$$\psi \sim e^{i\Phi} \Longleftrightarrow \begin{cases} \psi^2(x)=0 \\ \{\psi(x), \psi^\dagger(y)\} = \delta(x-y) \end{cases} \Big| \text{at } \boxed{x=y}$$

$$\psi^\dagger(x)\,\psi(x) \sim e^{-i\Phi} e^{i\Phi} = 1$$

$$\psi^2(x) = e^{i\Phi}\cdot e^{i\Phi} = e^{2i\Phi} \neq 0$$

$$\frac{F}{\sqrt{2\pi a}} \leftarrow \text{cutoff}$$

① ψ (fermi 场) 反对易关系.

② Hilbert 空间 -- 对应

③ 关联函数一样 $\Rightarrow$ 两体关联一样 $\Leftarrow$ 多体关联一样 $\Leftarrow$ 实验观测也一样.

近似 ~ 严格　　$\sigma_i \to a_i$

$\Downarrow$ 直观图像　$\Phi = \underbrace{\alpha \int_{-\infty}^{x} \rho(x)\,dx}_{\text{string(改变统计性质)}} + \underbrace{\theta(x)}_{\text{local phase fluctuation}}$

$$\begin{cases} \alpha = \pi \Rightarrow \text{fermion} \\ \alpha \neq \pi \Rightarrow \text{anyon.} \end{cases}$$

① Klein factor

② Hilbert space

③ Normal ordering $:\_:$ 三个等价定义.

* Mahan书

(1) $:A: = A - \langle 0|A|0\rangle$

(2) $\psi^\dagger(x)\,\psi(x) = \lim_{a\to 0} \psi^\dagger(x+a)\psi(x) - \langle 0|\psi^\dagger(x+a)\psi(x)|0\rangle$

(3) $:e^{i\Phi}: = e^{i\Phi_+} e^{i\Phi_-}$, $(\Phi = \Phi_+ + \Phi_-)$ 产生湮灭

证明:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}, \quad ([A,B] = \text{const})$$

$[\rho(x), \psi(y)] = -\psi(y)\delta(x-y)$ 对易/反对易关系.

$\Leftrightarrow [\rho_q, \psi(x)] = -e^{-iqx}\psi(x)$

$\Leftrightarrow [b_q, \psi(x)] = -\sqrt{\frac{2\pi}{L|q|}}\, e^{-iqx}\psi(x) = -\alpha_q(x)\psi(x)$

$[b_q, \psi(x)]|N\rangle_0 = -\alpha_q(x)\psi(x)|N\rangle_0$

$b_q\psi(x)|N_0\rangle - \cancel{\psi(x) b_q |N_0\rangle} = -\alpha_q(x)\psi(x)|N\rangle_0$

$$\boxed{b_q \underbrace{\psi(x)|N_0\rangle}_{H_{N-1}} = -\alpha_q(x)\psi(x)|N\rangle_0}$$

H_{N-1}, b_q的本征态 联想

相干态⟺经典近似. $a|\alpha\rangle = \alpha|\alpha\rangle$ $\begin{cases} a \to \alpha \\ a^\dagger \to \alpha^* \end{cases}$

$|\alpha\rangle \sim e^{\alpha a^\dagger}|0\rangle$ 由 $e^{\alpha^* a}|0\rangle = |0\rangle$

$\sim e^{\alpha a^\dagger} e^{\alpha^* a}|0\rangle$ ∵ $[\alpha^\dagger, a] = 0$.

$\sim e^{\alpha a^\dagger + \alpha^* a}|0\rangle$

$b_q \leftrightarrow -\alpha_q$, $\underbrace{\psi(x)|N\rangle_0}_{H_{N-1}} \sim e^{-\sum_q \alpha_q(x) b_q^\dagger}|N-1\rangle_0$

$\sim e^{-\sum_q \alpha_q^* b_q}|N-1\rangle_0$

$\psi(x) = F e^{i\Phi}$ 代入

$F e^{i\Phi(x)}|N\rangle_0 \sim e^{-\sum_q \alpha_q(x) b_q^\dagger}|N-1\rangle_0$

其中 ($F|N\rangle_0 = |N-1\rangle_0$) $e^{i\Phi} \Leftrightarrow \Phi = \Phi^\dagger$ 厄密场

(1)(0) 得 $e^{i\Phi(x)}\cancel{|N-1\rangle_0} \sim e^{-\sum_q \alpha_q b_q^\dagger}\cancel{|N-1\rangle_0}$

证明: $\boxed{\psi(x) = \frac{F}{\sqrt{2\pi a}} e^{i\Phi}}$ 无可调自由度

令 $\underbrace{\psi(x)|N\rangle_0}_{N-1} = \Lambda \cdot e^{\sum_q \alpha_q(x) b_q^\dagger}|N-1\rangle_0$ $F^\dagger F = FF^\dagger = 1$

$= \Lambda \cdot e^{\sum_q \alpha_q(x) b_q^\dagger} \cdot \underbrace{F|N\rangle_0}_{N-1}$

${}_0\langle N|F^\dagger = {}_0\langle N-1|$ ${}_0\langle N|F^\dagger\psi(x)|N\rangle_0 = \Lambda\, {}_0\langle N|\cancel{F^\dagger} e^{\sum_q \alpha_q(x) b_q^\dagger}\cancel{F}|N\rangle_0$

$\boxed{\psi(x) = \frac{1}{\sqrt{L}}\sum_k e^{ikx} c_k}$ 只有 $q = \frac{2\pi N}{L}$ 有贡献.

$\langle N|F^\dagger\psi(x)|N\rangle_0 = \frac{1}{\sqrt{L}} e^{i\frac{2\pi N}{L}x}$

$= \Lambda\langle N| e^{\sum_q \alpha_q(x) b_q^\dagger}|N\rangle_0 = \Lambda$

~~等价于求 $\langle 0| e^{\alpha a^\dagger}|0\rangle$~~ $\therefore \Lambda = \frac{1}{\sqrt{L}} e^{i\frac{2\pi N}{L}}$

$\psi(x)|N\rangle_0 = \frac{1}{\sqrt{L}} e^{i\frac{2\pi N}{L}} e^{b_+}|N-1\rangle_0$

$= \frac{F}{\sqrt{L}} e^{i\frac{2\pi N}{L}} e^{b_+(x)}|N\rangle_0$

$= \frac{F}{\sqrt{L}} e^{i\frac{2\pi N}{L}} e^{b_+(x)} e^{b_-(x)}|N\rangle_0$ 湮灭作用到基态

$= \frac{F}{\sqrt{L}} e^{i\frac{2\pi N}{L}} e^{b_+ + b_- + \boxed{\frac{1}{2}[b_+(x), b_-(x)]}}|N\rangle_0$

该常数发散，可以选取适当截断

$\frac{1}{2}[b_+(x), b_-(x)] = \frac{1}{2}\sum_q \alpha_q^*(x)\alpha_q(x)[b_q^\dagger, b_q]$

$= -\frac{1}{2}\sum_q \alpha_q^*(x)\alpha_q(x)$

$\alpha_q = \sqrt{\frac{2\pi}{L|q|}} e^{iqx}$

$= -\frac{1}{2}\sum_q \left(\sqrt{\frac{2\pi}{L|q|}}\right)^2 e^{-iqx+iqx}$

$q = \frac{2\pi n}{L}$

$= -\frac{1}{2}\sum_q \frac{2\pi}{L|q|}$

$= -\frac{1}{2}\sum_{n\geq 1} \frac{1}{n} \to \infty$

选取规范场 $\alpha_q = \sqrt{\frac{2\pi}{L|q|}} e^{iqx - \frac{a}{2}q}$ $a>0$

$= -\frac{1}{2}\sum_q \frac{2\pi}{L|q|} e^{-aq}$

$\ln(1-x) = \sum_{n=1}^{+\infty} \frac{x^n}{n}$

$= -\frac{1}{2}\sum_q \frac{1}{n} e^{-\frac{2\pi a}{L}n}$

$= \frac{1}{2}\ln(1-e^{-\frac{2\pi a}{L}})$

$= \ln\left|\sqrt{\frac{2\pi a}{L}}\right|$

代回 $\psi(x)|N_0\rangle = \frac{F}{\sqrt{L}} e^{i\frac{2\pi N}{L}} e^{b_+ + b_-} \sqrt{\frac{L}{2\pi a}}$

$= \frac{F}{\sqrt{2\pi a}} e^{i\varphi}$

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等价表示：1) $\psi \sim e^{i\Phi}$ ， $\frac{F}{\sqrt{2\pi a}}$ 省略

2) $\psi = \frac{F}{\sqrt{2\pi a}} e^{i\Phi}$ 最完整表示，或 $= \frac{1}{\sqrt{2\pi a}} e^{i\Phi}$

3) Normal ordering $\psi = \frac{F}{\sqrt{L}} :e^{i\Phi}: = \frac{F}{\sqrt{L}} e^{i\Phi_+} e^{i\Phi_-}$ (…

关联函数证明： $\psi = \frac{F}{\sqrt{2\pi a}} e^{i\Phi(x)} \Rightarrow \psi^2(x)$ 如何定义/计算

$\rho = \psi^+(x) \cdot \psi(x)$?

$\Phi = \pi \int_{-\infty}^{x} \rho(x) dx + \theta(x) = \underbrace{\phi(x)}_{\text{密度场构成}} + \underbrace{\theta(x)}_{\text{相位涨落}}$

$e^{-i\Phi(x)} e^{i\Phi(x)} = 1$

$$\begin{cases} \{\psi(x), \psi(y)\} = 0, & \text{if } x \to y \\ \{\psi(x), \psi^+(y)\} = \delta(x-y) & \text{if } x \to y \end{cases}$$

此时F抵消，不起作用.

$$\begin{cases} \varphi = \frac{i}{\sqrt{L}} \sum_{q>0} \frac{e^{iqx}}{\sqrt{q}} b_q \\ b^+ = \frac{i}{\sqrt{L}} \sum_{q>0} \frac{e^{-iqx}}{\sqrt{q}} b_q^+ \end{cases} \qquad \boxed{\Phi = b + b^+}$$

$$\{\psi(x), \psi^+(y)\} = \psi(x)\psi^+(y) + \psi^+(y)\psi(x)$$

$$= \frac{1}{2\pi a}\left(e^{i\Phi(x)} e^{-i\Phi(y)} + e^{-i\Phi(y)} e^{i\Phi(x)}\right)$$

$$= \frac{1}{2\pi a} e^{i\Phi(x) - i\Phi(y)} \{ e^{\frac{1}{2}[\Phi(x), \Phi(y)]} + e^{\frac{1}{2}[\phi(y), \phi(x)]} \}$$

要求 $[\Phi(x), \Phi(y)] = i\pi\, \mathrm{sgn}(x-y)$

$$= \left(\frac{i}{\sqrt{L}}\right)^2 \left[\frac{e^{iqx - \frac{a}{2}q}}{\sqrt{q}} b_q - \frac{e^{-iqx - \frac{a}{2}q}}{\sqrt{q}} b_q^+, \frac{e^{iq'y - \frac{a}{2}q'}}{\sqrt{q'}} b_{q'} - \frac{e^{-iq'y - \frac{a}{2}q'}}{\sqrt{q'}} b_{q'}^+ \right]$$

$q' = q$ 才有意义

$$= \left(\frac{i}{\sqrt{L}}\right)^2 \sum_{q>0} \left(\frac{e^{iq(x-y) - aq}}{q} - \frac{e^{-iq(x-y) - aq}}{q} \right)$$

$$= \frac{1}{L} \sum_{q>0} \frac{1}{q} \left(e^{q(i(x-y) - a)} - e^{q(-i(x-y) - a)} \right)$$

$\sum_n \frac{1}{n} e^{na} = -\ln(1-e^{a})$

$[\Phi(x),\Phi(y)] = \frac{1}{L}\sum_{n\geq1}\frac{L}{2\pi n}\left(e^{\frac{2\pi n}{L}(i(x-y)-a)} - e^{\frac{2\pi n}{L}(-i(x-y)-a)}\right)$

$\ln(1-e^{a}) = \ln a$ $\quad = \frac{1}{2\pi}\left[\ln(1-e^{\frac{2\pi}{L}(i(x-y)-a)}) - \ln(1-e^{\frac{2\pi}{L}(-i(x-y)-a)})\right]$

可证明：$\{\psi(x),\psi^\dagger(y)\} = \frac{1}{2\pi a} e^{i\Phi(x)-i\Phi(y)} \times \{c(x-y)+c(y-x)\}$

其中 $c(x-y) = e^{\frac{1}{2}[\Phi(x),\Phi(y)]}$

$\sim e^{\{\ln(\frac{2\pi}{L}(i(x-y)-a)) - \ln(\frac{2\pi}{L}(-i(x-y)-a))\}}$

$\sim e^{\ln(i(x-y)-a)-\ln(-i(x-y)-a)}$

$\sim e^{\ln\left(\frac{i(x-y)-a}{-i(x-y)-a}\right)} \longleftrightarrow \boxed{\mathrm{sign}(x-y)}$

$\Longrightarrow [\Phi(x),\Phi(y)] \sim \boxed{i\pi\,\mathrm{sign}(x-y)} \sim$

Density 如何计算？

$\rho = \psi^\dagger(x)\psi(x) \longrightarrow \infty$ 发散.

split operator

$= \lim_{\varepsilon\to0} \psi^\dagger(x+\varepsilon)\psi(x) - \langle 0|\psi^\dagger(x+\varepsilon)\psi(x)|0\rangle$

$= \frac{1}{2\pi a} e^{-i\Phi(x+\varepsilon)} e^{i\Phi(x)} - \frac{1}{2\pi a}\langle 0|e^{-i\Phi(x+\varepsilon)} e^{i\Phi(x)}|0\rangle$

$\langle 0|e^{\alpha a^\dagger+\alpha^* a}\, e^{\beta a^\dagger+\beta^* a}|0\rangle$

mathematica

$\frac{\Sigma}{q}$

$\propto \langle 0|e^{(\alpha+\beta)a^\dagger+(\alpha^*+\beta^*)a}|0\rangle$

$\propto \langle 0|e^{(\alpha+\beta)a^\dagger}\cdot e^{(\alpha^*+\beta^*)a}|0\rangle$

$\propto$ 常数 × 1 $\quad\swarrow$ 场 … slow filed

求常数 $\frac{1}{2\pi a}\langle 0|e^{-i\Phi(x+\varepsilon)}\cdot e^{i\Phi(x)}|0\rangle$

$= \frac{1}{2\pi a}\underbrace{e^{\frac{1}{2}[\Phi(x+\varepsilon),\Phi(x)]}}_{\text{发散}}\langle 0|e^{-i\Phi(x+\varepsilon)+i\Phi(x)}|0\rangle$

$\Phi(x) = \Phi_+(x) + \Phi_-(x)$

$= \frac{1}{2\pi a} e^{\frac{1}{2}[\ \ ]}\langle 0|e^{-i(\Phi_+(x+\varepsilon)-\Phi_+(x))+i(\Phi_-(x+\varepsilon)-\Phi_-(x))}|0\rangle$

分开 $e^{xa^\dagger+ya} = e^{xa^\dagger}e^{ya}e^{+\frac{1}{2}xy}$

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$$e^{\frac{1}{2}[\Phi_+(x+\varepsilon)-\Phi_+(x),\ \Phi_-(x+\varepsilon)-\Phi_-(x)]}$$

$$\therefore \rho \sim \frac{1}{2\pi a} e^{-i\Phi(x+\varepsilon)+i\Phi(x)}$$

$$\sim \frac{1}{2\pi a} e^{-i(\frac{\partial\Phi}{\partial x})\varepsilon}$$

$$\doteq \frac{1}{2\pi a}\left(1 - i\frac{\partial\Phi}{\partial x}\varepsilon + \frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2\varepsilon^2 + \cdots\right)$$

$$= \frac{1}{2\pi a} - \frac{1}{2\pi}(\partial_x\Phi)$$

意味着 $:\psi^+\psi: \sim \partial\Phi$

$:\psi^+\partial\psi: \sim (\partial\Phi)^2$

Dirac eq　　Bose-filed.

下次课：xxz model　$\boxed{\text{KPZ eq}}$