$$\frac{1}{2\pi a} e^{-i\Phi(x+\varepsilon)} e^{i\Phi(x)} \sim \infty \cdot \frac{1}{2\pi a} e^{-i\Phi(x+\varepsilon)+i\Phi(x)}$$

$$\Rightarrow \rho(x) \sim \frac{1}{2\pi a} e^{-i\Phi(x+\varepsilon)+i\Phi(x)} \cdot \infty - \infty$$

$$\sim \frac{1}{2\pi a} e^{-i\frac{\partial \Phi}{\partial x}\cdot \varepsilon} \infty - \infty$$

$$\sim \frac{1}{2\pi a}\left(1 - i\frac{\partial \Phi}{\partial x}\varepsilon + \frac{(-i)^2}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 \varepsilon^2 \dots\right)\infty - \infty$$

if canceled & $a=\varepsilon$

$$\sim -\frac{i}{2\pi}\frac{\partial \Phi}{\partial x} + \dots$$

i.e. $\psi^\dagger(x)\psi(x) \sim \partial_x \Phi$

high-order $\psi^\dagger(x)\partial_x\psi(x) \sim (\partial_x \Phi)^2$

Review: why $\psi = \frac{F}{\sqrt{2\pi a}} e^{i\Phi}$ is right?

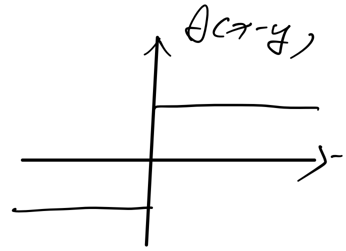
1) $\{\psi, \psi'\} = 0$

2) Hillbert space: $e^{i\Phi}|N\rangle_0 \in \mathcal{H}_N$

3) $Z_F = Z_B$

4) $\{\psi(x), \psi^\dagger(y)\} = \delta(x-y)$ $\to \Phi^\dagger(x) = \Phi(x)$

5) 关联函数 $\langle \psi^\dagger(x)\psi(y)\rangle = \frac{1}{2\pi a}\langle e^{-i\Phi^\dagger(x)} e^{i\Phi(y)}\rangle$



forsee: $[\Phi(x), \Phi(y)] = \ln\frac{x-y-ia}{y-x-ia}$

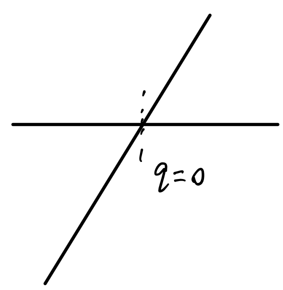
$\sim [\Phi_+(x), \Phi_-(y)] = \ln(x-y-ia)$

$[\Phi_+(y), \Phi_-(x)] = \ln(y-x-ia)$

4.25.

Aim 1) 关联函数相同

2) $[\Phi(x), \Phi(y)]$ 表达式.



$$\langle \psi^\dagger(x)\psi(y)\rangle = \frac{1}{L}\sum_q e^{-iq(x-y)}\langle n|c_q^\dagger c_q|n\rangle$$

$$= \frac{1}{L}\sum_{q<0} e^{-iq\cdot(x-y)+aq}, \quad \int_{-\infty}^{0} e^{Aq}dq \to \frac{1}{A}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} dq\; e^{[a-i(x-y)]q}$$

$$= \frac{1}{2\pi}\,\frac{1}{a-i(x-y)}$$

$$= \frac{1}{2\pi a}\langle e^{-i\phi(x)}e^{i\phi(y)}\rangle$$

$$= \frac{1}{2\pi a}e^{K}, \quad e^{K} = \frac{ia}{x-y+ia}$$

联想：$(\partial_x^2+\partial_y^2)\,g = \delta^2(\vec{x}) \Rightarrow g \sim \frac{1}{2\pi}\ln r$，下由 $\psi = \frac{F}{\sqrt{2\pi a}}\,e^{i\Phi}$ 推导之.

$$\Phi = i\sum_{q>0}\sqrt{\frac{2\pi}{Lq}}\left(e^{-iq\cdot x}b_q^{\dagger} + e^{iq\cdot x}b_q\right)$$

Remarks: $x \sim a + a^{\dagger}$

$p \sim i(a - a^{\dagger})$

$\to \rho \sim \frac{\partial\Phi}{\partial x} \sim \sum_{q>0}\rho_q e^{-iq\cdot x} + \rho_q^{\dagger}e^{iq\cdot x}$

求 $[\Phi(x), \Phi(y)] = \sum_{q>0}\frac{2\pi}{Lq}\left[e^{-iq\cdot x}b_q^{\dagger} + e^{iq\cdot x}b_q,\; e^{-iq\cdot y}b_q^{\dagger} + e^{iq\cdot y}b_q\right]$

$$= -\frac{2\pi}{L}\sum_{q>0}\left(\frac{1}{q}e^{-iq\cdot(x-y)-aq} - \frac{1}{q}e^{iq\cdot(x-y)-aq}\right)$$ 收敛因子

① $q = n\cdot\frac{2\pi}{L}$ ② 求和化积分

② $$= -\int_{0}^{+\infty} dq\cdot\frac{1}{q}\left(e^{-iq\cdot(x-y)-aq} - e^{iq\cdot(x-y)-a}\right)$$

$= mma\ldots$

① $$= -\sum_{n\geq 1}\frac{1}{n}\left(e^{(-i(x-y)-a)\frac{2\pi}{L}n} - e^{(i(x-y)-a)\frac{2\pi}{L}n}\right)$$

Identity: $\ln(1-x) = -\sum_{n\geq 1}\frac{1}{n}x^{n}$

$$= \ln\left[1 - e^{\frac{2\pi}{L}(-i(x-y)-a)}\right] - \ln\left[1 - e^{\frac{2\pi}{L}(i(x-y)-a)}\right]$$

$$\overset{L\text{大}}{=} \ln\left[\frac{2\pi}{L}(i(x-y)+a)\right] - \ln\left[\frac{2\pi}{L}(i(y-x)+a)\right]$$

$$= \ln\frac{x-y-ia}{y-x-ia} \longrightarrow i\pi\left[H(x-y) - H(y-x)\right]$$

$$\Leftrightarrow [\Phi_+(x), \Phi_-(y)] = \ln(x-y-ia)$$

则有其关联函数

$\frac{1}{2\pi a}\langle e^{-i\Phi(x)} e^{i\Phi(y)}\rangle$ Normal ordering: $:A: = A - \langle 0|A|0\rangle$

or $:e^{\Phi}: = e^{\Phi_+}e^{\Phi_-}$

$\Rightarrow \langle :e^{\Phi}: \rangle = \langle 0|e^{\Phi_+}e^{\Phi_-}|0\rangle = 1$

$e^A = e^{A_+ + A_-}$

$= e^{A_+}e^{A_-}e^{-\frac{1}{2}[A_+, A_-]}$

$\begin{cases} A_+ = -i\Phi_+(x) + i\Phi_+(y) \\ A_- = -i\Phi_-(x) + i\Phi_-(y) \end{cases}$

$= \frac{1}{2\pi a}\langle 0| e^{-i(\Phi_+(x)+\Phi_-(x))} e^{i(\Phi_+(y)+\Phi_-(y))}|0\rangle$

$= \frac{1}{2\pi a}\langle 0| e^{-i\Phi(x)+i\Phi(y)+\frac{1}{2}[\Phi(x),\Phi(y)]}|0\rangle$

$= \frac{1}{2\pi a} e^{\frac{1}{2}[\Phi(x),\Phi(y)]}\langle 0| e^{-i\Phi(x)+i\Phi(y)}|0\rangle$

$= \frac{1}{2\pi a} e^{\frac{1}{2}[\Phi(x),\Phi(y)]} e^{-\frac{1}{2}[A_+, A_-]}\cdot 1$

$\equiv \frac{1}{2\pi a} e^{K}$

$K = \frac{1}{2}[\Phi(x),\Phi(y)] - \frac{1}{2}[A_+, A_-]$

$= \frac{1}{2}[\Phi(x),\Phi(y)] + \frac{1}{2}[\Phi_+(x)-\Phi_+(y), \Phi_-(x)-\Phi_-(y)]$, $[\Phi_+(x), \Phi_-(y)] = \ln(x-y-ia)$

$= \frac{1}{2}\left\{\ln\frac{x-y-ia}{y-x-ia} + 2\ln(-ia) - \ln(x-y-ia) - \ln(y-x-ia)\right\}$, $[\Phi_+(x), \Phi_-(x)] = \ln(-ia)$

$= \ln(-ia) - \ln(y-x-ia) = \ln\frac{ia}{x-y+ia} \Longrightarrow$ 与上个结果相同.

then, $\frac{1}{2\pi a}e^{K} = \frac{1}{2\pi a}e^{\ln\frac{ia}{x-y+ia}} = \frac{i}{2\pi}\frac{1}{x-y+ia} = \langle\psi^\dagger(x)\psi(y)\rangle$

Similarly, $\langle\psi(x)\psi(y)\rangle \propto (x-y-ia)\dots$

then $\langle\{\psi(x),\psi(y)\}\rangle \propto -2ia \to 0.$

现有 $\langle\psi^\dagger(x)\psi(y)\rangle = \frac{i}{2\pi}\frac{1}{x-y+ia}$,

同理 $\langle\psi(y)\psi^\dagger(x)\rangle = \frac{i}{2\pi}\frac{1}{y-x+ia}$

则 $\langle\{\psi(y),\psi^\dagger(x)\}\rangle = \delta(x-y)$

W-D identity:

$= \frac{i}{2\pi}\left(\frac{1}{y-x+ia} + \frac{1}{x-y+ia}\right)$, 有 $\frac{1}{x-ia} = P\frac{1}{x} + i\pi\delta(x)$

$= \frac{i}{2\pi}\left[\cancel{P\frac{1}{y-x}} - i\pi\delta(y-x) + \cancel{P\frac{1}{x-y}} - i\pi\delta(x-y)\right]$

$= \delta(x-y)$

即采用 $\psi(x) = \frac{F}{\sqrt{2\pi a}}e^{i\Phi(x)}$ 可得 Fermionic 对易关联.

Question: $H = \psi^\dagger\left(-\frac{\hbar^2}{2m} - \mu\right)\psi + V_{int} \longrightarrow \rho_q = \sum_n c_n^\dagger c_{n+q}$ near μ.

$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + A\cos\phi$

cosΦ如何得到?

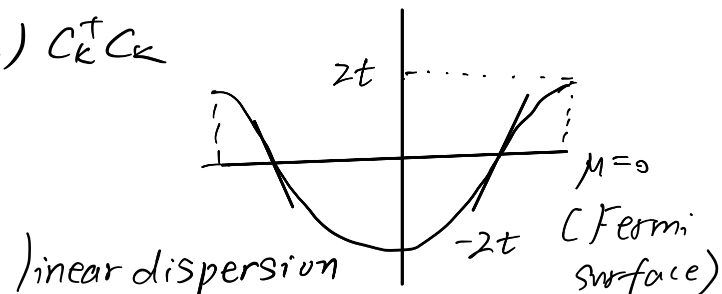
Model: XXZ model

$$H = -J\sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + \Delta \sum_i \sigma_i^z \sigma_{i+1}^z$$

Jordan-Wigner Transformation

$$H = \underline{-t\sum_i (C_i^\dagger C_{i+1} + h.c.)} + \mu C_i^\dagger C_i + U C_i^\dagger C_i C_{i+1}^\dagger C_{i+1}$$

$$H_0 \to H_{0,k} = (-2t\cos ka + \mu)\, C_k^\dagger C_k$$



$$H_0 = V_F \psi_R^\dagger\left(-i\frac{\partial}{\partial x}\right)\psi_R - V_F \psi_L^\dagger\left(-i\frac{\partial}{\partial x}\right)\psi_L$$

$$\overset{F\text{-trans}}{\sim}\ V_F k\, C_k^\dagger C_k - V_F k\, d_k^\dagger d_k$$

$$(-2t\cos ka)' = 2at\sin ka\Big|_{ka=\pm\frac{\pi}{2}} = \pm 2at = \pm V_F$$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_t \psi_R)^2 - V_F^2(\partial_x \psi_R)^2\right] + \frac{1}{2}\left[(\partial_t \psi_L)^2 - V_F^2(\partial_x \psi_L)^2\right]$$

How to achieve this? $\sum_i C_i^\dagger C_{i+1} \sim \int dx\, \psi^\dagger\left(\frac{\partial}{\partial x}\right)^2 \psi$

choose $C_i \sim \psi(x=ia)$ then $\sum_i C_i^\dagger C_i = \int \psi^\dagger(x)\psi(x)dx$

coeff.: $C_i = \sqrt{a}\,\psi(x=ia)$

$$\Rightarrow \sum_i -t\, C_i^\dagger C_{i+1} + h.c. = -ta\sum_i \psi^\dagger(x=ia)\,\psi(x=(i+1)a) + h.c.$$

$$= -t\int dx\left[\psi^\dagger(x)\psi(x+a) + \psi^\dagger(x+a)\psi(x)\right]$$

$$= -t\int dx\left[(1-\tfrac{a}{2}\partial_x)\psi^\dagger(x+\tfrac{a}{2})(1+\tfrac{a}{2}\partial_x)\psi(x+\tfrac{a}{2}) + (1+\tfrac{a}{2}\partial_x)\psi^\dagger(x+\tfrac{a}{2})(1-\tfrac{a}{2}\partial_x)\psi(x+\tfrac{a}{2})\right]$$

$$= -t\int dx\left\{\psi^\dagger\psi + \tfrac{a}{2}\cancel{\left[-(\partial_x\psi^\dagger)\psi + \psi^\dagger\partial_x\psi\right]} - \tfrac{a^2}{4}\partial_x\psi^\dagger\partial_x\psi + \underbrace{\psi^\dagger\psi}_{\text{const}} + \tfrac{a}{2}\cancel{\left[(\partial_x\psi^\dagger)\psi - \psi^\dagger\partial_x\psi\right]}\right.$$

$$\left.-\tfrac{a^2}{4}\partial_x\psi^\dagger\partial_x\psi\right\}$$

$$= -t\frac{a^2}{2}\int dx\, \psi^\dagger(x)\,\partial_x^2\,\psi(x)$$

interaction: $\sum_i C_i^\dagger C_i C_{i+1}^\dagger C_{i+1}$

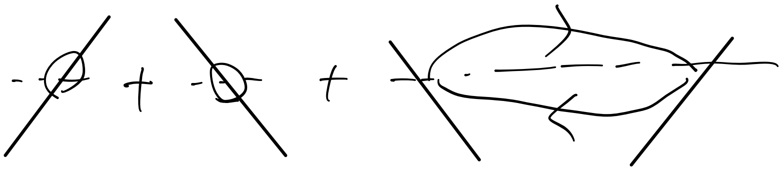
$$= \sum_i a^2\, \psi^\dagger(ia)\psi(ia)\psi^\dagger(ia+a)\psi(ia+a)$$

$$= a\int dx\, \psi^\dagger(x)\psi(x)\psi^\dagger(x+a)\psi(x+a),\quad \psi = e^{ik_F x}\psi_R + e^{-ik_F x}\psi_L$$

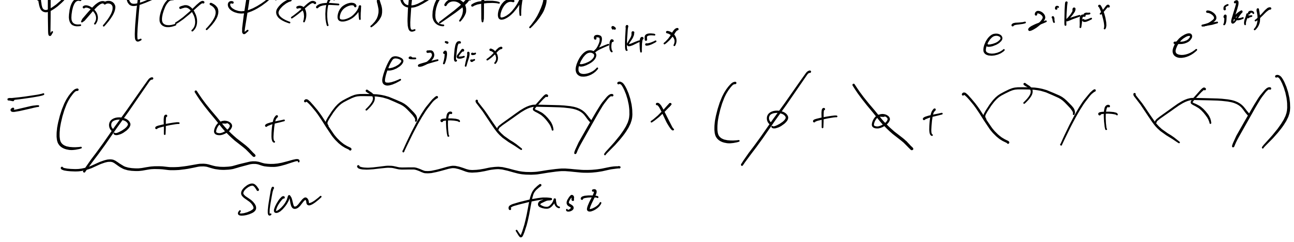
快 慢

$$\psi^\dagger(x)\psi(x) = \psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L + e^{-2ik_F x}\psi_R^\dagger\psi_L + h.c.$$

$$= \frac{F}{\sqrt{2\pi a}}\, e^{i\Phi}$$



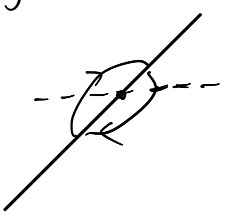
$\psi^\dagger(x)\psi(x)\psi^\dagger(x+a)\psi(x+a)$



$= (S+L)\times(S+L)$

$= SS + \underbrace{SL + LS}_{\to 0} + LL$, $\quad e^{-2ik_F x}\cdot e^{2ik_F x} \to$ finite term

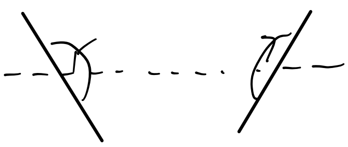
forward (向前) scattering



$\sim (\partial\phi)^2 \sim \rho^2$

density

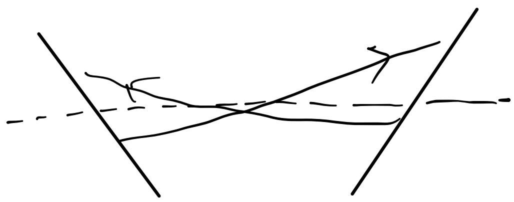
$\leftrightarrow S\cdot S$



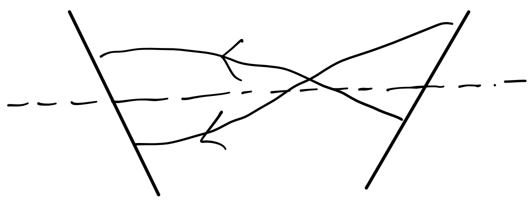
$\partial\phi \sim \rho$, $\partial\phi \sim \rho$

k-conservation

backward scattering (背散射)



$\leftrightarrow L\cdot L$



$\Delta k \sim 4k_F$ 需补偿.

$\hookrightarrow e^{-i4k_F x} = e^{-i\cdot n\,\underbrace{4k_F a}_{\sim 2m\pi \text{时}}}$

density wave ← 有意义

Sum: interaction: $S\cdot S$: $\rho\cdot\rho$ 作用

$L\cdot L$: like $\cos\phi$

$$h.c. + \psi_L^\dagger\psi_R = \frac{F_L^\dagger F_R}{2\pi a}e^{-i\Phi_L + i\Phi_R} + h.c. \qquad , L, R \text{ independent}, F^\dagger = F$$

$$= \frac{F_L F_R}{2\pi a}e^{-i\Phi_L + i\Phi_R} + \frac{F_R F_L}{2\pi a}e^{i\Phi_L - i\Phi_R}$$

$$= \frac{F_L F_R}{2\pi a}\left(e^{-i\Phi_L + i\Phi_R} - e^{i\Phi_L - i\Phi_R}\right)$$

$$\sim \frac{iF_L F_R}{\pi a}\sin(\Phi_R - \Phi_L)$$ / 的会导致对称性破坏

规范要求：L, R混合，相位差 $e^{i\theta} \longrightarrow$ 无 ∂^2 things.