2022.4.2

Bosonization (玻色化)

$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + A\cos(\phi)$

1) o—o—o—o—o

$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{v^2}{2}(\partial_x\phi)^2$

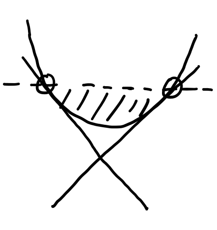
流体/声子振动 描述

$\mathcal{E}_k = v|k|$

红外发散 $\Lambda \to a$

* 一维情况下，仍然可以存在 boson、fermion、anyon
但由于不能通过 brading/交换体现其统计性质！
所以无法在统计性质上加以分辨！

2) Fermion $\Rightarrow$ Dirac equation



$\psi = \frac{1}{\sqrt{2}}(e^{ik_F x}\psi_R + e^{-ik_F x}\psi_L)$ (fast: $e^{ik_F x}$, slow: ψ_R)

$H = v_F\psi_R^\dagger(-i\frac{\partial}{\partial x})\psi_R - v_F\psi_L^\dagger(-i\frac{\partial}{\partial x})\psi_L$

$\mathcal{E}_n = v_F k$

$k = (\frac{n\pi}{L})$

3) Fermion - Boson Duality

$N_F = 1$

$H_F = v_F\sum_n n f_n^\dagger f_n (\frac{\pi}{L})$ $\xrightarrow{+\text{Interaction}}$ Thirring model

和 $H_B = v_F b_i^\dagger b_i (\frac{\pi}{L}) + E_0$

Thirring model $\downarrow$ map/exact

$N_F = 2$

和 $H_B = v_F(b_1^\dagger b_1 + 2b_2^\dagger b_2) + E_0$ $\leftarrow$ $\frac{1}{2}(\partial_\mu\phi)^2 + A\cos(\phi) \Rightarrow$ Sine - Gordon model

$\boxed{Z_F = Z_B}$

Haldane $\to$ Bosonization

Jordan - Wigner 变换

意义： spin - fermion 之间的关系
二者之间具有完全不同的统计

$\begin{cases} c_i c_j = -c_j c_i \\ \sigma_i\sigma_j = +\sigma_j\sigma_i \end{cases}$

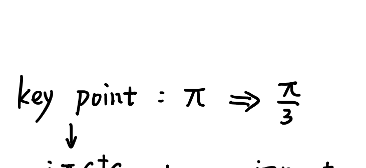
如何改变统计性质？

要将从 $-\infty$ 到 x 的所有贡献都加进来

定义 "string" (弦)

Bosonization 中类似定义 $\phi = \pi\int_{-\infty}^{x}\rho(x)dx$，改变了统计性质

$\boxed{\sigma_i^\dagger = \exp[i\pi\sum_{j=-\infty}^{i-1} c_j^\dagger c_j]\, c_i^\dagger = U c_i^\dagger}$



证明： $\sigma_i^\dagger\sigma_{i'}^\dagger = \sigma_{i'}\sigma_i$

$\sigma_i^\dagger = U(i)c_i^\dagger$

$U(i)c_i^\dagger c_{i'}^\dagger U(i') = U(i')c_{i'}^\dagger U(i) c_i^\dagger$

$\underbrace{c_i^\dagger U(i) c_{i'}^\dagger U(i')}_{\sigma_i^\dagger\sigma_{i'}^\dagger} = c_{i'}^\dagger U(i') c_i^\dagger U(i)$

$= c_{i'}^\dagger c_i^\dagger U(i')U(i)$

$= -c_i^\dagger c_{i'}^\dagger U(i')U(i)$

key point : $\pi \Rightarrow \frac{\pi}{3}$

$e^{i\pi c^\dagger c}c^\dagger = e^{i\pi n}c^\dagger = x c^\dagger e^{i\pi n}$

$(a+bn)|0\rangle = a|0\rangle = |0\rangle \Rightarrow a = 1$

$(a+bn)|1\rangle = (a+b)|1\rangle = -|1\rangle \Rightarrow a+b = -1 \Rightarrow b = -2$

$(a+bn)c^\dagger = (1-2n)c^\dagger = -c^\dagger(1-2n)$

$\boxed{e^{i2\pi n}c^\dagger = -c^\dagger e^{i\pi n}}$

- 1d XY model

$H = -J\sum_i(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) + h\sigma_i^z + \underline{J'\sigma_i^z\cdot\sigma_{i+1}^z}$ (XXZ model)

$\downarrow$ $\sigma_i^\dagger = \exp[\sum_{j=-\infty}^{i-1} i\pi c_j^\dagger c_j]\, c_j^\dagger$

$H = -\frac{J}{2}\sum_j(c_j^\dagger c_{j+1} + h.c.)$

自由 fermion

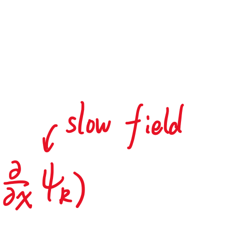
$c_j = \frac{1}{\sqrt{L}}\sum_k e^{ikj}c_k$

$\Rightarrow$ $H = -J\sum_k \cos k\, c_k^\dagger c_k$

$\mathcal{E}_k = -J\cos(k) = 0$

$\Updownarrow$

$k = \pm\frac{\pi}{2}$



XXZ model 中会对应 fermion 间的散射 (Fermi surface 上两点之间)

如何加入相互作用 $\begin{cases} \text{不强，无能隙，超流态} \\ \text{很强，打开 gap，绝缘态} \end{cases}$

何种相互作用会打开 gap？

1. 超导配对

$\int dx\, \psi^\dagger(x)(-i\frac{\partial}{\partial x})\psi^\dagger(x)$ $\Leftrightarrow \psi^2 = 0$

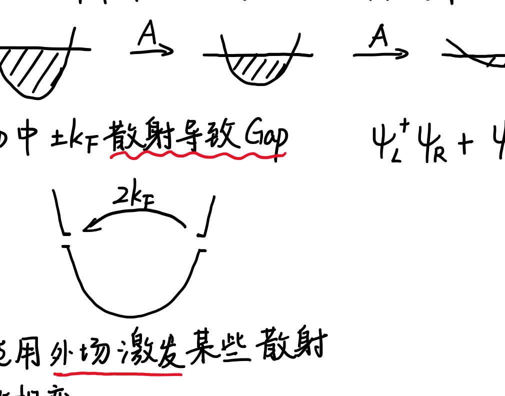
$\psi = \frac{1}{\sqrt{2}}(e^{ik_F x}\psi_R + e^{-ik_F x}\psi_L)$ $\quad k_F e^{ik_F x}\psi_R + e^{ik_F x}(-i\frac{\partial}{\partial x}\psi_R)$ (slow field)

$= \int dx\, \frac{1}{2}(e^{-ik_F x}\psi_R^\dagger + e^{ik_F x}\psi_L^\dagger)(k_F e^{-ik_F x}\psi_R^\dagger - k_F e^{ik_F x}\psi_L^\dagger)$

$\frac{k_F}{2}(\psi_R^\dagger\psi_L^\dagger - \psi_L^\dagger\psi_R^\dagger) = k_F\psi_R^\dagger\psi_L^\dagger$

2. 能带

$[-\frac{d^2}{dx^2} - \mu + A\cos(x)]\psi(x) = E\psi(x)$



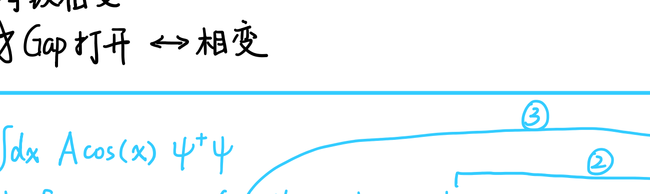
$k^2 - \mu$

$A = 0$

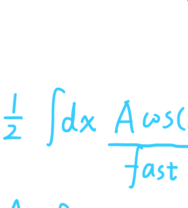
Gapped system

μ

1) μ 不在 Gap 中，无论如何调 A，都是 gapless



2) Gap 中 $\pm k_F$ 散射导致 Gap $\quad \psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L$



3) 可能用外场激发某些散射
导致相变

☆ Gap 打开 $\leftrightarrow$ 相变

$\int dx\, A\cos(x)\psi^\dagger\psi$

$= \frac{1}{2}\int dx\, A\cos(x)(e^{-ik_F x}\psi_R^\dagger + e^{ik_F x}\psi_L^\dagger)(e^{+ik_F x}\psi_R + e^{-ik_F x}\psi_L)$

①+② $= \frac{1}{2}\int dx\, \underset{\text{fast}}{A\cos(x)}[\underset{\text{slow}}{\psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L}] \simeq 0$

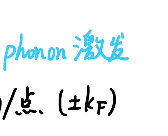
如 $\int e^{-ax^2}\cos(bx)dx \to 0$

③+④ $= \frac{A}{2}\int dx\, \cos(x)[\underset{\text{fast}}{e^{-2ik_F x}}\underset{\text{slow}}{\psi_R^\dagger\psi_L} + e^{2ik_F x}\psi_L^\dagger\psi_R]$

$= \frac{A}{2}\int[\psi_L^\dagger\psi_R(\frac{1}{2} + \text{fast}) + \psi_L^\dagger\psi_R(\frac{1}{2} + \text{fast})]dx$

If $2k_F = 1$，打开 Gap

If $2k_F \neq 1$，不能打开 Gap



$H = \begin{pmatrix} v_k & 0 \\ 0 & -v_k \end{pmatrix} + \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix}$ ($g(\psi_R^\dagger\psi_L + h.c.)$)

$= \begin{pmatrix} v_k & g \\ g & -v_k \end{pmatrix} \Leftrightarrow E = \pm\sqrt{v^2k^2 + g^2}$

相互作用 Boson system：

1) phonon 激发
风吹湖上涟漪 —— 外场产生 phonon 激发

2) 与前面系统区别 $\begin{cases} \text{无 Fermi 面/点 } (\pm k_F) \\ \text{有 Interaction} \end{cases}$

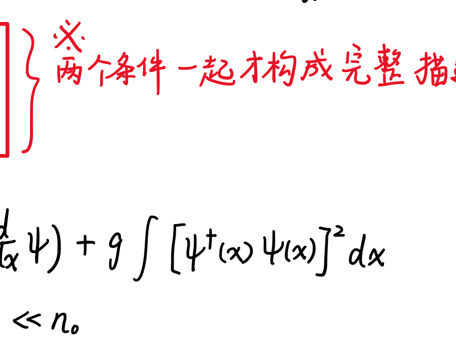
补充：

常见 BEC 中 $\to$ 声子激发谱 ($d=3$) $\quad \varphi = \underset{\text{condensate}}{\varphi_0} + \underset{\text{exciation}}{\delta\varphi}$

$d=1$ 时，$\psi = e^{i\theta}\sqrt{n}$ $\quad$ density - phase 激发

意义： ① Interaction 改变声速 ($E = v|k|$)
② n 和 θ 表象下处理 $\Rightarrow$ 流体描述

近似： $n = n_0 + \rho$



且 $\psi = e^{i\theta}\sqrt{n}$ 中
- n 和 θ 是数 (半经典描述) $\to$ 给出连续谱
- 要保证 ψ 是 boson，即 $[\psi(x), \psi^\dagger(y)] = \delta(x-y) \Rightarrow [n(x), \theta(y)] = i\delta(x-y)$

$\boxed{\begin{array}{l} \psi = e^{i\theta}\sqrt{n} \\ [n(x), \theta(y)] = i\delta(x-y) \end{array}}$ ※ 两个条件一起才构成完整描述

$H = \int dx\, \frac{\hbar^2}{2m}(\frac{d}{dx}\psi^\dagger)(\frac{d}{dx}\psi) + g\int[\psi^\dagger(x)\psi(x)]^2dx$

极限 $n = n_0 + \rho$，$|\rho| \ll n_0$

$\simeq \int dx\, \frac{\hbar^2 n_0}{2m}(\partial_x\theta)^2 + g\int(n_0 + \rho)^2dx$ $\quad$ $\int n_0\rho\, dx = 0$ (涨落抵消)

$= \underset{g\int n_0^2 dx}{\text{const}} + \int dx\, \frac{\hbar^2 n_0}{2m}(\partial_x\theta)^2 + g\int\rho^2 dx$

显然 $\boxed{L=1}$

$\rho(x) = \sum_q e^{iqx}\rho_q$

$\theta(x) = \sum_q e^{iqx}\theta_q$

$[\rho(x), \theta(y)]$ 恒等式

$= \sum_{qq'}[\rho_q, \theta_{q'}]e^{iqx + iq'y}$

$= i\delta(x-y)$

$= i\sum_p e^{ip(x-y)}$

$H = \frac{\hbar^2 n_0}{2m}\sum_q q^2\theta_q\theta_{-q} + g\sum_q\rho_q\rho_{-q}$

$\Leftrightarrow \boxed{[\rho_q, \theta_{-q}] = i}$ $\quad$ 很类似 $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$, $[x, p] = i\hbar$

$q^2\theta_q\theta_{-q}$
$= (-q)^2\theta_{-q}\theta_q$

$[\rho_q, \rho_{q'}] = 0$

$[\theta_q, \theta_{q'}] = 0$

$H = \sum_{q>0}\frac{\hbar^2 n_0 q^2}{m}\theta_q^*\theta_q + 2g\rho_q^*\rho_q$ (动量) $\Leftrightarrow \boxed{\omega_q = c|q|}$

其中 $[\rho_{q'}, \theta_q^*] = i$

结论： g 不能打开 Gap，只改变 v.

下节课讨论关联函数

$\langle\psi^\dagger(x)\psi(y)\rangle \sim \frac{1}{|x-y|}$