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Remarks on Bloch's Method of Sound Waves applied to Many-Fermion Problems

Sin-itiro TOMONAGA

Institute for Advanced Study
Princeton, N. J.
U.S.A.*

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Abstract

The fact implied by Bloch several years ago that in some approximate sense the behavior of an assembly of Fermi particles can be described by a quantized field of sound waves in the Fermi gas, where the sound field obeys Bose statistics, is proved in the one-dimensional case. This fact provides us with a new possibility of treating an assembly of Fermi particles in terms of the equivalent assembly of Bose particles, namely, the assembly of sound quanta. The field equation for the sound wave is found to be linear irrespective of the absence or presence of mutual interaction between particles, so that this method is a very useful means of dealing with many-Fermion problems. It is also applicable to the case where the interparticle force is not weak. In the case of force of too short a range this method fails.

§ 1. Introduction and summary.

The well-known method of Thomas and Fermi provides us with a very practical approximate treatment of many-Fermion problems. Because in this approximation, however, each particle is supposed to move independently, the effect of interaction between particles being simply replaced by an average field of force, one cannot speak of correlations between particles in this rough approximation. This simple method does not apply to problems in which inter-particle correlation plays an important role. A step toward the improvement of the method so as to include the correlation was taken by Euler[1], who calculated the effect of inter-particle interaction, which causes the correlation, by perturbation theory assuming the interaction to be small.

The calculation of such a type is carried out in the following manner: In the zeroth approximation each particle is in some one-particle quantum state, which we shall call a "level". Let us consider the lowest state for the sake of definitness. In this state all levels up to some highest one, which shall be called the "Fermi maximum", are each filled by one particle. The perturbation energy, the energy due to the inter-particle force, has non-vanishing matrix

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* Now returned to the Tokyo University of Education (Tokyo Bunrika Daigaku), Tokyo.

elements which cause virtual transitions to states in which two particles are excited simultaneously to levels higher than the Fermi maximum.

The state which results from the inter-particle force is thus such one which is a superposition of the zero-order state and various excited states in which holes and particles are present in some levels below and above the Fermi maximum respectively. In the lowest approximation of the perturbation calculation, the numbers of holes and excited particles are two; but, if the calculation is carried out to higher order, which is necessary when the inter-particle force is not small, there appear states in which a considerable number of holes and excited particles are present. The amplitudes of such highly excited states will be appreciable if the inter-particle force is strong.

Such a mixture of excited states gives rise to correlation between particles. We can in this way include the effects of correlations in treating certain problems. In the case of strong inter-particle forces, however, it is necessary to carry through the perturbation calculation up to a very high order, and this is too involved to be practicable. Recently, Nogami[2] proposed a method which applies also to cases of rather strong inter-particle forces. Still his method requires some kind of weakness of interaction because he had to neglect the interactions of holes and of excited particles with each other as well as the interactions between holes and exited particles. This neglect cannot be justified when too many holes and excited particles are present.

In such a situation it is desirable to find some approximate method of dealing with many-Fermion problems different from the perturbation method.

In his famous work on the stopping power of charged particles, Bloch[3] has treated the excited states of the Fermi gas not as states with holes and excited particles, but as states in which the gas oscillates. In this work it was not necessary to treat the oscillation quantum-theoretically. In a later paper[4], he also dealt with a problem in which the quantum aspect of the oscillation was essential. This was a problem in which the density fluctuation played a role. He showed in this work that the density fluctuation of a degenerate Fermi gas (in his theory it was sufficient to treat the gas without inter-particle force) can be calculated in two different ways giving the same result. The one was the orthodox method whereby one calculates directly the expectation value of the operator having expressed in terms of the quantized field variables ψ and ψ^* describing the assembly of the Fermi particles. The other method was to calculate the zero-point amplitudes of sound waves in the gas, the equations of motion for the sound waves being properly chosen. He showed that the correct value of the fluctuation was obtained in this way if the choice of the equations of motion for the sound waves was properly made and if the zero-point amplitudes of the waves had such values as would be expected for a sound field obeying Bose statistics.

If it is proved that the excited states of an assembly of Fermi particles can be in fact described as an excitation of sound waves, where the sound can be described by a Bose field, it will provide us with a new method of treating many-Fermion problems, not dealing directly with the assembly of the Fermi particles, but dealing with the equivalent assembly of Bose particles, i. e. the assembly of sound quanta.

There is a prospect that in this latter method the assumption of the weakness of inter-pariicle forces will not be essential. This prospect lies in the following situation: It is expected that the equations of motion for the sound waves will be linear in the field variables describing the sound field; otherwise we could not speak of waves at all. The field variables for the sound field will be the density ρ of the gas and its properly defined canonical conjugate. Now the essential point is that the linearity of the problem will not be destroyed even when an inter-particle force is present, because the interaction energy between particles is bilinear in ρ. This fact is in marked contrast to the circumstance that in terms of ψ and ψ^* the interaction energy contains four ψ's, or, more precisely, two ψ's and twe ψ^*'s; thus the field equations for ψ and ψ^* are no longer linear when an inter-particle force is present. This circumstance made it very difficult to treat many-Fermion problems with inter-particle forces. This difficulty will disappear when we deal with the problem in terms of the ρ field but not of the ψ field.

The purpose of this paper is to show that these expectations are really fulfilled. It will be shown that in some approximation, which does not necessarily require the weakness of the interaction, excitated states of Fermi gas are in fact equivalent to corresponding excitations of sound waves, and that the sound is describable by a Bose field whose field equations are linear in the field variables irrespective of the absence or presence of interparticle forces. Thus, Bloch's method of sound waves will be a very useful method for many-Fermion problems.

The possibility of this new method was found independently by Bohm[5], who discusses a very interesting phenomenon of plasma-like oscillations in a degenerate electron gas from a very similar point of view. He, too, gives a proof of the fact that these oscillations are describable by a Bose field in some approximation. He utilizes further the linearity of the field equations to study such a pure correlation phenomenon as plasma oscillations of an electron gas.

The present paper is of a rather more mathematical nature than physical in that it aims mainly at the analysis of the mathematical structure of the method, clarifying the underlying assumptions and the limit of applicability. A mathematically closed and clear-cut presentation of the theory is achieved, however, at the expense of physical usefulness, because, thus far, the author has succeded only in giving a complete formulation for a one-dimensional assembly of particles.

It is rather certain that a similar method applies to the three dimensional case too—this, indeed, has been done by Bohm—but the situation is more complicated in this case.

The mathematicl relation between the field of sound quanta on the one hand and the original field of Fermi particles on the other is very similar to the relation between the field of light quanta and the field of neutrinos in the neutrino theory of light[6]. One will find everywhere a marked parallelism between our theory and the neutrino theory of light.

The discussions will be performed in several steps: In § 2 we shall prove that the sound can be described by a Bose field under some assumptions imposed on the states under consideration. Then we shall set up in § 3 the Hamiltonian for the sound field. We shall see that the Hamiltonian is in fact bilinear in the field variables, so that the whole problem is liner. In § 4 we shall show that this Hamiltonian is really equal to the original Hamiltonian for the assembly of Fermi particles, and, therefore, the assembly of sound quanta is equivalent to the original assembly of Fermi particles. In § 5 we will go over to the solution of the eigenvalue problem. This can be done very simply by finding the normal coordinates for the sound field, because the field equations are linear. In § 6 we shall give several general formulae of physical interest which are derived directly by our method. A criterion for the applicability of the method will also be given in this section. In the last section we shall briefly mention the bearing of our results on the plasma oscillations treated by Bohm. Also, the relation between the two kinds of descriptions of the system, one as an assembly of Fermi particles and the other as an assembly of sound quanta, is discussed briefly.

In § 6 we shall see that our method does not work if the inter-particle force is of short range. The range of force must be larger than four times the mean distance between particles. Since sound waves with wave length shorter than the mean distance between particles have no meaning, it is quite conceivable that the method of sound waves fails in describing the event occuring in a small space region comparable with the mean distance between particles. This is the reason why the method fails in the case of short range force. The method is, on the other hand, very suitable for dealing with the case of long-range force in which a considerable number of holes and excited particles are present in the neighborhood of the Fermi maximum. Thus the present method covers a field where the known methods have failed.

§ 2. Approximate commutation relations for the density field.

Let us consider an assembly of Fermi particles in a one-dimensional "box" of length L. Let $\psi(x)$ and $\psi^*(x)$ be the quantized wave functions describing the assembly. The density $\rho(x)$ of the particles is then given by

$$\rho(x) = \psi^*(x)\psi(x). \tag{2.1}$$

We introduce the Fourier transforms ψ_n and ψ_n^* of $\psi(x)$ and $\psi^*(x)$ respectively. They are defined by

$$\begin{cases} \psi(x) = \dfrac{1}{\sqrt{L}} \sum_n \psi_n \exp\left(\dfrac{2\pi i}{L} nx\right), \\ \psi^*(x) = \dfrac{1}{\sqrt{L}} \sum_n \psi_n^* \exp\left(-\dfrac{2\pi i}{L} nx\right), \qquad n=0, \pm 1, \pm 2, \cdots. \end{cases} \tag{2.2}$$

The Fourier transform of $\rho(x)$, which is defined by

$$\rho(x) = \frac{1}{L} \sum_n \rho_n \exp\left(\frac{2\pi i}{L} nx\right), \tag{2.3}$$

is evidently given by

$$\rho_n = \sum_{n'} \psi_{n'}^* \psi_{n'+n} = \sum_{\bar n} \psi_{\bar n - \frac{n}{2}}^* \psi_{\bar n + \frac{n}{2}}, \tag{2.4}$$

where $\bar n$ takes integral values when n is even and half-odd integral values when n is odd.

Our purpose is to describe the system in terms of the density field $\rho(x)$ instead of describing it by the wave field $\psi(x)$. The first task is to find the commutation relations between the field quantity $\rho(x)$ and its properly defined canonical conjugate. For this purpose we separate each ρ into two parts ρ_n^+ and ρ_n^- by means of

$$\begin{cases} \rho_n^+ = \sum_{\bar n > 0} \psi_{\bar n - \frac{n}{2}}^* \psi_{\bar n + \frac{n}{2}}, \\ \rho_n^- = \sum_{\bar n < 0} \psi_{\bar n - \frac{n}{2}}^* \psi_{\bar n + \frac{n}{2}}. \end{cases} \tag{2.5}$$

We have evidently

$$\rho_n = \rho_n^+ + \rho_n^- \tag{2.6}$$

(in case of even n the term with $\bar n = 0$ is absent in (2.6), but this does not cause any serious error.) We have further

$$\rho_n^+ = \rho_{-n}^{+*}, \qquad \rho_n^- = \rho_{-n}^{-*}. \tag{2.7}$$

It will be seen later that the separation of ρ_n by (2.6) corresponds to the separation of the field into parts with positive and negative frequencies, which is the usual procedure in field theory. (See (3.1) and (3.2)).

We now examine the commutation relations of the ρ's. The commutation relations between the ψ's and ψ^*'s are

$$\begin{cases} [\psi_n^*, \psi_{n'}]_+ = \delta_{n,n'}, \\ [\psi_n^*, \psi_{n'}^*]_+ = [\psi_n, \psi_{n'}]_+ = 0. \end{cases} \tag{2.8}$$

From these relations we get the commutation relations of the ρ's. For the ρ^+'s they are

$$[\rho_n^+, \rho_{n'}^+] = \begin{cases} \sum\limits_{-\frac{n}{2} < \bar n \leqq \frac{|n'|}{2}} \psi_{\bar n - \frac{n}{2} - \frac{n'}{2}}^* \psi_{\bar n + \frac{n}{2} + \frac{n'}{2}} & \text{for } n > 0,\ n' \leqq -n, \\ \sum\limits_{\frac{n'}{2} < \bar n \leqq \frac{n}{2}} \psi_{\bar n - \frac{n}{2} - \frac{n'}{2}}^* \psi_{\bar n + \frac{n}{2} + \frac{n'}{2}} & \text{for } n > 0,\ -n \leqq n \leqq n, \\ -\sum\limits_{\frac{n}{2} < \bar n \leqq \frac{n'}{2}} \psi_{\bar n - \frac{n}{2} - \frac{n'}{2}}^* \psi_{\bar n + \frac{n}{2} + \frac{n'}{2}} & \text{for } n > 0,\ n' \geqq n, \end{cases} \tag{2.9}$$

which, for the special cases of $n'=-n$, reduce to

$$[\rho_n^+,\ \rho_{-n}^+]=\sum_{-\frac{n}{2}<\bar{n}\leqq\frac{n}{2}}\psi_{\bar{n}}^*\psi_{\bar{n}} \qquad \text{for } n>0. \tag{2.9'}$$

In the same way, we get for the ρ^-'s

$$[\rho_n^-,\ \rho_{n'}^-]=\begin{cases} -\sum\limits_{\frac{n'}{2}\leqq\bar{n}<\frac{n}{2}}\psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}}\psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } n>0,\ n'\geqq -n,\\ -\sum\limits_{-\frac{n}{2}\leqq\bar{n}<-\frac{n'}{2}}\psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}}\psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } n>0,\ -n\leqq n'\leqq n,\\ \sum\limits_{-\frac{n'}{2}\leqq\bar{n}<-\frac{n}{2}}\psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}}\psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } n>0,\ n'\geqq n, \end{cases} \tag{2.10}$$

and

$$[\rho_n^-,\ \rho_{-n}^-]=-\sum_{-\frac{n}{2}\leqq\bar{n}<\frac{n}{2}}\psi_{\bar{n}}^*\ \psi_{\bar{n}} \qquad n>0. \tag{2.10'}$$

The commutation relations between the ρ^+'s and ρ^-'s are found to be

$$[\rho_n^-,\ \rho_{n'}^+]=\begin{cases} \sum\limits_{-\frac{n}{2}<\bar{n}<\frac{n'}{2}}\psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}}\psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } -n<n',\\ -\sum\limits_{\frac{n}{2}<\bar{n}<-\frac{n'}{2}}\psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}}\psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } n<-n', \end{cases} \tag{2.11'}$$

and

$$[\rho_n^-,\ \rho_{-n}^+]=0. \tag{2.11'}$$

We now show that the commutation relations (2.9)—(2.11′) can be replaced by simpler ones if the states under consideration satisfy, at least approximately, some conditions which will be specified below. These simplified commutation relations tell us that the density field can be regarded as a Bose-field under these restricting conditions.

We first consider the case of the ideal Fermi gas in which there are no interactions between particles. If the gas is not excited too highly, only particles in the neighborhood of the Fermi maximum are raised to higher levels. There exist holes and excited particles only in the neighborhood of the surface of the Fermi sea. Now, in the case of a non-ideal Fermi gas, the inter-particle forces cause virtual transitions of particles. Thus extra holes and exited particles appear. But, if the range of the inter-particle force is not too short and the force itself is not too strong, these virtual holes and excited particles are still present only in the neighborhood of the Fermi maximum. Such are the states to which we will confine ourselves.

If we confine ourselves to states of such type, we can simplify the commutation relations (2.9)—(2.11′) in the following manner.

Let us consider, for instance, the first commutation relations $[\rho_n^+,\ \rho_{n'}^+]$ for which $n>0$ and $-n<n'<n$. We notice that the expression on the right-

hand side is a sum of operators each bringing one particle from the level $\bar{n}+\frac{n}{2}+\frac{n'}{2}$ to the level $\bar{n}-\frac{n}{2}-\frac{n'}{2}$. Because the summation over $\bar{n}$ is extended only between $\frac{n'}{2}$ and $\frac{n}{2}$, the final levels $\bar{n}-\frac{n}{2}-\frac{n'}{2}$ lie in a limited interval between $-\frac{n}{2}$ and $-\frac{n'}{2}$. Now, let $n_{\max}$ denote the value of $|n|$ at the Fermi maximum. Then, if n and $|n'|$ are both sufficiently small compared with $n_{\max}$ (the discussion about how small they should be will be given below), the levels $-\frac{n}{2}$ and $-\frac{n'}{2}$ both lie deep in the bottom of the Fermi sea where there are holes. In such a case the operator $\psi^+_{\bar{n}-\frac{n}{2}-\frac{n'}{2}}\psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}}$ will give a vanishing result because the final level is occupied. Thus, for the states under considration our commutators $[\rho_n^+, \rho_{n'}^+]$ are equivalent to zero.

We next consider $[\rho_n^+, \rho_{-n}^+]$. In this case, the right-hand side is

$$\sum_{-\frac{n}{2}<\bar{n}\leq\frac{n}{2}} \psi_{\bar{n}}^*\psi_{\bar{n}} = \sum_{-\frac{n}{2}<\bar{n}\leq\frac{n}{2}} N_{\bar{n}} \;, \tag{2.9''}$$

where $N_{\bar{n}}$ is the occupation number of the level $\bar{n}$. Since the level $\bar{n}$, which lies between $-\frac{n}{2}$ and $\frac{n}{2}$, lies deep in the Fermi sea if n is small compared with $n_{\max}$, it is occupied by one particle. Then the sum $\sum N_{\bar{n}}$ is simply equal to the number of levels between $-\frac{n}{2}$ and $\frac{n}{2}$, which is just n. So we find that $[\rho_n^+, \rho_{-n}^+]$ is equivalent to n.

A similar consideration applies to the remaining commutators. We can see that the following commutation relations hold in the sense of equivalence:

$$\begin{aligned} &[\rho_n^+, \rho_{n'}^+]=n\delta_{n,-n'} \\ &[\rho_n^-, \rho_{n'}^-]=-n\delta_{n,-n'} \\ &[\rho_n^+, \rho_{n'}^-]=0. \end{aligned} \tag{2.12}$$

In order that these simpler commutation relations can be used instead of the original ones, it was necessary that $|n|$ and $|n'|$ be sufficiently small compared with $n_{\max}$. We now discass this point more quantitatively.

First we notice that the total number N of the particles is related to $n_{\max}$ by

$$N=2n_{\max}+1, \tag{2.13}$$

which we shall make use of in later considerations.

We now assume that in the states under consideration there are no holes under the level specified by $|n|=an_{\max}$, a being a positive number less than unity. Then, we can easily see that

$$\frac{3}{2}|n|,\ \frac{3}{2}|n'| < a\, n_{\max} \tag{2.14}$$

is the required condition for the validity of the simplar commutation relations. (The factor 3/2 is required in order that (2.11) be equivalent to the third relation of (2.12)).

So far as the present consideration is concerned, there is no restriction on a. But we shall see later that a must be 3/4 in order that the whole treatment work consistently. The required conditions will be found to be the following:

(I) In the region $|n| < \frac{3}{4} n_{\max}$, there should be no holes.

(II) In the region $|n| > \frac{5}{4} n_{\max}$, there should be no excited particles.

(III) The absolute values of n in ρ_n^+ and ρ_n^- should not exceed $\frac{1}{2} n_{\max}$:

$$|n| < \frac{1}{2} n_{\max}. \tag{2.15}$$

Our method works when and only when these conditions are satisfied. The conditions (I) and (II) restrict the states; therefore, it is always necessary in applying our method to verify whether the states obtained as an answer do really satisfy these conditions. The condition (III) requires that no sound waves having shorter wave length than $2L/n_{\max}$ play a role in the problem. This condition will be satisfied if the range of inter-particle force is sufficiently long. The reason for the the necessity of the special choice of $a=3/4$ will be given later.

§ 3. Equation of motion and Hamiltonian for the density field.

We first consider the case of non-interacting particles. In this case the change with time of ρ_n^+ and ρ_n^- can be obtained easily because we know the change with time of ψ_n and ψ_n^*. In the general case we them find a very complicated time dependence of the ρ's. The motions of ρ_n^+ and ρ_n^- are by no means simple harmonic because each term of $\sum \psi^*_{\bar{n}-\frac{n}{2}}\, \psi_{\bar{n}+\frac{n}{2}}$ has a frequency which depends not only on n but also on $\bar{n}$. This fact means that the density does by no means behave like waves.

However, we may here make use of our conditions imposed on the states. According to them holes and excited particles are present in a narrow interval only, from $(3/4)n_{\max}$ to $(5/4)n_{\max}$, in the neighborhood of $n_{\max}$. This results in non-vanishing matrix elements in $\sum \psi^*_{\bar{n}-\frac{n}{2}}\, \psi_{\bar{n}+\frac{n}{2}}$ being contributed solely by terms with $\bar{n} \pm \frac{n}{2}$ nearly equal to $n_{\max}$ or $-n_{\max}$ according as we are dealing with ρ_n^+ or ρ_n^-. We may then approximate the frequencies of ψ^* and ψ by their expansion in the neighborhood of $\bar{n}\pm\frac{n}{2}=n_{\max}$ or $\bar{n}\pm\frac{n}{2}=-n_{\max}$ respectively. The frequency of $\psi_{\bar{n}+\frac{n}{2}}$ is, for instance,

$$\frac{1}{2\pi\hbar}\frac{1}{2m}\left(\frac{2\pi\hbar}{L}\right)^2\left(\bar{n}+\frac{n}{2}\right)^2,$$

m being the mass of the particles. If we here expand $\left(\bar{n}+\frac{n}{2}\right)^2$ in the neighborhood of $\bar{n}+\frac{n}{2}=n_{\max}$, we get

$$\left(\bar{n}+\frac{n}{2}\right)^2=\left\{n_{\max}+\left(\bar{n}+\frac{n}{2}-n_{\max}\right)\right\}^2$$

$$=n_{\max}^2+2n_{\max}\left(\bar{n}+\frac{n}{2}-n_{\max}\right)+\cdots.$$

Neglecting the small term $\left(\bar{n}+\frac{n}{2}-n_{\max}\right)^2$, the time dependence of $\psi_{\bar{n}+\frac{n}{2}}$ becomes

$$\psi_{\bar{n}+\frac{n}{2}}\approx\exp i\left[+\frac{1}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}^2}{2m}-\frac{1}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}\left(\bar{n}+\frac{n}{2}\right)\right]t.$$

In the same way we have

$$\psi_{\bar{n}-\frac{n}{2}}\approx\exp i\left[-\frac{1}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}^2}{2m}+\frac{1}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}\left(\bar{n}-\frac{n}{2}\right)\right]t.$$

Then, combining these two, we get a time dependence for ρ_n^+ of the form

$$\rho_n^+\approx\exp\left[-\frac{i}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}n\right]t. \tag{3.1}$$

By the same consideration using the expansion in the neighborhood of $\bar{n}\pm\frac{n}{2}=-n_{\max}$, we get

$$\rho_n^-\approx\exp\left[+\frac{i}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}n\right]t. \tag{3.2}$$

From (3.1) and (3.2) we see that ρ^+ and ρ^- satisfy the equations of motion

$$\begin{cases}\dot{\rho}_n^+=-\frac{i}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}n\rho_n^+\\ \dot{\rho}_n^-=+\frac{i}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}n\rho_n^-.\end{cases} \tag{3.3}$$

We notice here that the commutation relations (2.12) lead to the fact that the canonically conjugate momentum for

$$\rho_n=\rho_n^++\rho_n^- \tag{3.4}$$

is given by

$$\pi_n = \frac{i}{2}\frac{1}{n}\left(\rho_{-n}^{+} - \rho_{-n}^{-}\right). \tag{3.5}$$

As can be easily verified, we see that

$$[\rho_n, \pi_{n'}] = i\delta_{n,n'}. \tag{3.6}$$

By the equation of motion we then find

$$\pi_n = \frac{\hbar}{2}\left(\frac{L}{2\pi\hbar}\right)^2 \frac{m}{n_{\max}} \frac{1}{n^2}\dot{\rho}_{-n}. \tag{3.5'}$$

The last relation gives a physical meaning to our canonical momenta: As usual, momenta are proportional to the time derivatives of the coordinates. Though the equations of motion of the form of (3.3) hold only for the case of non-interacting particles, we shall see later that (3.5′) holds more generally.

We shall now set up the Hamiltonian for the ρ field. The Hamiltonian is determined by the requirement that

$$\frac{\hbar}{i}\dot{\rho}_n^{\pm} = [\mathfrak{H}, \rho_n^{\pm}] \tag{3.7}$$

yield the equations of motion (3.3) wtih the help of the commutation relations (2.12). We then find that the Hamiltonian has a very simple form:

$$\mathfrak{H} = \left(\frac{2\pi\hbar}{L}\right)^2 \frac{n_{\max}}{m} \sum_{n>0} (\rho_{-n}^{+}\rho_n^{+} + \rho_n^{-}\rho_{-n}^{-}). \tag{3.8}$$

The order of non-commuting factors is here specified by the condition that $\mathfrak{H}$ has no zero point value.

Though we have set up the Hamiltonian in this way, it is by no means self-evident that this Hamiltonian gives in fact the energy of the system (or eventually energy plus some additive constant). That this is actually the case will be proved in the next section.

So far we have neglected the mutual interaction between particles. It is quite simple to introduce the interaction into the theory, when the interaction force is of the ordinary type, neither of exchange type nor velocity dependent. In this case the interaction energy has the form

$$H_{\text{int}} = \frac{1}{2}\iint \rho(x)\rho(x')J(|x-x'|)dxdx' - \frac{1}{2}\int \rho(x)J(0)dx, \tag{3.9}$$

$J(|x-x'|)$ being the potential of the inter-particle force. The term $\frac{1}{2}\int \rho(x)J(0)dx$ is subtracted in order to remove the interaction of a particle with itself. In terms of the Fourier transform we find

$$H_{\text{int}} = \frac{1}{2}\sum_n{}' \rho_n\rho_{-n}J_n + \frac{1}{2}\rho_0^2 J_0 - \frac{1}{2}\rho_0 J(0), \tag{3.10}$$

where J_n is the matrix element of the potential defined by

$$J_n=\frac{1}{L}\int J(x)\exp\left(\frac{2\pi i}{L}nx\right)dx=J_{-n}. \tag{3.11}$$

The prime on the $\sum'$ symbol means that the term with $n=0$ should be omitted. ρ_0 is the quantity defined by

$$\rho_0=\sum_{\bar{n}}\psi_{\bar{n}}^*\psi_{\bar{n}}=N, \tag{3.12}$$

i.e. the total number of particles.

Since $H_{\rm int}$ commutes with each ρ_n, which is the sum $\rho_n^+ + \rho_n^-$, the equation of motion for ρ_n is not affected by the interaction, while the equations of motion for ρ_n^+ and ρ_n^- separately are affected by the interaction. This fact means that the canonically conjugate momenta π_n are still given by (3.5′) irrespective of the presence or absence of the interaction.

In terms of ρ^+ and ρ^- the interaction Hamiltonian can be expressed as

$$H_{\rm int}=\sum_{n>0}J_n(\rho_{-n}^+\rho_n^+ + \rho_n^-\rho_{-n}^- + \rho_n^+\rho_{-n}^- + \rho_n^-\rho_{-n}^+)$$
$$+\sum_{n>0}nJ_n+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0). \tag{3.13}$$

The term $\sum_{n>0} nJ_n$ appeared when we performed the rearrangement of factors in $\rho_n^+\rho_{-n}^+$ and $\rho_{-n}^-\rho_n^-$ into the correct order.

It is the essential point of our method that the energy, the "kinetic part" $\mathfrak{H}$ as well as the "potential part" $H_{\rm int}$, is bilinear in the $\rho^\pm$'s. This fact is in marked contrast to the fact that, in terms of ψ and ψ^*, only the kinetic part is bilinear; the potential part contains four ψ's. This latter fact makes it necessary to solve a complicated non-linear problem if one wishes to deal with a many-particle problem with inter-particle force. Now that we have found the Hamiltonian to be bilinear in the ρ's, irrespective of the presence or absence of the inter-particle force, we have no such difficulty if we deal with the problem in terms of the ρ field not in terms of the ψ field. The problem is simply to perform the principal-axes transformation of the bilinear form, i. e. to find the normal coordinates for the ρ field.

Before we enter into this problem, we shall show that the Hamiltonian $\mathfrak{H}$ is in fact equal to the kinetic energy of the system minus some constant which depends only on the total number of particles.

§ 4. Energy and momentum of the system.

In this section we give the proof that the Hamiltonian $\mathfrak{H}$ really gives the kinetic energy of the system. This can be done by a straightforward calculation in the following manner.

For this purpose we first examine $\sum \rho_{-n}^+ \rho_n^+$. We examine the diagonal and the non-diagonal parts of $\sum \rho^+_{-n}\rho^+_n$ separately, where the matrix is supposed to be referred to in the representation in which the occupation numbers of the particles are diagonal.

We have

$$\sum_{\frac{n_{\max}}{2}>n>0} \rho^+_{-n}\rho^+_n = \sum_{\frac{n_{\max}}{2}>n>0}\ \sum_{\bar{n}>0}\ \sum_{\bar{n}'>0} \psi^*_{\bar{n}+\frac{n}{2}}\psi_{\bar{n}-\frac{n}{2}}\psi^*_{\bar{n}'-\frac{n}{2}}\psi_{\bar{n}'+\frac{n}{2}}$$

$$= \sum_{\frac{n_{\max}}{2}>n>0} \Big(\sum_{\bar{n}>0}\ \sum_{\bar{n}'>0}\Big)' \psi^*_{\bar{n}+\frac{n}{2}}\psi_{\bar{n}-\frac{n}{2}}\psi^*_{\bar{n}'-\frac{n}{2}}\psi_{\bar{n}'+\frac{n}{2}}$$

$$+ \sum_{\frac{n_{\max}}{2}>n>0}\ \sum_{\bar{n}>0} (1-N_{\bar{n}-\frac{n}{2}})N_{\bar{n}+\frac{n}{2}}. \tag{4.1}$$

The prime on $(\sum\sum)'$ means that terms with $\bar{n}=\bar{n}'$ are omitted. By this omission the first sum on the right-hand side of (4.1) gives the non-diagonal part and the second sum the diagonal part of $\sum \rho^+_{-n}\rho^+_n$. The summation over n is to be extended only up to $n_{\max}/2$ because of our condition (2.15) (III).

We first observe the diagonal part

$$D^+ = \sum_{\frac{n_{\max}}{2}>n>0}\ \sum_{\bar{n}>0} (1-N_{\bar{n}-\frac{n}{2}})N_{\bar{n}+\frac{n}{2}}. \tag{4.2}$$

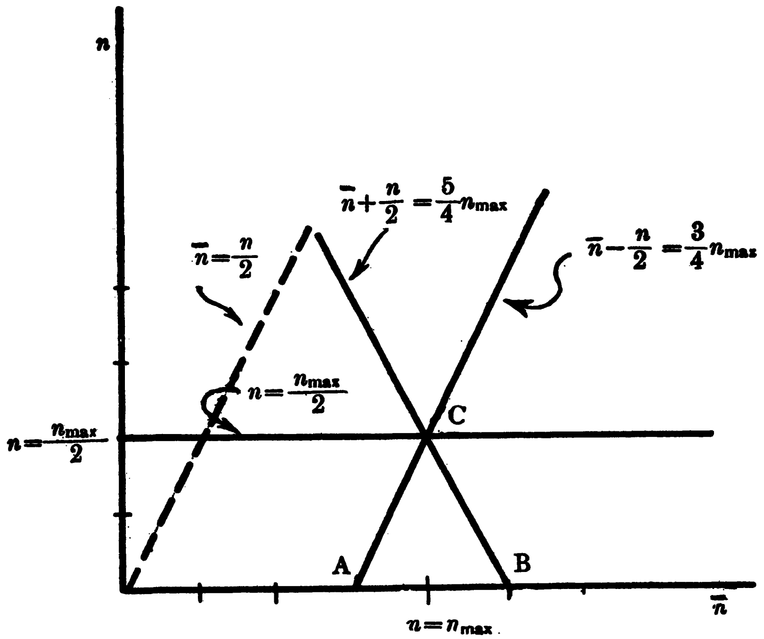

Fig. 1.

We make use of the fact that, because of the first two conditions of (2.15), the summand $(1-N_{\bar{n}-\frac{n}{2}})N_{\bar{n}+\frac{n}{2}}$ has a non-vanishing value only at points $(n, \bar{n})$ lying inside the triangle ABC (see Fig. 1) enclosed by the three straight lines

$$\begin{cases} \bar{n}+\dfrac{n}{2}=\dfrac{5}{4}n_{\max} \\ \bar{n}-\dfrac{n}{2}=\dfrac{3}{4}n_{\max} \\ n=0\ . \end{cases}$$

Since the summand vanishes outside this triangle, two of the boundaries of the domain of summation

$$n=\frac{n_{\max}}{2} \qquad \text{and} \qquad \bar{n}=0 \tag{4.3}$$

can be changed at our will so long as the changed boundaries lie outside the triangle. So we may use the boundaries

$$n=\infty \qquad \text{and} \qquad \bar{n}=\frac{n}{2} \tag{4.3'}$$

instead of the original ones. Then we have

$$\begin{aligned} D^{+}&=\sum_{\infty>n>0}\ \sum_{\bar{n}\geq\frac{n}{2}}(1-N_{\bar{n}-\frac{n}{2}})N_{\bar{n}+\frac{n}{2}} \\ &=\sum_{n>0}\ \sum_{\bar{n}'\geq n}(1-N_{\bar{n}'-n})N_{\bar{n}'}. \end{aligned} \tag{4.4}$$

We now notice that

$$\sum_{n>0}\ \sum_{\bar{n}'\geq n}N_{\bar{n}'}=\sum_{\bar{n}'\geq 0}\bar{n}'N_{\bar{n}'} \tag{4.5}$$

and

$$\begin{aligned} \sum_{n>0}\ \sum_{\bar{n}'\geq n}N_{\bar{n}'-n}\,N_{\bar{n}'}&=\frac{1}{2}\Big(\sum_{\bar{n}\geq 0}N_{\bar{n}}\Big)^{2}-\frac{1}{2}\sum_{\bar{n}\geq 0}N_{\bar{n}} \\ &=\frac{1}{2}n_{\max}\,(n_{\max}+1). \end{aligned} \tag{4.5'}$$

We then get from (4.4)

$$D^{+}=\sum_{n\geq 0}nN_{n}-\frac{1}{2}n_{\max}\,(n_{\max}+1). \tag{4.6}$$

In the same way we get the corresponding result for $\sum\rho_{\bar{n}}\rho_{-\bar{n}}$

$$D^{-}=\sum_{n\leq 0}|n|\,N_{n}-\frac{1}{2}n_{\max}\,(n_{\max}+1). \tag{4.7}$$

We now go over to the non-diagonal part:

$$C^{+}=\sum_{\frac{n_{\max}}{2}>n>0}\Big(\sum_{\bar{n}>0}\ \sum_{\bar{n}'>0}\Big)'\,\psi^{*}_{\bar{n}+\frac{n}{2}}\,\psi_{\bar{n}-\frac{n}{2}}\,\psi^{*}_{\bar{n}'-\frac{n}{2}}\,\psi_{\bar{n}'+\frac{n}{2}} \tag{4.8}$$

and show that C^{+} vanishes.

By writing $-n$ instead of n, $\bar{n}'$ instead of $\bar{n}$ and $\bar{n}$ instead of $\bar{n}'$, we see that C^+ can be written also in the following form:

$$C^+ = \sum_{-\frac{n_{\max}}{2}<n<0} \Big(\sum_{\bar{n}>0}\ \sum_{\bar{n}'>0}\Big)' \psi^*_{\bar{n}+\frac{n}{2}}\ \psi_{\bar{n}-\frac{n}{2}}\ \psi^*_{\bar{n}'-\frac{n}{2}}\ \psi_{\bar{n}'+\frac{n}{2}}, \tag{4.8}$$

where we have used the fact that $\psi^*_{\bar{n}+\frac{n}{2}}\ \psi_{\bar{n}-\frac{n}{2}}$ commutes with $\psi^*_{\bar{n}'-\frac{n}{2}}\ \psi_{\bar{n}'+\frac{n}{2}}$ since $\bar{n} \neq \bar{n}'$. From (4.8) and (4.8′) we get

$$C^+ = \frac{1}{2} \sum'_{-\frac{n_{\max}}{2}<n<\frac{n_{\max}}{2}} \Big(\sum_{\bar{n}>0}\ \sum_{\bar{n}'>0}\Big)' \psi^*_{\bar{n}+\frac{n}{2}}\ \psi_{\bar{n}-\frac{n}{2}}\ \psi^*_{\bar{n}'-\frac{n}{2}}\ \psi_{\bar{n}'+\frac{n}{2}}. \tag{4.8″}$$

Now a considerasion similar to the above one shows that, because of the condition (2.15), we can change the boundaries of summation

$$n = \pm\frac{n_{\max}}{2}, \quad \bar{n}=0 \quad \text{and} \quad \bar{n}'=0 \tag{4.9}$$

into new ones

$$n = \pm\infty, \qquad \bar{n} = -\frac{n}{2}, \qquad \bar{n}' = \frac{n}{2}. \tag{4.9′}$$

We then get

$$C^+ = \frac{1}{2} \sum'_{-\infty<n<\infty} \Big(\sum_{\bar{n}\geqq-\frac{n}{2}}\ \sum_{\bar{n}'\geqq\frac{n}{2}}\Big)' \psi^*_{\bar{n}+\frac{n}{2}}\ \psi_{\bar{n}-\frac{n}{2}}\ \psi^*_{\bar{n}'-\frac{n}{2}}\ \psi_{\bar{n}'+\frac{n}{2}}; \tag{4.10}$$

but this C^+ vanishes as is shown in the following manner.

By rearranging factors in (4.10) we get

$$C^+ = -\frac{1}{2} \sum'_{-\infty<n<\infty} \Big(\sum_{\bar{n}\geqq-\frac{n}{2}}\ \sum_{\bar{n}'\geqq\frac{n}{2}}\Big)' \psi^*_{\bar{n}'-\frac{n}{2}}\ \psi_{\bar{n}-\frac{n}{2}}\ \psi^*_{\bar{n}+\frac{n}{2}}\ \psi_{\bar{n}'+\frac{n}{2}}. \tag{4.10′}$$

If we put here

$$\begin{cases} l = \bar{n}' - \bar{n} \\ \bar{l}' = \frac{1}{2}(\bar{n}' + \bar{n} + n) \\ \bar{l} = \frac{1}{2}(\bar{n}' + \bar{n} - n), \end{cases}$$

we can write (4.10′) in the form:

$$C^+ = -\frac{1}{2} \sum'_{-\infty<l<\infty} \Big(\sum_{\bar{l}\geqq-\frac{l}{2}}\ \sum_{\bar{l}'\geqq\frac{l}{2}}\Big)' \psi^*_{\bar{l}+\frac{l}{2}}\ \psi_{\bar{l}-\frac{l}{2}}\ \psi^*_{\bar{l}'-\frac{l}{2}}\ \psi_{\bar{l}'+\frac{l}{2}}. \tag{4.11}$$

That the domains of summation over l, $\bar{l}$ and $\bar{l}'$ are really as specified by the inequalities under the summation symbols can be easily shown. One has only to notice the identities

$$\begin{cases} \bar{n}+\frac{n}{2}=\bar{l}'-\frac{l}{2} \\ \bar{n}'-\frac{n}{2}=\bar{l}+\frac{l}{2}. \end{cases}$$

Comparing (4.11) with (4.10) we find $C^+=-C^+$, which means that

$$C^+=0. \tag{4.12}$$

A similar consideration applies to $\sum \rho_n^- \rho_{-n}^-$. We find that the non-diagonal part of $\sum \rho_n^- \rho_{-n}^-$ vanishes:

$$C^-=0. \tag{4.13}$$

This rather lengthy proof that $C^+=C^-=0$ can be summarized in the following way. Each term in C^+ or C^- corresponds to a simultaneous transition of two particles. If there is a term corresponding to a transition $a\rightarrow b$, $c\rightarrow d$, there is always a term corresponding to the transition $a\rightarrow d$, $c\rightarrow b$. Because of the Fermi statistics, the latter term has a sign which is opposite to the former so that they cancel each other when added together.

Substituting (4.6), (4.7), (4.12) and (4.13) into $\mathfrak{H}$, we obtain

$$\mathfrak{H}=\left(\frac{2\pi\hbar}{L}\right)^2 \frac{n_{\max}}{m}\{\sum_n |n|\, N_n - n_{\max}(n_{\max}+1)\}$$

$$=\left(\frac{2\pi\hbar}{L}\right)^2 \frac{n_{\max}}{m}\{\sum_{|n|\leqq n_{\max}} |n|\, (N_n-1)+\sum_{|n|>n_{\max}} |n|\, N_n\}. \tag{4.14}$$

On the other hand the kinetic energy of the system is evidently

$$H_{\text{kin}}=\left(\frac{2\pi\hbar}{L}\right)^2 \frac{1}{2m} \sum_n n^2 N_n. \tag{4.15}$$

In the state of perfect degeneracy, i.e., in the state where all levels between $-n_{\max}$ and $n_{\max}$ are occupied and all other levels are empty, this energy becomes

$$H_0=\left(\frac{2\pi\hbar}{L}\right)^2 \frac{1}{2m} \sum_{|n|\leqq n_{\max}} n^2. \tag{4.16}$$

We shall now show that, in the same approximation, which is allowed by our condition (2.15), our Hamiltonian $\mathfrak{H}$ is equal to the energy (4.15) minus the constant energy (4.16) : $\mathfrak{H}=H_{\text{kin}}-H_0$.

We calculate $H_{\text{kin}}-H_0$ in the following way: From (4.15) and (4.16) we get

$$H_{\text{kin}}-H_0=\left(\frac{2\pi\hbar}{L}\right)^2 \frac{1}{2m}\{\sum_{|n|\leqq n_{\max}} n^2(N_n-1)+\sum_{|n|>n_{\max}} n^2 N_n\}. \tag{4.17}$$

Making use of the assumption that holes and excited particles are present only in the neighborhood of $\pm n_{\max}$, we see that N_n-1 and N_n differ from zero only when n lies in the neighborhood of $\pm n_{\max}$. This allows us to use instead of n^2 its expansion:

$$\begin{aligned} n^2 &= \{\pm n_{\max} + (n \mp n_{\max})\}^2 \\ &\approx n_{\max}^2 \pm 2n_{\max}(n \mp n_{\max}) \\ &= -n_{\max}^2 + 2n_{\max}|n|. \end{aligned}$$

Substituting this in (4.17), we find

$$H_{\text{kin}} - H_0 = -\left(\frac{2\pi\hbar}{L}\right)^2 \frac{n_{\max}^2}{m}\{N-(2n_{\max}+1)\} + \mathfrak{H}. \tag{4.17'}$$

Then the use of (2.13) gives immediately the required result:

$$H_{\text{kin}} - H_0 = \mathfrak{H}. \tag{4.18}$$

This result means that our Hamiltonian $\mathfrak{H}$ is really the kinetic energy of the assembly minus a constant which depends only on the total number of particles. This constant is the energy of the assembly in the case of perfect degeneracy and thus $\mathfrak{H}$ can be interpreted as the deviation of the kinetic energy from this standard value. We express this fact by simply calling $\mathfrak{H}$ " the excitation kinetic energy ".

The total energy of the system is evidently

$$H = \mathfrak{H} + H_{\text{int}} + H_0 \tag{4.19}$$

when there are inter-particle forces.

The relations (4.6) and (4.7) can be used to express the momentum of the assembly in terms of ρ. Let the momentum be denoted by G. Then

$$G = \left(\frac{2\pi\hbar}{L}\right) \sum_{n>0} (\rho_{-n}^{+} \rho_n^{+} - \rho_n^{-} \rho_{-n}^{-}). \tag{4.20}$$

The expression for G applics irrespective of the presence or absence of the inter-particle force.

One sees that for the proof of the relation (4.18) the possibility of change of domains of summation, the replacement of (4.3) by (4.3′) and the replacement of (4.9) by (4.9′), were essential. These are possible only when the three conditions in (2.15) are simultaneously satisfied. This is the reason why we had to choose the special value of $a=3/4$. It is easy to see that another choice of a would make it impossible to change the summation domains so that we would not get the relation (4.18).

§ 5. Solution of the eigen-value problem.

We now go over to the solution of the eigen-value problem of our Hamiltonian. As mentioned before, this can be done by performing the principal-axes transformation of the bilinear form representing the Hamiltonian.

Adding (3.8), (3.13) and H_0, the total Hamiltonian is obtained:

$$H=\mathfrak{H}+H_{\rm int}+H_0$$
$$=\sum_{n>0}\left\{\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\rm max}}{m}+J_n\right\}(\rho_{-n}^+\rho_n^+ +\rho_n^-\rho_{-n}^-)$$
$$+\sum_{n>0}J_n(\rho_n^+\rho_{-n}^- +\rho_n^-\rho_{-n}^+)$$
$$+\sum_{n>0}nJ_n+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0)+H_0. \qquad (5.1)$$

We introduce real coordinates Q_n and real momenta P_n by means of

$$\begin{cases}\rho_n^+=\sqrt{\frac{n}{2}}(Q_n+iP_n)\\ \rho_{-n}^+=\sqrt{\frac{n}{2}}(Q_n-iP_n)\\ \rho_n^-=\sqrt{\frac{n}{2}}(Q_{-n}-iP_{-n})\\ \rho_{-n}^-=\sqrt{\frac{n}{2}}(Q_{-n}+iP_{-n}),\end{cases}$$

where the suffix n is considered positive. Then in terms of P's a ' Q's the Hamiltonian is expressed as

$$H=\sum_n|n|\left\{\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\rm max}}{m}+J_n\right\}\left(\frac{1}{2}P_n^2+\frac{1}{2}Q_n^2-\frac{1}{2}\right)$$
$$+\frac{1}{2}\sum_n|n|J_n(Q_nQ_{-n}-P_nP_{-n})$$
$$+\frac{1}{2}\sum_n|n|J_n+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0)+H_0. \qquad (5.3)$$

In this expression the suffix n takes both positive and negative values and the summation $\sum_n$ is extended over positive as well as negative valus of n.

The commutation relations for the P's and Q's are

$$\begin{cases}[Q_n, P_{n'}]=i\,\delta_{n,n'}\\ [Q_n, Q_{n'}]=[P_n, P_{n'}]=0.\end{cases} \qquad (5.4)$$

The transformation of H into principal form can be performed by defining new variables q_n and p_n by means of the canonical transformation

$$P_n=\frac{1}{2}\left\{\left(\frac{T_n+2U_n}{T_n}\right)^{1/4}+\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}\right\}p_n+\frac{1}{2}\left\{\left(\frac{T_n+2U_n}{T_n}\right)^{1/4}-\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}\right\}p_{-n}$$
$$Q_n=\frac{1}{2}\left\{\left(\frac{T_n+2U_n}{T_n}\right)^{1/4}+\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}\right\}q_n-\frac{1}{2}\left\{\left(\frac{T_n+2U_n}{T_n}\right)^{1/4}-\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}\right\}q_{-n}$$
$$(5.5)$$

with the abbreviations

$$\begin{cases} T_n=\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}|n|, \\ U_n=|n|J_n. \end{cases} \tag{5.6}$$

It can be easily seen that the p_n and q_n satisfy

$$\begin{cases} [q_n, p_{n'}]=i\delta_{n,n'}, \\ [q_n, q_{n'}]=[p_n, p_{n'}]=0, \end{cases} \tag{5.7}$$

and that H is transformed into

$$H=\sum_n\sqrt{T_n}\left\{\sqrt{T_n+2U_n}\left(\frac{1}{2}p_n^2+\frac{1}{2}q_n^2-\frac{1}{2}\right)+\frac{1}{2}\left(\sqrt{T_n+2U_n}-\sqrt{T_n}\right)\right\}$$

$$+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0)+H_0\,. \tag{5.8}$$

The momentum G, expressed in terms of q's and p's, is

$$G=\left(\frac{2\pi\hbar}{L}\right)\sum_n n\left(\frac{1}{2}p_n^2+\frac{1}{2}q_n^2-\frac{1}{2}\right). \tag{5.9}$$

The expressions (5.8) and (5.9) show that our system is in fact equivalent to an assembly of uncoupled harmonic oscillators, or, better, to an assembly of uncoupled sound quanta. Each sound quantum has momentum and energy given by

$$\text{momentum}=\left(\frac{2\pi\hbar}{L}\right)n \qquad n=0,\ \pm 1,\ \pm 2,\cdots \tag{5.10}$$

and

$$\text{energy}=\sqrt{T_n(T_n+2U_n)}$$

$$=\left[\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}\left\{\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}+2J_n\right\}\right]^{1/2}|n|, \tag{5.10'}$$

and the number of sound quanta with the momentum $\left(\frac{2\pi\hbar}{L}\right)n$ is represented by the operator

$$M_n=\frac{1}{2}p_n^2+\frac{1}{2}q_n^2-\frac{1}{2}. \tag{5.11}$$

In terms of M_n the energy and the momentum are

$$H=\sum_n\sqrt{T_n}\left\{\sqrt{T_n+2U_n}\,M_n+\frac{1}{2}\left(\sqrt{T_n+2U_n}-\sqrt{T_n}\right)\right\} \tag{5.8'}$$

$$+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0)+H_0\,,$$

and

$$G=\left(\frac{2\pi\hbar}{L}\right)\sum_n n\, M_n \tag{5.9'}$$

respectively.

In the wave language the relations (5.10) and (5.11′) state that the sound wave with the wave length

$$\lambda_n=\frac{L}{|n|} \tag{5.12}$$

has the frequency

$$\nu_n=\frac{1}{2\pi}\left[\left(\frac{2\pi}{L}\right)^2\frac{n_{\max}}{m}\left\{\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}+2J_n\right\}\right]^{1/2}|n|\,. \tag{5.12'}$$

The phase velocity of the sound wave is

$$v_n'=\left[\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}\left\{\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}+2J_n\right\}\right]^{1/2}. \tag{5.13}$$

The phase velocity given by (5.13) is dependent on n and hence on λ_n when J_n does not vanish. This means that, when the inter-particle force is present, our Fermi gas is dispersive. (5.13) shows further that the phase velocity of the sound waves increases or decreases from its value in the ideal gas according as the inter-particle force is repulsive or attractive.

The formula (5.8′) enables us immediately to calculate the energy in various stationaly states. In the lowest state, for instance, where there are no sound quanta, we get

$$E=\frac{1}{2}\sum_n\sqrt{T_n}\,(\sqrt{T_n+2U_n}-\sqrt{T_n})$$
$$+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0)+H_0\,. \tag{5.14}$$

In the case of an attractive inter-particle force it may happen that E, considered as a function of L, has a minimum for some value of L. Then this will mean that the assembly is capable of forming a stable aggregate.

In the case where J_n is very small, we can expand $\sqrt{T_n+2U_n}$ and thus express E as a power series in J_n. We shall then find that the term linear in J_n together with the term H_0 gives the energy in the usual Tomas-Fermi ap-approximation including the effect of exchange. The energy quadratic in J_n corresponds to the energy obtained by Euler as the second order term of a perturbation calculation.

§ 6. Correlation between particles. Transitions caused by an external perturbing force. Criterion for applicability of the method.

In this section we shall mention some general results of physical interest which can be obtained directly by our theory. We shall give also a criterion for the applicability of our method.

(I) Correlation of position of particles.

As is known there is no correlation between particles in an ideal gas besides that due to the exclusion principle. The interaction between particles causes an extra correlation, which can be calculated very simply.

Mathematically the correlation of position can be exppessed by the quantity

$$C(\xi)=\frac{1}{N^2}\left\langle\int\rho(x)\rho(x+\xi)dx\right\rangle_{Av}, \tag{6.1}$$

$C(\xi)d\xi$ giving the probability of finding another particle at a distance between ξ and $\xi+d\xi$ from a particle. This correlation function can be calculated in the following manner.

Expressing (6.1) in terms of the Fourier transform of $\rho(x)$, we have

$$C(\xi)=\frac{1}{L}+\frac{1}{2N^2L}\sum_{n>0}\langle\rho_n\rho_{-n}\rangle_{Av}\cos\left(\frac{2\pi n}{L}\xi\right), \tag{6.2}$$

so that our task is to find $\langle\rho_n\rho_{-n}\rangle_{Av}$. in terms of the q's and p's defined by (5.2) and (5.5) the operator $\rho_n\rho_{-n}$ is

$$\rho_n\rho_{-n}=2n\sqrt{\frac{T_n}{T_n+2U_n}}\left\{\frac{1}{2}(p_n^2+q_n^2)+\frac{1}{2}(p_{-n}^2+q_{-n}^2)+q_nq_{-n}-p_np_{-n}\right\}. \tag{6.3}$$

It then follows that

$$\langle\rho_n\rho_{-n}\rangle_{Av}=2n\sqrt{\frac{T_n}{T_n+2U_n}}(M_n+M_{-n}+1). \tag{6.4}$$

which gives

$$C(\xi)=\frac{1}{L}+\frac{1}{N^2L}\sum_n\left(M_n+\frac{1}{2}\right)\sqrt{\frac{T_n}{T_n+2U_n}}|n|\cos\left(\frac{2\pi n}{L}\xi\right). \tag{6.5}$$

The correlation function $C(\xi)$ consists of two parts. The one is the part expressing the correlation which already exists in the absence of the inter-particle force. This part is due solely to the exclusion principle. This part of the correlation can be obtained by putting $U_n=0$ in (6.5). Denoting this part of $C(\xi)$ by $C_0(\xi)$, we have

$$C_0(\xi)=\frac{1}{L}+\frac{1}{N^2L}\sum_n\left(M_n+\frac{1}{2}\right)|n|\cos\left(\frac{2\pi n}{L}\xi\right). \tag{6.6}$$

The second part of $C(\xi)$ is that part of the correlation which is purely due to inter-particle forces. Denoting this second part by $C_1(\xi)$, we get

$$C_1(\xi)=\frac{1}{N^2L}\sum_n\left(M_n+\frac{1}{2}\right)\left(\sqrt{\frac{T_n}{T_n+2U_n}}-1\right)|n|\cos\left(\frac{2\pi n}{L}\xi\right). \tag{6.7}$$

(6.7) can also be written as

$$C_1(\xi)=\frac{\partial}{\partial\xi}\left\{\frac{1}{2\pi N^2}\sum_n\left(M_n+\frac{1}{2}\right)\left(\sqrt{\frac{T_n}{T_n+2U_n}}-1\right)\sin\left(\frac{2\pi|n|}{L}\xi\right)\right\}. \quad (6.7')$$

Then the quantity

$$D_1(\xi)=\frac{1}{2\pi N^2}\sum_n\left(M_n+\frac{1}{2}\right)\left(\sqrt{\frac{T_n}{T_n+2U_n}}-1\right)\sin\left(\frac{2\pi|n|}{L}\xi\right) \quad (6.7'')$$

can be interpreted as the change of probability due to the inter-particle force of finding another particle within the distance ξ from a particle.

(II) Matrix element of an external perturbing force.

Suppose an external perturbing force is impressed upon the system. Then the perturbation will cause transitions of the system. As is well known, the transition probabilities depend essentially on the matrix elements of the perturbing energy. Now, it is very simple to calculate these matrix elements.

Let the potential of the perturbing force be $V(x)$. Then the perturbation energy is

$$\begin{aligned} H'&=\int\rho(x)V(x)\,dx \\ &=\sum_{n>0}\{V_n(\rho_n^++\rho_n^-)+V_{-n}(\rho_{-n}^++\rho_{-n}^-)\}, \end{aligned} \quad (6.8)$$

where V_n and V_{-n} are the matrix elements of $V(x)$ defined by

$$\begin{cases} V_n=\frac{1}{L}\int V(x)\exp\left(\frac{2\pi i}{L}nx\right)dx \\ V_{-n}=V_n^*. \end{cases} \quad (6.9)$$

In terms of the q's and p's it is found that

$$\begin{aligned} H'=\sum_{n>0}\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}\Bigg[&\sqrt{\frac{n}{2}}V_n\{(q_n+ip_n)+(q_{-n}-ip_{-n})\} \\ &+\sqrt{\frac{n}{2}}V_{-n}\{(q_{-n}+ip_{-n})+(q_n-ip_n)\}\Bigg]. \end{aligned} \quad (6.10)$$

Now, the matrix elements of $q_{\pm n}+ip_{\pm n}$ as well as $q_{\pm n}-ip_{\pm n}$ are all known. They are the matrix elements for harmonic oscillators. Our result (6.10) shows that the matrix elements of H' are given simply by multiplying these universal matrix elements by the factor $(T_n/T_n+2U_n)^{1/4}\sqrt{n/2}\,V_{\pm n}$. When the inter-particle force is absent, the factors are simply $\sqrt{n/2}\,V_{\pm n}$. Our result shows that the effect of the inter-particle force is simply to replace $\sqrt{n/2}\,V_{\pm n}$ by $(T_n+2U_n)^{1/4}\times\sqrt{n/2}\,V_{\pm n}$. The effect of the inter-particle force can thus be dealt with as an apparent change of the perturbing potential by the factor $(T_n/T_n+2U_n)^{1/4}$, the effective potential being expressed as

$$V_{\pm n}^{\text{eff}} = \left(\frac{T_n}{T_n+2U_n}\right)^{1/4} V_{\pm n}. \tag{6.11}$$

The factor $\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}$ is larger or smaller than unity according as the inter-particle force is attractive or repulsive. This means that in the case of an attractive inter-particle force the effect of an external perturbing force becomes larger, and in the case of a repulsive one the perturbing force becomes less effective as compared with the case of an ideal gas.

(III) Criterion for the applicability of the method.

Because our theory is based on the fundamental conditions of (2.15), it is necessary to see under what circumstances we can be sure that these conditions are satisfied, at least approximately.

It is first clear that the range of the inter-particle force should not be too short in order that the third condition of (2.15) be satisfied. This condition requires that sound waves with wave lengths shorter than $2L/n_{\max}$ play no role. Let us fist consider the lowest state. Then this condition requires that in the sum $\sum_n \sqrt{T_n}(\sqrt{T_n+2U_n}-\sqrt{T_n})$ of (5.14) no contribution should arise from terms with $n>n_{\max}/2$. This requires that J_n is negligibly small for $n>n_{\max}/2$. Thus, roughly speaking, the range of the force should be longer than $2L/n_{\max}$ or larger than four times the mean distance of particles.

In the excited states we have to impose a further condition: There should not be any sound quanta with wave length shorter than $2L/n_{\max}$.

In order to see whether or not the first two conditions of (2.15) are satisfied, it is necessary to examine the eigen function which is obtained and see how the holes and excited particles are distributed among the levels. If this eigen function is such that the probability distribution of holes and excited particles is essentially limited between the levels $(3/4)n_{\max}$ and $(5/4)n_{\max}$, the solution obtained will be a good approximation. There is a rough but simple criterion to see under what circumstances such will be the case.

It is to calculate the mean excitation kinetic energy per particle. If this energy is too large, we must expect that there will be a large number of highly excited particles and many holes lying deep in the Fermi sea. If this energy is small, we are sure that the excited particles and holes are present only in the neighborhood of the Fermi maximum.

The excitation kinetic energy per particle is given by

$$\frac{1}{N}\langle \mathfrak{H} \rangle_{Av} = \frac{1}{2n_{\max}+1}\left(\frac{2\pi\hbar}{L}\right)^2 \frac{n_{\max}}{m} \sum_{n>0} \langle \rho_{-n}^+ \rho_n^+ + \rho_n^- \rho_{-n}^- \rangle_{Av}, \tag{6.12}$$

where we have

$$\rho_{-n}^{+}\rho_{n}^{+}+\rho_{n}^{-}\rho_{-n}^{-}=\frac{n}{2}\Big[\Big(\sqrt{\frac{T_n+2U_n}{T_n}}+\sqrt{\frac{T_n}{T_n+2U_n}}\Big)\Big\{\frac{1}{2}(p_n^2+q_n^2)+\frac{1}{2}(p_{-n}^2+q_{-u}^2)\Big\}$$

$$+\Big(\sqrt{\frac{T_n+2U_n}{T_n}}-\sqrt{\frac{T_n}{T_n+2U_n}}\Big)(p_np_{-n}-q_nq_{-n})-2\Big].$$

Taking the expectation value, we then get

$$\frac{1}{N}\langle\mathfrak{H}\rangle_{Av}=\Big(\frac{2\pi\hbar}{L}\Big)^2\frac{1}{2m}\sum_n\Big\{\Big(\sqrt{\frac{T_n+2U_n}{T_n}}+\sqrt{\frac{T_n}{T_n+2U_n}}\Big)M_n$$

$$+\frac{1}{2}\Big(\sqrt{\frac{T_n+2U_n}{T_n}}+\sqrt{\frac{T_n}{T_n+2U_n}}-2\Big)\Big\}\frac{|n|}{2}; \tag{6.13}$$

in particular, for the lowest state

$$\frac{1}{N}\langle\mathfrak{H}\rangle_{Av}=\Big(\frac{2\pi\hbar}{L}\Big)^2\frac{1}{8m}\sum_n\Big(\sqrt{\frac{T_n+2U_n}{T_n}}+\sqrt{\frac{T_n}{T_n+2U_n}}-2\Big)|n|. \tag{6.13'}$$

We must now determine the value of this mean excitation energy below which our method works. In order to find this critical value, we consider a special state in which all particles in the levels between $(3/4)n_{\max}$ and $n_{\max}$ are raised to the levels between $n_{\max}$ and $(5/4)n_{\max}$. In this state the mean excitation energy per particle is $\Big(\frac{2\pi\hbar}{L}\Big)^2\frac{1}{m}\frac{n_{\max}^2}{4^2}$. If the mean excitation energy is larger than this amount in some state, it is certain that in this state some particles are excited to levels higher than $(5/4)n_{\max}$ or some holes are present in levels below $(3/4)n_{\max}$. So our method cannot work consistently in such a case. We find in this way that the necessary condition for the applicability of our method is

$$\frac{1}{N}\langle\mathfrak{H}\rangle_{Av}<\Big(\frac{2\pi\hbar}{L}\Big)^2\frac{1}{m}\frac{n_{\max}^2}{16}. \tag{6.14}$$

In the lowest state (6.14) is

$$\sum_n\Big(\sqrt{\frac{T_n+2U_n}{T_n}}+\sqrt{\frac{T_n}{T_n+2U_n}}-2\Big)|n|<\frac{n_{\max}^2}{2}. \tag{6.15}$$

Of course the condition (6.14) or (6.15) is only necessary but not sufficient. There may be cases in which this condition is satisfied and still our method does not work. But such circumstances will occur rather exceptionally.

The condition (6.15) is certainly satisfied if the inter-particle force is so small that we can expand the square root of the form $\sqrt{T_n+2U_n}$ as a power series in J_n and use only a few terms. This condition is just the one required by the validity of the perturbation theory. But the condition (6.15) is weaker than the condition arising in the perturbation theory, because (6.15) does not require that J_n be uniformly small for all values of n. It requires only the smallness of the sum as a whole, and therefore our method has a wider domain of applicability than the perturbation theory. Because of the factor n outside the bracket, terms with n small do not contribute much to the sum on the left-hand side of (6.15), so that J_n with small n may even be very large, provided that T_n+2U_n does not become negative. This fact means that our method is especially suitable for long-range repulsive interactions. As has been first noticed and worked out by Bohm, the method of sound waves provides a very promising means for the study of an assembly of electrons interacting with each other through a repulsive Coulomb force.

§ 7. Concluding remarks.

In concluding the paper, we shall give a few disconnected remarks.

(I) Plasma-like oscillations.

In order to relate the results of our considerations to the work of Bohm, we shall mention briefly the bearing of our results on plasma-like oscillations of a degenerate electron gas. In this case the inter-particle force is the Coulomb repulsion. By Coulomb force is meant here the fact that J_n is proportional to to $1/n^2$. (Notice that the Poisson equation for a point source is $n^2 J_n=$const. when expressed in terms of the Fourier transform of the potential). According to (5.12′) we find that in this case V_n is independent of n for small n, i. e. for n so small that we can neglect $\left(\frac{2\hbar\pi}{L}\right)^2 \frac{n_{\max}}{m}$ as compared with $2J_n=\text{const}/n^2$. This means that the frequency of the wave is independent of the wave length, which is characteristic of plasma oscillations.

Another remarkable fact which occurs in the case of Coulomb repulsion is the following. According to (6.11) we find that the matrix elements of the external perturbing force become very small for small n. This means that the plasma oscillations are disturbed very little by external perturbing forces. This fact has been anticipated by Bohm in his paper about superconductivity.[7]

(II) Relation between the two kinds of descriptions.

As we have seen, the system can be described either as an assembly of Fermi particles, or as an assembly of sound quanta. Then the question arises: What relation will exist between the states, in one of which the occupation numbers of the Fermi particles have some specified values, and in the other of which the occupation numbers of the sound quanta have some specified values.[8] As the operator $\frac{1}{2}(p_n^2+q_n^2)-\frac{1}{2}$, which represents the number of sound quanta, and the operator $\psi_n^*\psi_n$, which represents the number of Fermi particles, do not commute with each other, it is clear that one cannot assign numerical values simultaneously to both kinds of occupation numbers. In other words, a state in which the occupation numbers of sound quanta have some definite values is a complicated superposition of various states, each of which is specified by a different set of which is specified by a different set of values of occupation numbers of Fermi particles, and *vice versa*. The statistical relation between the two kinds of occupation numbers will be obtained if one can determine the amplitude of each state in this superposition. We shall describe here briefly the general prescription to get this statistical relation.

First consider the case of a system of non-interacting particles. In this case, we can use $\frac{1}{2}(P_n^2+Q_n^2)-\frac{1}{2}$ instead of $\frac{1}{2}(p_n^2+q_n^2)-\frac{1}{2}$. The operator $\mathfrak{H}$, representing the energy of the system, is a function of the number of sound quanta. At the same time, it is a function of the number of the Fermi particles too. This means that a state in which the occupation numbers of sound quantn have some specified values, and a states in which the occupation numbers of Fermi particles have some specified values, are both some eigen-states of the energy operator. This fact results in all states in the superposition mentioned above belonging to the same eigenvalue of the energy. Now, because the number of linearly independent states belonging to one eigenvalue of $\mathfrak{H}$ is finite, one has only to solve a secular equation of finite degree in order to determine the coefficients in the superposition under consideration. So it is always possible, at least in principle, to answer the question about the statistical relation between the two kinds of occupation numbers.

In the case of interacting particles, we must first express the state in which the occupation numbers of the sound quanta have definite values as a superposition of various states, each having some definite set of values of $\frac{1}{2}(P_n^2+Q_n^2)-\frac{1}{2}$. The problem is none other than to expand Hermite functions of q's in terms of other Hermite functions of Q's. We then apply the method mentioned above to each term of this superposition. In this way we find the required statistical relation for the general case.

In a stationary state the occupation numbers of the sound quanta have some definite values, but the occupation numbers of the Fermi particles do not. The stationary state is a very complicated superposition of states with various numbers of holes and excited particles at various levels. This means that the stationary state is very far from a state where only one particle is excited to some higher level. This fact corresponds to Bohr's statement[9] that the one-particle model is a very bad approximation for the stationary states of such an assembly.

(III) A possible application to the case of exchange forces.

Though our method is applicable only to the case of ordinary forces, there see. to be a hope of applying it also to the case of exchange forces. It is to combine it with Ritz's procedure using a trial function whose form is suggested by our method when applied to ordinary forces. This procedure means physically to replace the exchange force by an equivalent ordinary force in such a way as to gives the best approximation.

Unfortunately, the mathematical structure of the method when generalized to the three-dimensional case is not simple and the author has not yet accomplished it. So in this paper we must be content only to present considerations of a rather mathematical nature without entering into real physical problems.

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