2022.2.24 第1周第2节课.

上节课回顾: $\phi(x,t)$

$$\frac{\partial^2}{\partial t^2}\phi - \frac{\partial^2}{\partial x^2}\phi + m^2\phi = 0. \Rightarrow$$ 波函数.

$\phi(x,t)$表示 x点t时刻振幅. $\longleftrightarrow$ q_α or x_α一样.

量子化 $[x,p]=i\hbar \Rightarrow [x_i, p_j] = i\hbar\delta_{ij} \longrightarrow$ 类比.

$$[\phi(x), \pi(y)] = i\hbar\delta(x-y)$$

$$\mathcal{L} = \frac{m}{2}\sum_\alpha \dot{q}_\alpha^2 - \frac{k}{2}(q_\alpha - q_{\alpha+1})^2 :$$

Remark. $\frac{m}{2}$. $\frac{k}{2}\phi^2$. $\frac{g}{3!}\phi^3$ or $\frac{\lambda}{4!}\phi^4$.
为了方便求导之后消去系数.

$$k q_\alpha q_{\alpha+1}$$

$$q_\alpha - q_{\alpha+1} = \sqrt{\Delta x}\,[\phi(x_\alpha) - \phi(x_{\alpha+1})]$$
$$= (\sqrt{\Delta x}\, a)\frac{\partial\phi}{\partial x}$$

$$\mathcal{L} = \frac{m}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{ka^2}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \Delta^2\phi^2.$$ Klein-Gordon Eq.

作用量: $$S = \int dx\,dt\left(\frac{m}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{ka^2}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \Delta^2\phi^2\right)$$

ϕ位移 $\Leftrightarrow$ $\frac{\partial\phi}{\partial t}$ 速度 $\qquad \frac{1}{2}m\dot{x}^2 - \frac{m}{2}\omega^2 x^2$

Eq of motion. $\Rightarrow \delta S = 0$.

$S[\phi+f] - S[\phi] = 0$ 对$\forall f$成立.

$$(\partial_t\phi - \partial_t f)^2 - (\partial_t\phi)^2 = 0$$

$$\underset{\substack{\uparrow\\ \text{小量}}}{(\partial_t f)^2} - 2\partial_t\phi\,\partial_t f = 0 \Rightarrow 2\partial_t\phi\,\partial_t f = 0$$

$$\underset{\substack{\uparrow\\ \text{全微分}}}{2\partial_t((\partial_t\phi)f)} - 2(\partial_t^2\phi)f = 0.$$
全微分 + 其它.

$$\int_{t_i}^{t_f} dt\,\partial_t(Q) = Q(t_f) - Q(t_i) = \underset{\searrow 0}{f(t_f)}(\cdots) - \underset{\searrow 0}{f(t_i)}(\cdots) = 0.$$

② ∂_t or $(\partial_t)^2$

$$\frac{m}{2}\cdot(-2)f\,(\partial_t^2\phi)-\frac{k^2a^2}{2}(-2)\,f\left(\frac{\partial^2\phi}{\partial x^2}\right)-\frac{\Delta^2}{2}(+2)f\,\phi=0$$ 对 $\forall f$ 成立

$$m\,\partial_t^2\phi-k^2a^2\frac{\partial^2\phi}{\partial x^2}+\Delta^2\phi=0$$

$$\Longleftrightarrow\qquad \phi(x,t)=\sum_{k,\omega}e^{i(k\cdot x-\omega t)}$$

$$-m\omega^2+k^2a^2k^2+\Delta^2=0.\quad\Rightarrow\quad m\omega^2=k^2a^2k^2+\Delta^2$$ 声子方程.

类比于 $E^2=k^2c^2+m^2$

or $E^2=k^2c^2+\Delta^2$. (Δ: gap).

质能方程.

$E=kc$.

关键性现象:

质能方程 $\Leftrightarrow$ 声子方程. (光子 $\leftrightarrow$ 声子).

推广: ϕ: 振幅.

1) ϕ: 标量 $\Rightarrow$ 声子. $\left(\frac{\partial\phi}{\partial t}\right)^2-\left(\frac{\partial\phi}{\partial x}\right)^2-\Delta^2\phi^2$ (gap)

2) 矢量: $\phi=\vec{A}$. $\left(\frac{\partial\vec{A}}{\partial t}\right)^2-\underbrace{\left(\frac{\partial}{\partial x}\vec{A}\right)^2}$

$\frac{\partial}{\partial x}\vec{A}\begin{cases}\nabla\cdot\vec{A}\\ \nabla\times\vec{A}.\end{cases}$

$$\left(\frac{\partial\vec{A}}{\partial t}\right)^2-(\nabla\times\vec{A})^2-\frac{\Delta^2}{2}\vec{A}^2$$ ← Meisser Effect. (SC) gap. 产生一个无能隙的规范玻色子.

$$=\vec{E}^2-\vec{B}^2$$

推广: NLσM. (Nonlinear Sigma Model).

$\phi=(\phi_1,\phi_2,\cdots,\phi_N)$.

$$\mathcal{L}=\sum_i\frac{1}{2}(\partial_t\phi_i)^2-\frac{1}{2}(\partial_x\phi_i)^2-\frac{\Delta^2}{2}\phi^\dagger\phi-\frac{\lambda}{4!}(\phi^\dagger\phi)^2$$ ($\sum_i\phi_i^2$)

推广: 如果 ϕ 是一个矩阵. $\phi\Rightarrow U(N)$. $O(N)$ $\cdots$ (NLσM / LσM).

$$\mathcal{L}=\mathrm{Tr}\left[(\partial_t U^{-1})(\partial_t U)-(\partial_x U^{-1})(\partial_x U)\right]$$

补充. Tensor 标记.

规定: $A^2 = \sum_i A^i A_i = A^i A_i$ (Einstein 规定).

$A^i = g_{ij} A^j$ 反之 $A_i = A_j g^{ij}$, $g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

$(\partial\phi)^2 = \partial^i \phi \, \partial_i \phi$;

$$\begin{cases} \partial^i = \partial_t - \partial_x \\ \partial_i = \partial_t + \partial_x \end{cases}$$

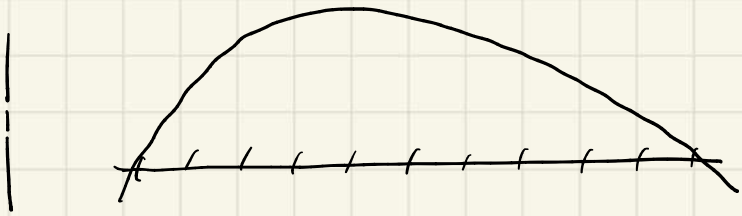
Path Integral.

为什么量子力学方程 $H\psi = E\psi = i\frac{\partial}{\partial t}\phi$. 不用 L. 也没有出现作用量?

Feynman (Lecture. III) (量子干涉 的理解).

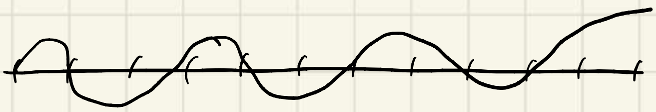
A.Zee. Chapter 1.2

- ★ 双缝干涉.
- ★ 故事.
- ★ 类比.



$\langle x | e^{-iHT} | y \rangle$ 单粒子.

$\langle x_1, x_2, \cdots, x_N | e^{-iHT} | y_1, y_2, \cdots, y_N \rangle$ 多粒子.



$N \to \infty$.

$\langle \phi(x) | e^{-iHT} | \phi(y) \rangle$ ← Path Integral of Field ← $N \to +\infty$.

$N=1$. $\psi = \psi_1$

$N=2$. $\psi = \psi_1 + \psi_2$

$N=3$. $\psi = \psi_1 + \psi_2 + \psi_3$

...

$N=\infty$. $\psi = \sum_i \psi_i$. 但 ψ_i 并不能表示. 如何求解?

⇓

Dirac 方法.

⇓

Feynman 实现.

$$H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\nabla^2 + V(x)$$

$$\langle x | e^{-iHT} | y \rangle = \langle x, T | y, 0 \rangle$$

$\langle x|$: T时刻 $\quad |y\rangle$: 0时刻.

$$= \langle x | e^{-iH\delta\varepsilon} e^{-iH\delta\varepsilon} e^{-iH\delta\varepsilon} \cdots e^{-iH\delta\varepsilon} | y \rangle \qquad (\delta\varepsilon \to 0).$$

$\hat{I}$ 单位阵插入. $\int dx |x_i\rangle\langle x_i| = \hat{I}$.

$$= \int \langle x | e^{-iH\delta\varepsilon} | x_1 \rangle \langle x_1 | e^{-iH\delta\varepsilon} | x_2 \rangle \langle x_2 | \cdots | x_n \rangle \langle x_n | e^{-iH\delta\varepsilon} | y \rangle \, dx_i$$

$$\langle x_i | e^{-iH\delta\varepsilon} | x_{i+1} \rangle$$

$$= \langle x_i | e^{-i\frac{p^2}{2m}\delta\varepsilon} e^{-iV(x)\delta\varepsilon} | x_{i+1} \rangle$$

$$= \langle x_i | e^{-i\frac{p^2}{2m}\delta\varepsilon} | x_{i+1} \rangle e^{-iV(x_{i+1})\delta\varepsilon}.$$

$$e^{A\delta\varepsilon + B\delta\varepsilon} = e^{A\delta\varepsilon} e^{B\delta\varepsilon} (1 + O(\delta\varepsilon^2))$$

$$\simeq e^{A\delta\varepsilon} e^{B\delta\varepsilon};$$

$$\int \langle x | e^{-i\frac{p^2}{2m}\delta\varepsilon} | k \rangle \langle k | x \rangle \, dk \quad \frac{1}{2\pi}$$

$$\langle k | x \rangle = e^{i\vec{k}\cdot\vec{x}}$$

$$= \int e^{-i\frac{p^2}{2m}\delta\varepsilon} e^{i\vec{k}\cdot(\vec{x}_{i+1} - \vec{x}_i)} \, dk,$$

$$\int e^{Ax^2 + Bx} dx = \int e^{Ax^2} dx \, e^{B^2/4A};$$

$$= \sqrt{\frac{\pi}{A}} e^{B^2/4A};$$

$$= \bigcirc \; e^{\frac{(x_{i+1} - x_i)^2}{4(\delta\varepsilon/2m)} i}$$

约当定理. $2\pi i \, P(x)$

~~经典对照~~: $P = m\frac{x_{i+1} - x_i}{\delta\varepsilon}$;

$$e^{i\frac{\delta\varepsilon}{m}\cdot m^2 \left(\frac{x_{i+1} - x_i}{\delta\varepsilon}\right)^2}$$

$$= e^{i\frac{(x_{i+1} - x_i)^2}{\delta\varepsilon} m}$$

$$\langle x | e^{-iHT} | y \rangle \propto \lim_{N\to\infty} \int dx_1 \cdots dx_N \left(e^{iS} = e^{i\int L \, dt} \right).$$

$$= \int Dx \, e^{iS[x]}$$

泛函.

($d = \infty$ integral.).

$$\bigcirc^N = \left(\frac{-im}{2\pi\hbar \, \delta t} \right)^{\frac{N}{2}};$$

场 $\int D\phi \, e^{iS[\phi]}$. 的积分 $(N \to \infty)$;