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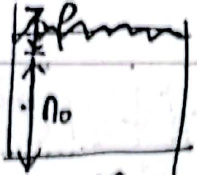
近似 $n=n_0+\rho$



和以前的处理不同 $\psi=e^{i\theta}\sqrt{n}$

textbook中 n和θ是数 ψ还是玻色子.

$$[\psi(x),\psi^+(y)]=\delta(x-y) \Longleftrightarrow \begin{cases}[n(x),\theta(y)]=i\delta(x-y)\\ \psi=e^{i\theta}\sqrt{n}\end{cases}$$ 构成完整描述

$$\hat{H}=\int dx\frac{\hbar^2}{2m}\left(\frac{d}{dx}\psi^+\right)\left(\frac{d}{dx}\psi\right)+g\int(\psi^+(x)\psi(x))^2dx$$

极限下 $n\doteq n_0+\rho$, $|\rho|<<n_0$

$$=\int dx\frac{\hbar^2 n_0}{2m}(\partial_x\theta)^2+g\int(n_0+\rho)^2dx$$ → 交叉项 $\int n_0\rho dx=0$

$$=(const)+\boxed{\int dx\frac{\hbar^2 n_0}{2m}(\partial_x\theta)^2+g\int\rho^2dx}.$$ 两个自由度 $\begin{cases}\text{相位场}\\ \text{密度场}\end{cases}$

$\int g n_0^2dx$ 背景.

显然 $\rho(x)=\frac{1}{\sqrt{L}}\sum_q e^{iqx}\rho_q$ 代回

$\theta(x)=\frac{1}{\sqrt{L}}\sum_q e^{iqx}\theta_q$

得 $H=\frac{\hbar^2 n_0}{2m}\sum_q q^2\theta_q\theta_{-q}+g\sum_q\rho_q\rho_{-q}$

很像 $H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2$ 条件: $[x,p]=i\hbar$

考虑 ρ与θ的对易关系.

$[\rho(x),\theta(y)]=\frac{1}{L}\sum_{qq'}[\rho_q,\theta_{q'}]e^{iqx+iq'y}$

令$L=1$ $=i\delta(x-y)=i\sum_p e^{ip(x-y)}$

唯一解: $q+q'=0$

$[\rho(x),\theta(y)]=\sum_q[\rho_q,\theta_{-q}]\cdot e^{iq(x-y)} \Longleftrightarrow$

$[\rho_q,\theta_{-q}]=i$

注: $q^2\theta_q\theta_{-q}=(-q)^2\theta_{-q}\theta_q$

$H=\sum_{q>0}\frac{\hbar^2n_0q^2}{m}\theta_q^*\theta_q+2g\rho_q^*\rho_q$ 坐标. 动量

其中 $[\rho_q,\theta_q^*]=i$

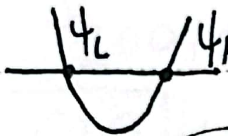
条件: $[\rho_q,\rho_{q'}]=0$, $[\theta_q,\theta_{q'}]=0$

$\Leftrightarrow \omega_q = c|q|$ (无能隙激发)

结论: g不能打开gap, 只改变 v(声速).

关联函数 $\langle\psi^\dagger(x)\psi(y)\rangle \sim \frac{1}{|x-y|}$ 与费米子关联函数相同.

总结:



ψ_L ψ_R $\Rightarrow$ Dirac eq

$\Rightarrow$ 打开gap $\psi_R^\dagger\psi_L^\dagger \Rightarrow$ 超导

$\int V\cos(2k_F x) + \psi^\dagger\psi \Rightarrow$ 能带

Fermi面附近

Luttinger Liquid

相变过程 $\longrightarrow (\frac{V}{2}\psi_R^\dagger\psi_L + h.c.)$

$\Rightarrow$ 1D BEC $\left|\begin{array}{l}\psi^\dagger=\sqrt{\rho}\,e^{-i\theta}\\ \psi=e^{i\theta}\sqrt{\rho}\end{array}\right.$ $\Leftrightarrow [\rho,\theta]=i\delta(x-y)$ 粒子数.相位

证据: 1d下 F/B/Anyon $\Rightarrow$ 相似性质

流体/Bose描述

Jordan-wigner $\Rightarrow$ 改性 $\Rightarrow$ string

$\sigma_i = \exp[i\sum_{j=-\infty}^{i-1}\pi c_j^\dagger c_j]\, c_i$ 连续化 $\rightarrow \pi\int_{-\infty}^{x}\rho(x)dx$

① string F $\Rightarrow$ spin, Bose $\Rightarrow \sqrt{\rho}\,e^{-i\theta}$ ← phase fluctuation

类比: $\psi_F(x) = \exp[i\pi\int_{-\infty}^{x}\rho(x)dx]\,e^{-i\theta(x)}\sqrt{\rho(x)}$

$\psi_F(x) = \sqrt{\rho_0}\,e^{\pm i\pi\int_{-\infty}^{x}\rho(x')dx' - i\theta(x)}$

$= \sqrt{\rho_0}\,e^{i\Phi}$ $\quad (\Phi = \underbrace{\pm\pi\int_{-\infty}^{x}\rho(x')dx'}_{改性} - \theta)$ 相位涨落 Bose field

$e^{i\pi c^\dagger c} = e^{-i\pi c^\dagger c}$

$e^{i\pi} = e^{-i\pi}$ ($\pm$ 对应左右行波)

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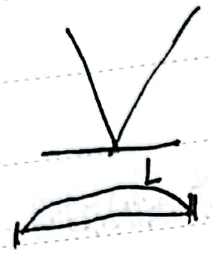
② $Z_B = Z_F$ （另一个证据：直接计算）

Fermion $\varepsilon_k = v|k|$, $k = \frac{n\pi}{L}$ 左右行波



$$Z_F = \left[\prod_{n=1}^{+\infty}\left(1+e^{-\beta v\frac{n\pi}{L}}\right)\right]^{2} = e^{-\beta F}$$

$$-\beta F = 2\sum_{n=1}^{+\infty}\ln\left(1+e^{-\beta\frac{\pi v}{L}n}\right)$$

令 $a = \beta\frac{\pi v}{L} \ll 1$

$$= 2\int_0^{+\infty}\ln(1+e^{-an})dn$$

$$= \frac{2}{a}\int_0^{+\infty}\ln(1+e^{-x})dx$$

$$= \frac{2}{a}\cdot\frac{\pi^2}{12} = \frac{\pi^2}{6a}$$

Boson field $\varepsilon_k = v|k|$

$$Z_B = \prod_{n=1}^{+\infty}\frac{1}{1-e^{-\frac{\beta v n\pi}{L}}} = e^{-\beta F}$$

$$-\beta F = -\sum_{n=1}^{+\infty}\ln(1-e^{-an})$$

$$= -\int_0^{+\infty}\ln(1-e^{-an})dn$$

$$= -\frac{1}{a}\int_0^{+\infty}\ln(1-e^{-x})dx$$

$$= \frac{\pi^2}{6a}.$$

⇒ 证明 C_v 所有性质都一样

1933-1934 F.Bloch Sound Wave 理论.

⇒ 1953年 Tomonaga 朝永振一郎

Interacting fermion ⇒ Bose 激发

缺点：过多近似

⇒ 1963 Luttinger

1) Mahan书

2) Tomonage 论文 ⇒ Remarks on Bloch sound waves applied to many fermion problem" 1950 重点，图像⇒公式

Tomonga-Luttinger liquid.

Introduction: 出发点 Fermi liquid Theory (单电子近似)

$m \Rightarrow m^*$ move independently, no correlation

而 [LL] 无单粒子激发，只有集体行为

主要思想: $N_F=1$, $v\sum_k k f_k^+ f_k \leftrightarrow \omega b^+ b$

$|1100000\rangle$ $v\frac{\pi}{L} \leftrightarrow \omega$

$|1010000\rangle$ $v\cdot\frac{2\pi}{L} \leftrightarrow 2\omega$

$|1001000\rangle$ $v\cdot\frac{3\pi}{L} \leftrightarrow 3\omega$

$\boxed{C_i^+C_j}|1100000\rangle$ 算子 $C_i^+C_j$

$C_i^+C_j$ — Fermion $\Rightarrow$ 一个空间到另一个空间; Boson $\Rightarrow$ 提升 } Tomonga 证明

费米 $\rightarrow \rho_q=\sum_n C_n^+C_{n+q} \propto b_q$ ← 近似

$\rho_q^+ \propto b_q^+$

Bose 等效于 particle-Hole pair

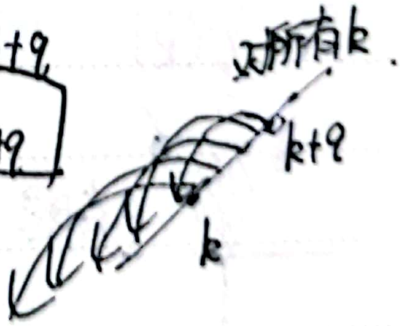
$\rho(x)=\psi^+(x)\psi(x)$, $\psi(x)=\frac{1}{\sqrt{L}}\sum_k e^{ikx}C_k$

$\rho(x)=\frac{1}{L}\sum_{kk'} e^{-ikx+ik'x} C_k^+C_{k'}$

令 $k'=k+q$ 则 $\rho(x)=\frac{1}{L}\sum_k\sum_q e^{iqx}C_k^+C_{k+q}$

$=\frac{1}{L}\sum_q e^{iqx}\boxed{\sum_k C_k^+C_{k+q}}$

$=\frac{1}{L}\sum_q e^{iqx}\rho_q$



$\rho_q=\sum_q C_k^+C_{k+q}$

明确意义 $\rightarrow \rho_q\propto b_q$, $\rho_q^+\propto b_q^+$

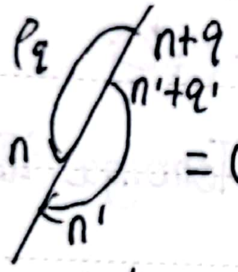
(2) 计算二者对易关系. $[\rho_q,\rho_{q'}]=?$

基本公式 $[C_i^+C_j, C_k^+C_l]$

$$[c_i^+c_j, c_k^+c_l]$$
$$= c_i^+\underline{c_j c_k^+}c_l - c_k^+\underline{c_l c_i^+}c_j$$
$$= c_i^+(\delta_{jk} \pm c_k^+c_j)c_l - c_k^+(\delta_{li} \pm c_i^+c_l)c_j$$
$$= \cancel{\pm c_i^+c_k^+c_jc_l \pm c_k^+c_i^+c_lc_j} + \delta_{jk}c_i^+c_l - \delta_{li}c_k^+c_j$$

讨论 $[\rho_q, \rho_{q'}] = \sum_{nn'} [c_n^+c_{n+q}, c_{n'}^+c_{n'+q'}]$

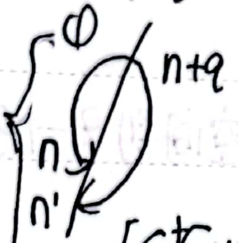
$$= \sum_n c_n^+c_{n'+q'}\Big|_{n'=n+q} - c_{n'}^+c_{n+q}\Big|_{n'+q'=n}$$
$$= \sum_n (c_n^+c_{n+q+q'} - c_{n-q'}^+c_{n+q})$$



$= 0$.

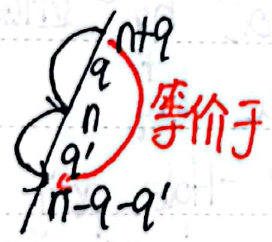
四个点 $= 0$

三个点 ①



$[c_n^+c_{n+q}, c_n^+c_{n+q}] = 0$.

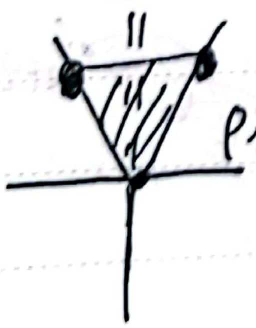
②



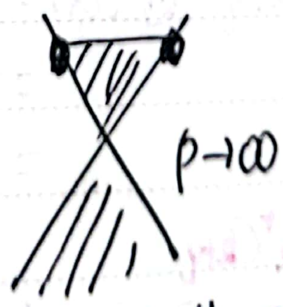
如果可以，挪动 $n-q' \Rightarrow n$.

Tomonaga (不可挪)

$\varepsilon_k = v|k|$

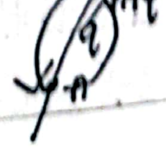


Luthinger Liquid (可挪)



$[\rho_q, \rho_{q'}] = 0$ 例外 $q' = -q$

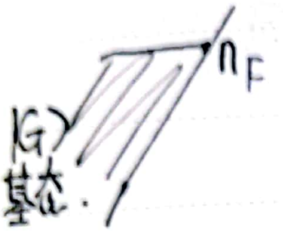
$[\rho_q, \rho_{-q}]$ 图像



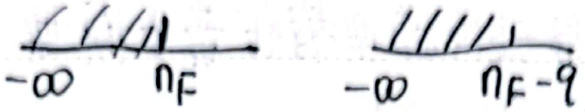
$= \sum_n c_n^+c_n - c_{n+q}^+c_{n+q}$

$[\rho_q, \rho_{-q}] = A$, $\langle G|[\rho_q,\rho_{-q}]|G\rangle = \langle A|G|A\rangle$

$= \sum_n \langle G| C_n^+ C_n - C_{n+q}^+ C_{n+q} |G\rangle$



$= \sum_{n\le n_F} 1 - \sum_{n+q\le n_F} 1$



$= -q \Rightarrow A \neq 0.$

$\simeq -q + \text{fluctuation}$.

$[\rho_q, \rho_{-q}] = -q$ 类比 $[b, b^+] = 1$ $\quad$ $q<0$

令 $\boxed{\rho_q = \sqrt{-q}\, b_q \ , \ \rho_{-q} = \sqrt{-q}\, b_q^+}$

对应于产生、湮灭玻色子.

$H_0 = v\sum_n \frac{n\pi}{L} C_n^+ C_n$, $\rho_q = \sum_m C_m^+ C_{m+q}$ $\begin{cases} F: m+q \to m \\ B: \text{p-H pair}. \end{cases}$

$[H_0^F, \rho_q] = v\sum_{n,m} \frac{n\pi}{L} [C_n^+ C_n, C_m^+ C_{m+q}]$

$\Downarrow$ 等价

$[H_0^B, \rho_q]$

$= v\sum_{n,m} \frac{n\pi}{L} \left(C_n^+ C_{m+q}\big|_{n=m} - C_m^+ C_n \big|_{m+q=n} \right)$

$= v\sum_n \frac{n\pi}{L} \left(C_n^+ C_{n+q} - C_{n-q}^+ C_n \right)$

$= v\sum_n \left(\frac{n\pi}{L} - \frac{(n+q)\pi}{L} \right) C_n^+ C_{n+q}$

$= -v\frac{\pi q}{L} \sum_n C_n^+ C_{n+q}$

$= -v\frac{\pi q}{L} \rho_q$

$\rho_q^+ = \rho_{-q}$

use $[\rho_q, \rho_q^+] = -q$, $[\rho_q \rho_{q'}] = 0$ if $q = -q'$

$H_0^B = \sum_p A_p \rho_p^+ \rho_p$

$[H_0^B, \rho_q] = \sum_p A_p [\rho_p^+ \rho_p, \rho_q]$

$= -A_q \rho_q q$

综上所述: $[H_0^F, \rho_q] = -v\frac{\pi q}{L} \rho_q$, $[H_0^B, \rho_q] = -A_q \rho_q q$

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核心：ρ_q　　令 $A_q = v(\frac{\pi}{L})$ 二者就无法区别.

$H_F = H_0^B + V_{//}$ ，$i\dot{\rho}_q = [H_F, \rho_q]$

$H_B = \sum_q A_q \rho_q^+ \rho_q + V_{//}$　$i\dot{\rho}_q = [H_B, \rho_a]$　二者结果相同.

可用玻色模型替代费米模型.

$i\dot{A} = [H_F, A] \neq [H_B, A]$.

任何物理量 $\langle\psi\psi\rangle$ 关联　$\langle O\rangle = \frac{\int D\phi\, O e^{-S}}{\int D\phi\, e^{-S}}$

$\rho_q = \sqrt{-q}\, b_q$　$n+q$　n　$\rho_q^+ = \rho_{-q} = \sqrt{-q}\, b_q^+$

$H_0^B = \sum_p A_p \rho_p^+ \rho_p = \sum_{q\in \mathbb{Z}(整)} \frac{N\pi}{L} q\, b_q^+ b_q \Leftrightarrow$ bose field.

时间节点：1933-1934 Bloch C_v / Free Energ 相等.

1950 Fomorage　1963 Latthinger　$\rho_q = \sum_n C_n^+ C_{n+q} \propto \boxed{\sqrt{q}\, b_q^+}$　↓对应　(量子化的结果.)

$H_F = V\psi^+(-i\frac{\partial}{\partial x})\psi \Rightarrow kVC_k^+C_k$

$\mathcal{L}_F = \psi^+(\text{Dirac eq})\psi \longleftrightarrow \mathcal{L}_B = \frac{1}{2}(\frac{\partial}{\partial t}\phi)^2 - \frac{V^2}{2}(\frac{\partial\phi}{\partial x})^2$

1975　用于 spin model, XXZ ...

1980　Haldane 直接从 $\mathcal{L}_F \longrightarrow \mathcal{L}_B[\phi]$

$\boxed{\psi = e^{i\phi}} = e^{i(\phi+\theta)}$

Witten　Non-Alebian Bosonization

$\rho_q = \sum_n C_n^+ C_{n+q}$　$\begin{cases} F: \\ B: \end{cases}$　$\int_0^{+\infty} \ln(1+e^{-x})dx = \frac{\pi^2}{12}$

$n+q$　n　　$\int_0^{+\infty} -\ln(1-e^{-x})dx = \frac{\pi^2}{6}$

类似于 cooper 对　$[H_F, \rho_q] = vq\rho_q$

$[H_B, \rho_q] = vq\rho_q$　　$\psi_F \sim e^{i\Phi_B}$

$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + A\cos(\phi)$.

↓ $\boxed{RG}$