$N_F = 2$.

$|11000\cdots\rangle \to 1+2$

$|101000\cdots\rangle \to 1+3$

$|011000\cdots\rangle \to 2+3$

$|010100\cdots\rangle \to 2+4$

$|001100\cdots\rangle \to 3+4$

$|1000100\cdots\rangle \to 1+5$

$H = E_0 + \omega b_1^\dagger b_1 + 2\omega b_2^\dagger b_2$ 对应. 基态: $E_0 = (1+2)\omega$.

n_1	n_2	
0	0	$E_0 = (1+2)\omega = 3\omega$.
1	0	$(1+3)\omega$
2	0	$(2+3)\omega$.
0	1	$(3+2)\omega$
3	0.	$6\ \omega$.
1	1.	$6\ \omega$
⋮		⋮

$N_F = n$ 情况也一样.

作业: Sachdev note: Eq 1 – Eq 11

$$\prod_{n=1}^{+\infty}(1+q^{2n-1})^2 = \left(\sum_{n=1}^{+\infty}\frac{1}{1-q^{2n}}\right)\left(\sum_{\nu=-\infty}^{+\infty} q^{\nu^2}\right).$$

Fermion: Z. Boson: Z , $q = e^{-\beta v\pi/L}$

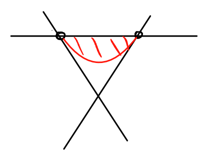
$\underbrace{C_i^\dagger C_j}|G\rangle$

电子空穴对 (Boson)

(4.1)

Bosonization. (玻色化)

$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + A\cos(\phi)$



1) –O–O–O–O–O–O–

$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{v^2}{2}(\partial_x\phi)^2$

流体 / 声子振动

$\varepsilon_k = v|k|$

红外发散, $\Lambda \Rightarrow a$

2) Fermion ⇒ Dirac eq

$\psi = \frac{1}{\sqrt{2}}\left(\underbrace{e^{ik_Fx}}_{\text{fast}}\underbrace{\psi_R}_{\text{slow}} + e^{-ik_Fx}\psi_L\right)$

$H = v_F\psi_R^\dagger(-i\frac{\partial}{\partial x})\psi_R - v_F\psi_L^\dagger(-i\frac{\partial}{\partial x})\psi_L$

$\varepsilon_n = v_F k = v_F(\frac{n\pi}{L})$

3) Fermion-Boson duality.

$N_F = 1$

$H_F = v_F \sum_n n f_n^+ f_n \left(\frac{\pi}{L}\right)$ $\xrightarrow{\text{Interaction}}$ Thirring model

map exactly

和 $H_B = v_b b_1^+ b_1 \left(\frac{\pi}{L}\right) + E_0$

$N_F = 2$

和 $H_B = v_F (b_1^+ b_1 + 2 b_2^\dagger b_2) + E_0 \leftarrow \frac{1}{2}(\partial_\mu \phi)^2 + A\cos(\phi)$ (Sine-Gordon model)

Haldane Bosonization. $Z_F = Z_B$ (条件：无能隙/相互作用)

Jordon-Wigner 变换

意义： 建立 spin-Fermion之间的关系.

但 spin和Fermion 的统计性质完全不同.

$c_i c_j = -c_j c_i$

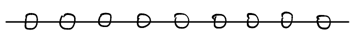
$\sigma_i \sigma_j = +\sigma_j \sigma_i$

改变统计性质： string (弦)

$-\infty$ ———————— x

$\phi = \pi \int_{-\infty}^{x} \rho(x)\, dx$

目的：改性



$$\sigma_i^+ = \exp\Big[i\pi \underbrace{\sum_{j=-\infty}^{i-1}}_{} c_j^+ c_j\Big] c_i^\dagger = U c_i^+$$

X点的粒子与 $-\infty \sim X-1$ 有关，是弦的性质.

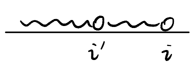
证明： $\sigma_i^+ \sigma_{i'}^+ = \sigma_{i'}^+ \sigma_i^+$

for: $i' < i$

$LHS = c_i^\dagger U(i) c_{i'}^\dagger U(i')$
$= -c_i^+ c_{i'}^+ U(i') U(i)$

$RHS = c_{i'}^\dagger U(i') c_i^\dagger U(i)$
$= c_{i'}^\dagger c_i^\dagger U(i') U(i)$
$= -c_i^\dagger c_{i'}^\dagger U(i') U(i)$

改性



$$e^{i\pi n_i} c_i^\dagger = -c_i^\dagger e^{i\pi n_i}$$

$\therefore LHS = RHS \longrightarrow \sigma_i^+ \sigma_{i'}^+ = \sigma_{i'} \sigma_i$

$$e^{i\pi n} \to e^{i\theta n} \Longrightarrow \psi_i\psi_j = e^{i\theta}\psi_j\psi_i \quad \text{(任意子)}$$

1d XY model

$$H = -J\sum_j(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) + h\sigma_j^z$$ 调节 Fermi 面位置

$$+J'\sigma_j^z\sigma_{j+1}^z$$ XXZ model

$$\sigma_j^+ = \exp\left[\sum_{k=-\infty}^{j-1} i\pi C_k^+ C_k\right] C_j^+$$

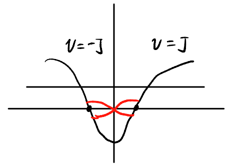
$$H = -\frac{J}{2}\sum_j(C_j^+C_j + h.c.)$$

自由 Fermion

$$C_j = \frac{1}{\sqrt{2}}\sum_k e^{ikJ}C_k$$

$$H = -J\sum \cos k\, C_k^+C_k$$

$$\varepsilon_k = -J\cos k = 0 \Rightarrow k = \pm\frac{\pi}{2}$$



什么样的相互作用能打开Gap?

1) 超导配对.

$$\int\psi^+(x)\psi^+(x)dx \Leftrightarrow \psi^2 = 0 \quad \times$$

$$\int dx\,\psi^+(x)(-i\tfrac{\partial}{\partial x})\psi^+(x)$$

$$\psi = \frac{1}{\sqrt{2}}(e^{ik_Fx}\psi_R + e^{-ik_Fx}\psi_L)$$

$$= \int dx\,\tfrac{1}{2}(e^{-ik_Fx}\psi_R^+ + e^{ik_Fx}\psi_L^+)(-i\tfrac{d}{dx})(e^{-ik_Fx}\psi_R^+ + e^{ik_Fx}\psi_L^+)$$

$$k_F e^{ik_Fx}\psi_R^+ + \cancel{e^{ik_Fx}(-i\tfrac{\partial}{\partial x})\psi_R^+}$$

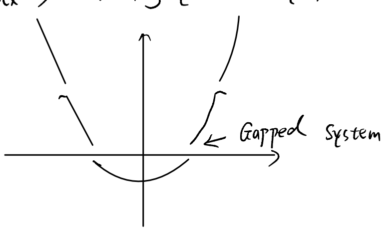
slow

$$\simeq \int dx\,\tfrac{1}{2}(e^{-ik_Fx}\psi_R^+ + e^{ik_Fx}\psi_L^+)(-k_F e^{-ik_Fx}\psi_R^+ + k_F e^{ik_Fx}\psi_L)$$

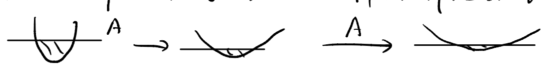
$$k_F(\psi_R^+\psi_L^+ - \psi_L^+\psi_R^+) = k_F\psi_R^+\psi_L^+ \quad \text{(打开 Gap)}$$

2) 固体物理

$$\left[-\frac{d^2}{dx^2} - \mu - A\cos x\right]\psi(x) = E\psi(x)$$



1) μ在不Gap中间,则无论怎么调节A,都是无能隙的



2) Gap 中 $\pm k_F$ 的散射导致 Gap

$$\psi_L^+\psi_R + h.c.$$

$$\int dx\, A\cos x\, \psi^\dagger \psi$$

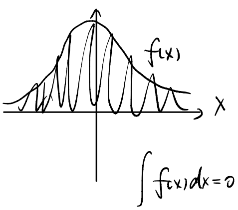
$$= \frac{1}{2}\int dx\, A\cos x\, \left(e^{-ik_Fx}\psi_R^\dagger + e^{ik_Fx}\psi_L^\dagger\right)\left(e^{ik_Fx}\psi_R + e^{-ik_Fx}\psi_L\right)$$

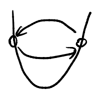
$$① + ② = \frac{1}{2}\int dx\, \underbrace{A\cos x}_{\text{fast}}\,(\underbrace{\psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L}_{\text{slow}}) = 0$$

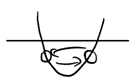
$$③ + ④ = \frac{1}{2}\int dx\, \underbrace{A\cos x}_{\text{fast}}\,[\underbrace{e^{-2ik_Fx}}_{\text{fast}}\underbrace{\psi_R^\dagger\psi_L}_{\text{slow}} + e^{2ik_Fx}\psi_L^\dagger\psi_R]$$

$$\cos x\, e^{-2ik_Fx} \xrightarrow{2k_F=1} = \cos^2 x - i\cos x\sin x$$
$$= \frac{1}{2}(1+\cos 2x) - \frac{i}{2}\sin 2x$$
$$= \frac{1}{2} + \frac{1}{2}\cos 2x - \frac{i}{2}\sin 2x$$

$$= \frac{A}{2}\int [\psi_L^\dagger\psi_R(\tfrac{1}{2}+\cancel{\text{fast}}) + \psi_L^\dagger\psi_R(\tfrac{1}{2}+\cancel{\text{fast}})]dx$$ ，打开 Gap.



if $2k_F = 1$,  有效散射，打开 Gap.

if $2k_F \neq 1$, 无效  不能开 Gap.

$$H = \begin{pmatrix} vk & 0 \\ 0 & -vk \end{pmatrix} + \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix} \Longleftrightarrow E = \pm\sqrt{v^2k^2+g^2}$$ ，打开 Gap.

$$g(\psi_R^\dagger\psi_L + h.c.)$$

3) 可能通过外场激发某些散射，从而导致相变（Gap 打开）

Interacting Boson model:

1) phonon 激发

2) 与前面结论差别 $\begin{cases} \text{无} \pm k_F \\ \text{有 Interaction} \end{cases}$

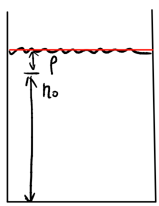
提供材料 (BEC) $\Rightarrow$ 声子激发谱 $(d=3)$ $\Rightarrow$ $\varphi = \underset{\substack{\uparrow \\ \text{conderstate}}}{\varphi_0} + \delta\varphi$ $\leftarrow$ excitation

今天: $d=1$. $\psi = e^{i\theta}\sqrt{n}$ density-phase 表象

意义: ① Interaction $\Rightarrow$ 改变声速 $\varepsilon = v|k|$

② n 和 θ 表象下处理 (对流体描述的自然方式)

近似: $n = n_0 + \rho$. 和以前处理有一点不同.



← 水波只在水面上

$$[\psi(x), \psi^\dagger(y)] = \delta(x-y)$$
$$\Downarrow$$
$$\begin{cases} [n(x), \theta(y)] = i\delta(x-y) \\ \psi = e^{i\theta}\sqrt{n} \end{cases}$$ 构成完整描述

$\psi = e^{i\theta}\sqrt{n}$
↑
教材中假设 n 和 θ 是数.
ψ 是 Boson.

密度和相位是共轭的.

$$\Rightarrow H = \int dx \frac{\hbar^2}{2m}\left(\frac{d}{dx}\psi^\dagger\right)\left(\frac{d}{dx}\psi\right) + g\int \left(\psi^\dagger(x)\psi(x)\right)^2 dx$$

极限: $n = n_0 + \rho$, $|\rho| \ll n_0$

$\int n_0 \rho \, dx = 0$ = 交叉项

$$= \int dx \frac{\hbar^2 n_0}{2m}(\partial_x\theta)^2 + g\int (n_0+\rho)^2 dx$$

$$= \underset{\substack{\uparrow \\ \int g n_0^2 dx}}{\text{const}} + \int dx \frac{\hbar^2 n_0}{2m}(\partial_x\theta)^2 + g\int \rho^2 dx$$ —— two DOF: ρ & θ.

有联系: $[n, \theta] = i\delta(x-y)$.