Chapter3 Bosonization-II

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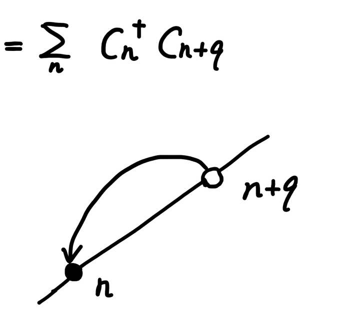
$\rho_q = \sum_n C_n^+ C_{n+q}$



$\begin{cases} F: & \text{Jordan-Wigner} \quad \psi = e^{i\Phi} \\ B: & \end{cases}$ $\quad \psi^+\psi = e^{-i\Psi} e^{i\Psi} = 1$

$\left.\begin{aligned}[H_F, \rho_q] &= -vq\rho_q \\ [H_B, \rho_q] &= -vq\rho_q\end{aligned}\right\}$ 关系相同

$\boxed{\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + A\cos(\phi)}$

$\downarrow RG$

...

$\int_0^{+\infty} \ln(1+e^{-x})\,dx = \frac{\pi^2}{12}$

$\int_0^{+\infty} -\ln(1-e^{-x})\,dx = \frac{\pi^2}{6}$

重要关系 ×2

BKT 相变

XY model $\qquad \mathcal{L} = \frac{1}{2}(\partial_x\phi)^2 + \frac{1}{2}(\partial_y\phi)^2 + A\cos(\beta\phi)$

相变（∞阶相变） $\qquad$ 经典问题 $\downarrow$

补充：Landau 相变 $\qquad \mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + A\cos(\beta\phi)$

Beyond Landau 理论

1) Anderson localization
2) BKT phase transition
3) QHE / FQHE

$\partial_t^2\phi = \partial_x^2\phi + A\beta\sin(\beta\phi)$

Sine - Gordon Eq
Soliton：时空 vortex
可解 model

$\mathcal{L}(\psi^+, \psi) \xrightarrow{\psi \sim e^{i\phi}} \mathcal{L} = \frac{1}{2}(\partial_\tau\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - A\cos(\beta\phi)$

研究这个 model 的相变 (BKT)

key point：A不是常数，和能标有关

$Z = \int D\phi\, e^{-S}$

$S = \int dt\,dx \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + A\cos(\beta\phi)\right]$

标度分析 $\boxed{d=2}$

$\int dt\,dx\,(\partial_t\phi)^2 = \int dt'dx'\,(\partial_{t'}\phi'(x',t'))^2$

取 $t' = \lambda t$, $x' = \lambda x$ $\qquad \lambda^2/\lambda^2 = 1$

$= \int dt\,dx\,\partial_t(\phi'(\lambda x, \lambda t))^2$

$\boxed{\phi'(\lambda x', \lambda t') = \phi(x,t)} \iff \boxed{[\phi] = 0}$

$\int dt\,dx\, A\cos(\beta\phi)$

$\Downarrow$

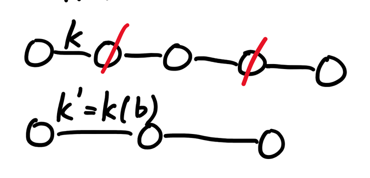
$\int dx\,dy\; y\Lambda^2\cos(\beta\phi)$

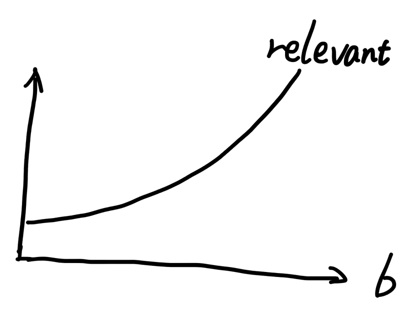
$\int dt\,dx\; y\frac{1}{a^2}\cos(\beta\phi)$

$\left.\begin{aligned} x &\to bx \\ t &\to bt\end{aligned}\right\}$ $\int A b^2\cos(\beta\phi)\,dt\,dx$

$= A(b)\int\cos(\beta\phi)\,dt\,dx$

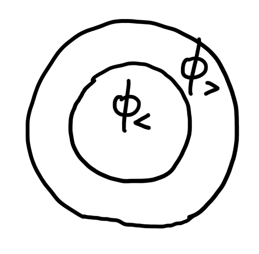
$A(b) = Ab^2$





$Z = \int D\phi\; e^{-\int\frac{1}{2}(\partial\phi)^2 + A\cos(\beta\phi)}$

方法：实空间处理 $\qquad = \int D\phi_< D\phi_>\; e^{-(S_< + S_> + \delta S)}$



其中 $S_< = \int dx\,\frac{1}{2}(\partial\phi_<)^2 + A\cos(\beta\phi_<)$

$S_> = \int dx\,\frac{1}{2}(\partial\phi_>)^2$

$\delta S = \int A\cos[\beta(\phi_< + \phi_>)] - A\cos(\beta\phi_<)$

$S_< = \int dx\,\frac{1}{2}(\partial_x\phi_<)^2$

$S_>$ 不变

$\delta S = \int A\cos[\beta(\phi_< + \phi_>)]$

• 第一种处理

$\cos[\beta(\phi_< + \phi_>)] - \cos\beta\phi_<$

$= \cos\beta\phi_<\,\cos\beta\phi_> - \sin\beta\phi_<\,\sin\beta\phi_> - \cos\beta\phi_<$

$= \cos\beta\phi_<\,(1 - \frac{1}{2}\beta^2\phi_>^2) + \dots$

$= \boxed{-}\,\frac{\beta^2}{2}\phi_>^2\cos(\beta\phi_<) - \sin(\beta\phi_<)(\beta\phi_>)$

$Z = \int D\phi_<\; e^{-S_<}\langle e^{-\delta S}\rangle_> \times \text{const}$

$\langle e^{-\delta S}\rangle = e^{-\langle\delta S\rangle_>^c + \frac{1}{2}\langle\delta S^2\rangle_>^c}$

$\delta S = -\frac{A}{2}\beta^2\int dx\,\cos(\beta\phi_<)\,\phi_>^2$

$\langle\delta S\rangle_>^c = -\frac{A\beta^2}{2}\int dx\,\cos(\beta\phi_<)\langle\phi_>^2(x)\rangle$

$= -\frac{A\beta^2}{2}\frac{1-b}{2\pi}\int dx\,\cos(\beta\phi_<)$

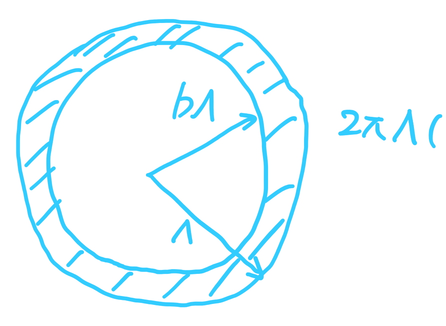
Remark
① ϕ^4 $\quad$ 1阶+2阶
② ϕ^3 $\quad$ $d=6$
③ SG $\quad$ 1阶

$\langle\psi^+(x)\psi(y)\rangle \sim g(x-y)$

$\langle\phi_>^2(x)\rangle = \langle\phi_>^2(0)\rangle$

$= \sum_{b\Lambda<|k|<\Lambda}\frac{1}{k^2} = \left(\frac{1}{2\pi}\right)^2$

$= \left(\frac{1-b}{2\pi}\right)\frac{2\pi\Lambda^2(1-b)}{\Lambda^2}$



$Z = \int D\phi_<\; e^{-\int dx\,\frac{1}{2}(\partial\phi_<)^2 + A\cos(\beta\phi_<) - \frac{A\beta^2}{2}\frac{1}{2\pi}(1-b)\cos(\beta\phi_<)}$

$= \int D\phi_<\; e^{-\int dx\,\frac{1}{2}(\partial\phi_<)^2 + A(1 - \frac{1-b}{4\pi}\beta^2)\cos(\beta\phi_<)}$

$x \to \frac{1}{b}x,\ t \to \frac{1}{b}t$

$= \int D\phi\; e^{-\int dx\,\frac{1}{2}(\partial\phi)^2 + Ab^{-2}(1 - \frac{1-b}{4\pi}\beta^2)\cos(\beta\phi)}$

此处 $b = 1 - dl = 1 - (d\Lambda/\Lambda) = 1 - d\ln\Lambda$

$A(b) = Ab^{-2}(1 - \frac{1-b}{4\pi}\beta^2)$

$= A(1 + 2dl)(1 - \frac{dl}{4\pi}\beta^2)$

$= A[1 + (2 - \frac{\beta^2}{4\pi})dl]$

$= A - \frac{dA}{d\ln\Lambda}dl$

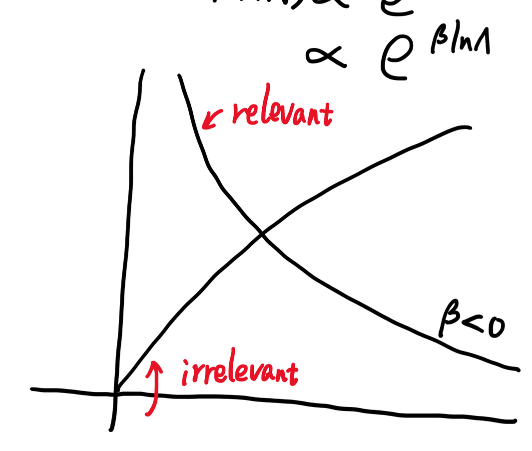
$= A - \beta\,dl$

$A(1-dl) = A - \frac{dA}{d\ln\Lambda}dl$

$$\boxed{\begin{aligned}\frac{dA}{d\ln\Lambda} &= \left(\frac{\beta^2}{4\pi} - 2\right)A \\ &= A\left(\frac{k_BT}{4\pi J} - 2\right)\end{aligned}}$$

$A(\Lambda) \propto e^{(\frac{\beta^2}{4\pi} - 2)\ln\Lambda}$

$\propto e^{\beta\ln\Lambda}$



相变点 $\boxed{\beta^2 = 8\pi}$

该结果与A无关（因为忽略了高阶效应）

Wen's book

$\langle e^{\int A\cos\beta(\phi_< + \phi_>)}\rangle_> = e^{\langle\int dx\,A\cos\beta(\phi_< + \phi_>)\rangle_>}$

$= e^{\frac{A}{2}\int dx\,(e^{i\beta\phi_<}\langle e^{i\beta\phi_>}\rangle_> + h.c.)}$

$= e^{\frac{A}{2}\int dx\,(e^{i\beta\phi_<}e^{-\frac{\beta^2}{2}\langle\phi_>^2\rangle} + h.c.)}$

$= e^{A\,e^{-\frac{\beta^2}{2}\langle\phi_>^2\rangle}\int\cos(\beta\phi_<)dx}$

以上计算过程

ref：① Shankar's book $\quad$ §18.4.2 P352 - P354

② X. G. Wen's book

高阶效应 Wen's book

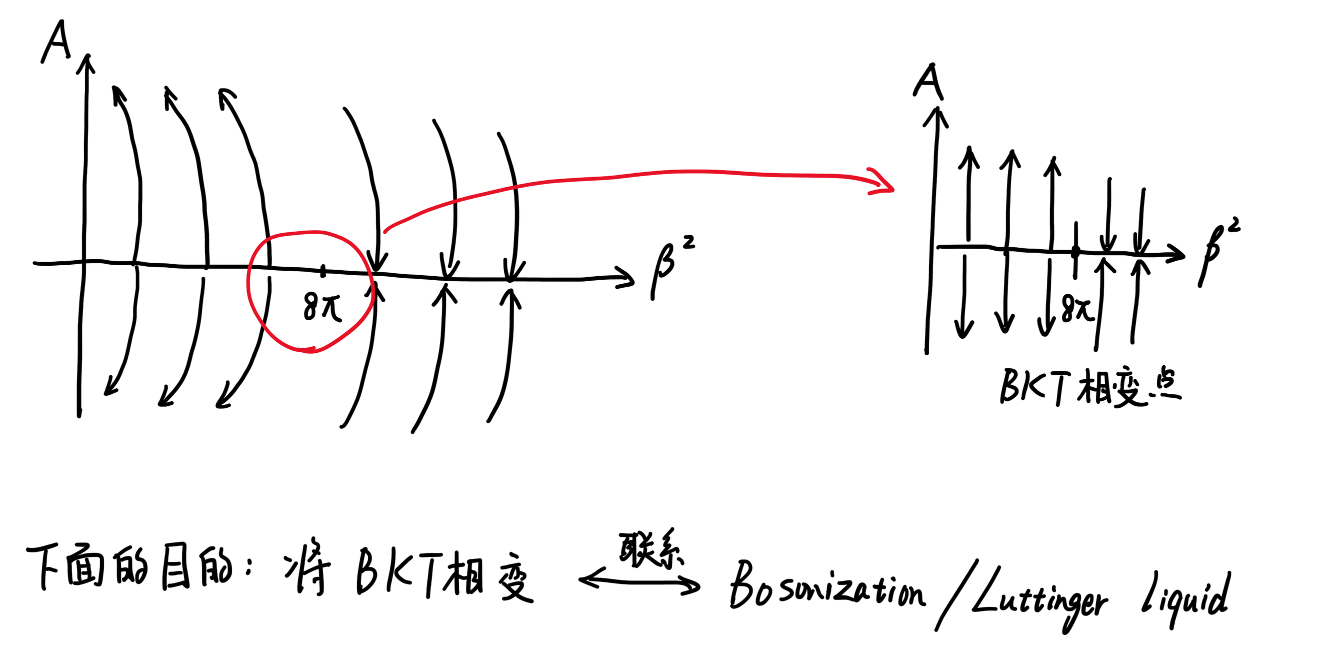
二阶：$\left\langle\int dx\,dy\,\cos(\beta\phi_< + \beta\phi_>)_x\,\cos[\beta(\phi_< + \phi_>)]_y\right\rangle_>$

$\langle\phi_>(x)\phi_>(y)\rangle \propto \frac{1}{(x-y)^{\square}}$

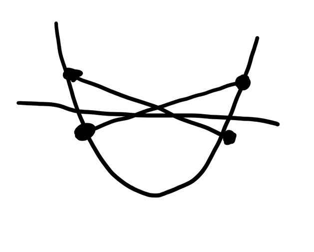
$y = x + z$

$= \int dx\,dz\,\langle\cos\beta\,\phi(x)\cos(\beta\phi(x+z))\rangle_>$

$\frac{dA}{d\ln\Lambda} = \left(\frac{\beta^2}{4\pi} - 2\right)A + (\cdots)A^2$



下面的目的：将 BKT 相变 $\xleftrightarrow{\text{联系}}$ Bosonization / Luttinger liquid



$H_F = \psi^+\left(\frac{d^2}{dx^2}\right)\psi + V_{int}$

两个过渡

H_B

$\mathcal{L}_F$

$\psi = e^{i\Phi}$

$\psi \sim e^{i\Phi}$

$\cos(\Phi)$

Φ^2

$\Phi \to \Phi + \text{const}$

$\Phi \sim \Phi + 2\pi$

$\mathcal{L}_B = \frac{1}{2}\left[(\partial_t\Phi)^2 - v^2(\partial_x\Phi)^2\right] + A\cos(\Phi)$