒等式方程

① $\dfrac{\int dx\ e^{-\frac{A}{2}x^2}x^2}{\int dx\, e^{-\frac{A}{2}x^2}} = \dfrac{1}{A}$ ② $\dfrac{\int dz d\bar{z}\, e^{-A\bar{z}z}\,\bar{z}z}{\int dz d\bar{z}\, e^{-A\bar{z}z}} = \dfrac{1}{A}$

发散问题

目的：说明 ∞ 无处不在

e.g. 点电荷产生电场的能量 $E \propto \int \frac{e^2}{r^4}d^3\vec{r} = 4\pi\left(\frac{1}{r}\right)\Big|_0^{+\infty}$

例1. 2dim $\delta(\vec{x})$ potential

ref. Nyeo. Am.J.Phys. 2000.