2.24 ← 这是一个场，非波函数

$\phi(x,t)$：表示x点t时刻振幅 (这里x是空间位置)

量子化：$[x,p]=i\hbar \Rightarrow [x_i,p_j]=i\hbar\delta_{ij}$ (这里的x是粒子的位移)

$$\mathcal{L}=\frac{m}{2}\sum_a \dot{q}_a^2 - K(q_a-q_{a+1})^2$$

def: $q_a - q_{a+1} = \sqrt{\Delta x}\,[\phi(x_a)-\phi(x_{a+1})]$

$$= (\sqrt{\Delta x}\,a)\left(\frac{\partial\phi}{\partial x}\right)\Big|_a$$

拉氏量密度

$$\mathcal{L} = \frac{m}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{k^2a^2}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \Delta^2\phi^2$$ Klein-Gorden方程

拉氏量 $L=\int\mathcal{L}\,dx$ 作用量 $S=\int\mathcal{L}\,dt\,dx$

求运动方程: $\delta S = 0$

$$S[\phi+f]-S[\phi]=0 \quad \text{for any } f \quad O(f)=0$$

$$(\partial_t\phi+\partial_t f)^2-(\partial_t\phi)^2$$

舍 →

$$= (\partial_t f)^2 + 2\partial_t\phi\,\partial_t f = 2\partial_t\big((\partial_t\phi)f\big)-2(\partial_t^2\phi)f$$

$$\frac{m}{2}(-2)f(\partial_t^2\phi) - \frac{k^2a^2}{2}(-2)f\frac{\partial^2\phi}{\partial x^2} - \frac{\Delta^2}{2}2f\phi = 0$$

$$\therefore\ m\partial_t^2\phi - k^2a^2\frac{\partial^2\phi}{\partial x^2} + \Delta^2\phi = 0$$

解之，令 $\Longleftrightarrow$ $\phi(x,t)=\sum_{k\omega} e^{i(k\cdot x-\omega t)}\varphi(k,\omega)$

$$-m\omega^2 + k^2a^2k^2 + \phi^2 = 0 \qquad \left(\frac{\phi}{\omega}\right) E = \sqrt{k^2c^2+m^2}$$

key results:

mass-energy Eq $\Leftrightarrow$ phonon Eq (photon $\Leftrightarrow$ phonon)

generalize: ϕ: amplitude

1) ϕ scalar $\Rightarrow$ phonon: $\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \delta^2\phi^2$ gap

2) vector $\phi \to \vec{A} \Rightarrow$ $\left(\frac{\partial\vec{A}}{\partial t}\right)^2 - (\nabla\times\vec{A})^2 - \frac{\Delta^2}{2}A^2$ Meissnereffe gap

$= \vec{E}^2 - \vec{B}^2$

nolinear σ Model

$\phi = (\phi_1, \phi_2, \phi_3, \cdots \phi_N)$

$\sum_i \phi_i^2$

$\mathcal{L} = \sum_1 \frac{1}{2}(\partial_t\phi_i)^2 - \frac{1}{2}(\partial_x\phi_i)^2 - \frac{\Delta^2}{2}\phi^\dagger\phi = \frac{\lambda}{4!}(\phi^\dagger\phi)^2$

for ϕ is a matrix, say, a unitary matrix $\phi = U(N)$

$\mathcal{L} = Tr\left[(\partial_t U^{-1}\partial_t U) - (\partial_x U^{-1})(\partial_x U)\right]$

some tricks:

Tensor

$(\partial_t\phi)^2 - (\partial_x\phi)^2 = (\partial\phi)^2$

def: $A^2 = A^i A_i \quad A^i = g^{ij}A_j \quad A_i = g_{ij}A^j$

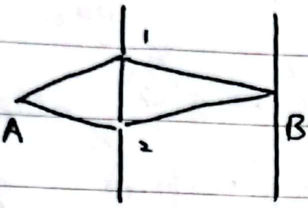
$g_{ij} = \mathrm{diag}(1, -1, -1, -1) \overset{2d}{=\!=} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

set $\partial_i = (\partial_t, \partial_x)$ $g^i = (\partial_t, -\partial_x)$

then $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$

Path integral (A. Zee. Chapter 1.2)

Young's double-slit



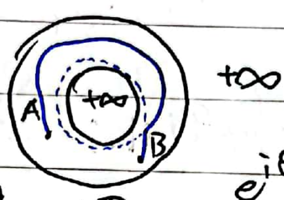
$\psi = \psi_1 + \psi_2 = \psi(e^{i\theta_1} + e^{i\theta_2})$

如果有N条缝：$\psi = \psi(e^{i\theta_1} + e^{i\theta_2} + \cdots + e^{i\theta_N})$

⇒如果有∞条缝？即板不存在时：

从A到B的概率幅是所有可能路径概率幅之和（积分）！

[topo structure:



from A to B：绕转一圈对相位有影响 $e^{i\theta}$]

$$H = \frac{p^2}{2m} + V(x) = \frac{\hbar^2}{2m}\nabla^2 + V(x)$$

t-displacement operator

$$\langle x,0|\, e^{iHT} \,|y,0\rangle = \langle x,t|y,0\rangle \qquad t = \delta\varepsilon + \cdots + \delta\varepsilon$$

$$= \langle x|\, e^{-iH\delta\varepsilon}\, e^{-iH\delta\varepsilon} \cdots e^{-iH\delta\varepsilon} \,|y\rangle$$

$$= \int \langle x|\, e^{-iH\delta\varepsilon} \,|x_1\rangle\langle x_1|\, e^{-iH\delta\varepsilon} \,|x_2\rangle\langle x_2|\, e^{-iH\delta\varepsilon} \cdots |y\rangle \, dx_1 \cdots dx_n$$

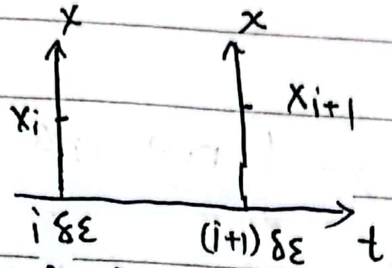
$$\left(\int |x_i\rangle\langle x_i|\, dx_i = \int |x_j\rangle\langle x_j|\, dx_j\right)$$

$$\langle x_i|\, e^{-iH\delta\varepsilon} \,|x_{i+1}\rangle = \langle x_i|\, e^{-i\frac{p^2}{2m}\delta\varepsilon}\, e^{-iV(x)\delta\varepsilon} \,|x_{i+1}\rangle$$

$$= \langle x_i|\, e^{-i\frac{p^2}{2m}\delta\varepsilon} \,|x_{i+1}\rangle\, e^{-iV(x)\delta\varepsilon}$$

下面计算 $\langle x_i| e^{-i\frac{p^2}{2m}\delta\varepsilon}|x_{i+1}\rangle\Big|_{\delta\varepsilon\to 0}$

直观解释：



$$p=\frac{x_{i+1}-x_i}{\delta\varepsilon}$$

$$\langle x_i|\, e^{-i\frac{\delta\varepsilon}{2m}\left(\frac{x_{i+1}-x_i}{\delta\varepsilon}\right)^2}|x_{i+1}\rangle = e^{-i\frac{\delta\varepsilon}{2m}(\ )^2}\langle x_i|x_{i+1}\rangle = e^{-i\frac{\delta\varepsilon}{2m}m^2(\ )}$$

严格计算

$$=\int \langle x_i|e^{-i\frac{p^2}{2m}\delta\varepsilon}|k\rangle\langle k|x_{i+1}\rangle\, dk\,\frac{1}{2\pi} \qquad \langle k|x\rangle = e^{i\vec{k}\cdot\vec{x}}$$

$$=\int\frac{1}{2\pi}e^{-i\frac{k^2}{2m}\delta\varepsilon}e^{ik(x_{i+1}-x_i)}dk$$ 这是一个 Fourier Transformat

$$=\square\frac{(x_{i+1}-x_i)^2}{4i(\delta\varepsilon/2m)}$$

有一系数

回到 $\langle x|e^{-iHT}|y\rangle \propto \int dx_1\cdots dx_n\, e^{-ii\sum_i\left[\frac{m}{2}\left(\overbrace{\frac{x_{i+1}-x_i}{\delta\varepsilon}}^{\dot{x}}\right)^2\delta\varepsilon - iV(x_{i+1})\delta\varepsilon\right]}$

$$\sum_i V(x_{i+1})\delta\varepsilon = \int V(x)dt$$

$$\propto\int dx_1dx_2\cdots dx_n\, e^{iS}$$

$$S=\int\left(\frac{m}{2}\dot{x}^2 - V(x)\right)dt = \int L\,dt$$

最终写成

$$\langle x|e^{-iHT}|y\rangle \propto \lim_{N\to\infty}\int dx_1\cdots dx_N\, e^{iS}. \qquad \infty\text{-dim integral}$$

$$=\int Dx\, e^{iS[x]}$$

因为场($\phi(x,t)$)同样是无穷多自由度，考虑将 $S[x]$ 变化到场论中

$$\langle q_1 q_2 | e^{-iHT} | q_1' q_2' \rangle = \int Dq_1 Dq_2 \, e^{iS[q_1, q_2]}$$

$$\lim_{N\to\infty} \langle q_1 \cdots q_N | e^{-iHT} | q_1' \cdots q_N' \rangle = \int D\phi \, e^{iS[\phi]}$$

只有变换到动量空间，才有可能继续做下去

2.28

$$\prod_i \int D\phi(x_i) = J \prod_k \int D\phi(k) \qquad \phi(k) = \sum_i e^{ikx_i} \phi(x_i)$$

变为 Gauss integral　Feynmann diagram

无穷维只有 Gauss 积分有解析形式

Path integral

① Boson Field ② Fermion Field ③ Gauge Field ④ Spin Field

$\int D\psi D\psi^{+}$　　$\int DA$

Boson field: Gauss Integral (Ref. A. Zee

$$\int dx \, e^{-Ax^2} = \sqrt{\frac{\pi}{A}}$$

$$\int e^{-Ax^2 + Jx} dx = \cdots = \sqrt{\frac{\pi}{A}} \, e^{\frac{J^2}{4A}} \rightarrow \text{左、右 Taylar series}$$

$$\sum_{n=0}^{+\infty} \int e^{-Ax^2} \frac{J^{2n}}{(2n)!!} x^{2n} dx$$

$\int e^{-Ax^2} x^{2n} dx =$ 右边展开 $\frac{J^{2n}}{(2n)!!}$ 前的系数

$$\langle x^{2n} \rangle = \frac{(2n)!}{n! \, 2^n} \left(\frac{1}{2A}\right)^n$$

可用 $2n$ 个点两两配对，连线为 $\frac{1}{2A}$ 得 $\frac{C_{2n}^2 C_{2n-2}^2 \cdots C_2^2}{n!} \left(\frac{1}{2A}\right)^n$