

# Fermions in one dimension: Tomonaga-Luttinger liquids

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## Abstract

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## I. NON-INTERACTING FERMIONS

We start with a continuum Fermi field,  $\Psi_F(x)$ , and expand it in terms of its right and left moving components near the two Fermi points:

$$\Psi_F(x) = e^{ik_F x} \Psi_R(x) + e^{-ik_F x} \Psi_L(x) \quad (1)$$

The Fermi wavevector,  $k_F$ , is related to the fermion density,  $\rho_0$ , by  $k_F = \pi\rho_0$ . We linearize the fermion dispersion about the Fermi points in terms of a Fermi velocity  $v_F$ , and then the dynamics of  $\Psi_{R,L}$  is described by the simple Hamiltonian

$$H_{FL} = -iv_F \int dx \left( \Psi_R^\dagger \frac{\partial \Psi_R}{\partial x} - \Psi_L^\dagger \frac{\partial \Psi_L}{\partial x} \right), \quad (2)$$

which corresponds to the imaginary time Lagrangean  $\mathcal{L}_{FL}$

$$\mathcal{L}_{FL} = \Psi_R^\dagger \left( \frac{\partial}{\partial \tau} - iv_F \frac{\partial}{\partial x} \right) \Psi_R + \Psi_L^\dagger \left( \frac{\partial}{\partial \tau} + iv_F \frac{\partial}{\partial x} \right) \Psi_L. \quad (3)$$

We will examine  $\mathcal{L}_{FL}$  a bit more carefully and show, somewhat surprisingly, that it can also be interpreted as a theory of free relativistic bosons. The mapping can be rather precisely demonstrated by placing  $\mathcal{L}_{FL}$  on a system of finite length  $L$ . We choose to place antiperiodic boundary conditions of the Fermi fields  $\Psi_{L,R}(x+L) = -\Psi_{L,R}(x)$ ; this arbitrary choice will not affect the thermodynamic limit  $L \rightarrow \infty$ , which is ultimately all we are interested in. We can expand  $\Psi_{L,R}$  in Fourier modes

$$\Psi_R(x) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \Psi_{Rn} e^{i(2n-1)\pi x/L}, \quad (4)$$

and similarly for  $\Psi_L$ . The Fourier components obey canonical Fermi commutation relations  $\{\Psi_{Rn}, \Psi_{Rn'}^\dagger\} = \delta_{nn'}$  and are described by the simple Hamiltonian

$$\tilde{H}_R = \frac{\pi v_F}{L} \sum_{n=-\infty}^{\infty} (2n-1) \Psi_{Rn}^\dagger \Psi_{Rn} - E_0, \quad (5)$$

where  $E_0$  is an arbitrary constant setting the zero of energy, which we adjust to make the ground state energy of  $\tilde{H}_R$  exactly equal to 0; very similar manipulations apply to the left-movers  $\Psi_L$ . The ground state of  $\tilde{H}_R$  has all fermions states with  $n > 0$  empty, while those with  $n \leq 0$  are occupied. We also define the total fermion number (“charge”),  $Q_R$ , of any state by the expression

$$Q_R = \sum_n : \Psi_{Rn}^\dagger \Psi_{Rn} :, \quad (6)$$

and similarly for  $Q_L$ . The colons are the so-called normal-ordering symbol – they simply indicate that the operator enclosed between them should include a  $c$ -number subtraction of its expectation

value in the ground state of  $\tilde{H}_R$ , which of course ensures that  $Q_R = 0$  in the ground state. Note that  $Q_R$  commutes with  $\tilde{H}_R$  and so we need only consider states with definite  $Q_R$ , which allows us to treat  $Q_R$  as simply an integer. The partition function,  $Z_R$ , of  $\tilde{H}_R$  at a temperature  $T$  is then easily computed to be

$$Z_R = \prod_{n=1}^{\infty} (1 + q^{2n-1})^2, \quad (7)$$

where

$$q \equiv e^{-\pi v_F / TL}. \quad (8)$$

The square in (7) arises from the precisely equal contributions from the states with  $n$  and  $-n + 1$  in (5) after the ground state energy  $E_0$  has been subtracted out.

We will provide an entirely different interpretation of the partition function  $Z_R$ . Instead of thinking in terms of occupation numbers of individual fermion states, let us focus instead on particle–hole excitations. We create a particle–hole excitation of “momentum”  $n > 0$  above any fermion state by taking a fermion in an occupied state  $n'$  and moving it to the unoccupied fermion state  $n' + n$ . Clearly the energy change in such a transformation is  $2n\pi v_F / L$  and is independent of the value of  $n'$ . This independence on  $n'$  is a crucial property and is largely responsible for the results that follow. It is a consequence of the linear fermion dispersion in (3), and of being in  $d = 1$ . We will interpret the creation of such a particle–hole excitation as being equivalent to the occupation of a state with energy  $2n\pi v_F / L$  created by the canonical boson operator  $b_{Rn}^\dagger$ . We can place an arbitrary number of bosons in this state, and we will now show how this is compatible with the multiplicity of the particle–hole excitations that can be created in the fermionic language.

The key observation is that there is a precise one-to-one mapping between the fermionic labeling of the states and those specified by the bosons creating particle–hole excitations. Take any fermion state,  $|F\rangle$ , with an arbitrary set of fermion occupation numbers and charge  $Q_R$ . We will uniquely associate this state with a set of particle–hole excitations above a particular fermion state we label  $|Q_R\rangle$ ; this is the state with the lowest possible energy in the sector of states with charge  $Q_R$ , that is,  $|Q_R\rangle$  has all fermion states with  $n \leq Q_R$  occupied and all others unoccupied. The energy of  $|Q_R\rangle$  is

$$\frac{\pi v_F}{L} \sum_{n=1}^{|Q_R|} (2n - 1) = \frac{\pi v_F Q_R^2}{L}. \quad (9)$$

To obtain the arbitrary fermion state,  $|F\rangle$ , with charge  $Q_R$ , first take the fermion in the “topmost” occupied state in  $|Q_R\rangle$ , (i.e., the state with  $n = Q_R$ ) and move it to the topmost occupied state in  $|F\rangle$  (see Fig. 1). Perform the same operation on the fermion in  $n = Q_R - 1$  by moving it to the next lowest occupied state in  $|F\rangle$ . Finally, repeat until the state  $|F\rangle$  is obtained. This order of occupying the boson particle–hole excitations ensures that the  $b_{Rn}^\dagger$  act in descending order in  $n$ . Such an ordering allows one to easily show that the mapping is invertible and one-to-one. Given any set of occupied boson states,  $\{n\}$ , and a charge  $Q_R$ , we start with the state  $|Q_R\rangle$  and act on

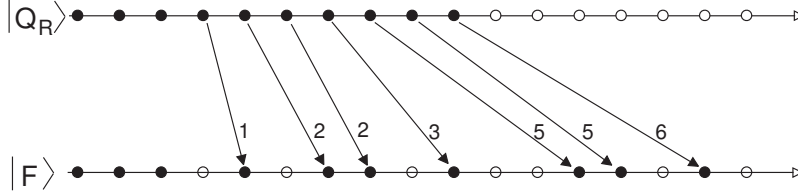


FIG. 1. Sequence of particle–hole excitations (bosons  $b_{Rn}$ ) by which one can obtain an arbitrary fermion state  $|F\rangle$  from the state  $|Q_R\rangle$ , which is the lowest energy state with charge  $Q_R$ . The filled (open) circles represent occupied (unoccupied) fermion states with energies that increase in units of  $2\pi v_F/L$  to the right. The arrows represent bosonic excitations,  $b_{Rn}$ , with the integer representing the value of  $n$ . Note that the bosons act in descending order in energy upon the descending sequence of occupied states in  $|Q_R\rangle$ .

it with the set of Bose operators in the same descending order; their ordering ensures that it is always possible to create such particle–hole excitations from the fermionic state, and one is never removing a fermion from an unoccupied state or adding it to an occupied state. The gist of these simple arguments is that the states of the many-fermion Hamiltonian  $\tilde{H}_R$  in (5) are in one-to-one correspondence with the many-boson Hamiltonian

$$\tilde{H}'_R = \frac{\pi v_F Q_R^2}{L} + \frac{2\pi v_F}{L} \sum_{n=1}^{\infty} n b_{Rn}^\dagger b_{Rn}, \quad (10)$$

where  $Q_R$  can take an arbitrary integer value. It is straightforward to compute the partition function of  $\tilde{H}'_R$  and we find

$$Z'_R = \left[ \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n})} \right] \left[ \sum_{Q_R=-\infty}^{\infty} q^{Q_R^2} \right]. \quad (11)$$

Our pictorial arguments above prove that we must have  $Z_R = Z'_R$ . That this is the case is an identity from the theory of elliptic functions. (The reader is invited to verify that the expressions (7) and (11) generate identical power series expansions in  $q$ .)

### A. Tomonaga-Luttinger liquid theory

The above gives an appealing picture of bosonization at the level of states and energy levels, but we want to extend it to include operators, and obtain expressions for the bosonized Hamiltonian in a continuum formulation. From the action of the  $b_{Rn}$  operator on the fermion states, we can anticipate that it may be proportional to the Fourier components of the fermion density operator. So we consider the operator  $\rho_R(x)$  representing the normal-ordered fermion density:

$$\rho_R(x) =: \Psi_R^\dagger(x) \Psi_R(x) := \frac{Q_R}{L} + \frac{1}{L} \sum_{n \neq 0} \rho_{Rn} e^{i2n\pi x/L}, \quad (12)$$

where the last step is a Fourier expansion of  $\rho_R(x)$ ; the zero wavevector component is  $Q_R/L$ , while nonzero wavevector terms have coefficient  $\rho_{Rn}$ . The commutation relations of the  $\rho_{Rn}$  are central to our subsequent considerations and require careful evaluation; we have

$$\begin{aligned} [\rho_{Rn}, \rho_{R-n'}] &= \sum_{n_1, n_2} [\Psi_{Rn_1}^\dagger \Psi_{Rn_1+n}, \Psi_{Rn_2}^\dagger \Psi_{Rn_2-n'}] \\ &= \sum_{n_2} (\Psi_{Rn_2-n}^\dagger \Psi_{Rn_2-n'} - \Psi_{Rn_2}^\dagger \Psi_{Rn_2+n-n'}). \end{aligned} \quad (13)$$

It may appear that a simple change of variables in the summation over the second term in (13) ( $n_2 \rightarrow n_2 + n$ ) shows that it equals the first, and so the combined expression vanishes. However, this is incorrect because it is dangerous to change variables on expressions that involve the summation over all integer values of  $n_2$  and are therefore individually divergent; rather, we should first decide upon a physically motivated large-momentum cutoff that will make each term finite and then perform the subtraction. We know that the linear spectrum in (5) holds only for a limited range of momenta, and for sufficiently large  $|n|$ , lattice corrections to the dispersion will become important. However, in the low-energy limit of interest here, the high fermionic states at such momenta will be rarely, if ever, excited from their ground state configurations. We can use this fact to our advantage by explicitly subtracting the ground state expectation value (“normal-order”) from every fermionic bilinear we consider; the fluctuations will then be practically zero for the high energy states in both the linear spectrum model (5) and the actual physical systems, and only the low energy states, where (5) is actually a good model, will matter. After such normal-ordering, the summation over both terms in (12) is well defined and we are free to change the summation variable. As a result, the normal-ordered terms then do indeed cancel, and the expression (13) reduces to

$$\begin{aligned} [\rho_{Rn}, \rho_{R-n'}] &= \delta_{nn'} \sum_{n_2} (\langle \Psi_{Rn_2-n}^\dagger \Psi_{Rn_2-n} \rangle - \langle \Psi_{Rn_2}^\dagger \Psi_{Rn_2} \rangle) \\ &= \delta_{nn'} n. \end{aligned} \quad (14)$$

This key result shows that the only nonzero commutator is between  $\rho_{Rn}$  and  $\rho_{R-n}$  and that it is simply the number  $n$ . By a suitable rescaling of the  $\rho_{Rn}$  it should be evident that we can associate them with canonical bosonic creation and annihilation operators. We will not do this explicitly but will simply work directly with the  $\rho_{Rn}$  as a set of operators obeying the defining commutation relation (14), without making explicit reference to the fermionic relation (12). We assert that the Hamiltonians  $\tilde{H}_R, \tilde{H}'_R$  are equivalent to

$$\tilde{H}_R'' = \frac{\pi v_F Q_R^2}{L} + \frac{2\pi v_F}{L} \sum_{n=1}^{\infty} \rho_{R-n} \rho_{Rn}. \quad (15)$$

This assertion is simple to prove. First, it is clear from the commutation relations (14) that the eigenvalues and degeneracies of (15) are the same as those of (10). Second, the definition (15) and

the commutation relations (14) imply that

$$[\tilde{H}_R'', \rho_{R-n}] = \frac{2\pi v_F n}{L} \rho_{R-n}. \quad (16)$$

Precisely the same commutation relation follows from the fermionic form (5) and the definition (12).

We can now perform the same analysis on the left-moving fermions. The expressions corresponding to (4), (5), (12), (14), and (15) are

$$\tilde{H}_L = -\frac{\pi v_F}{L} \sum_{n=-\infty}^{\infty} (2n-1) \Psi_{Ln}^\dagger \Psi_{Ln} - E_0, \quad (17)$$

$$\Psi_L(x) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \Psi_{Ln} e^{i(2n-1)\pi x/L}, \quad (18)$$

$$\rho_L(x) =: \Psi_L^\dagger(x) \Psi_L(x) := \frac{Q_L}{L} + \frac{1}{L} \sum_{n \neq 0} \rho_{Ln} e^{i2n\pi x/L}, \quad (19)$$

$$[\rho_{Ln}, \rho_{L-n'}] = -\delta_{nn'} n. \quad (20)$$

$$\tilde{H}_L'' = \frac{\pi v_F Q_L^2}{L} - \frac{2\pi v_F}{L} \sum_{n=1}^{\infty} \rho_{Ln} \rho_{L-n}. \quad (21)$$

We have now completed a significant part of the bosonization program. We have the “bosonic” Hamiltonian in (15) in terms of the operators  $\rho_{Rn}$ , which obey (14), and we also have the simple explicit relation (12) to the fermionic fields (along with the corresponding expressions for the left-movers above). Before proceeding further, we introduce some notation that will allow us to recast the results obtained so far in a compact, local, and physically transparent notation. We combine the operators  $\rho_{Rn}$  and  $\rho_{Ln}$  (the Fourier components of the left-moving fermions  $\Psi_L$ ) into two local fields  $\phi(x)$  and  $\theta(x)$ , defined by

$$\begin{aligned} \phi(x) &= -\phi_0 + \frac{\pi Q x}{L} - \frac{i}{2} \sum_{n \neq 0} \frac{e^{i2n\pi x/L}}{n} [\rho_{Rn} + \rho_{Ln}], \\ \theta(x) &= -\theta_0 + \frac{\pi J x}{L} - \frac{i}{2} \sum_{n \neq 0} \frac{e^{i2n\pi x/L}}{n} [\rho_{Rn} - \rho_{Ln}], \end{aligned} \quad (22)$$

where  $Q = Q_R + Q_L$  is the total charge,  $J = Q_R - Q_L$ , and  $\phi_0$  and  $\theta_0$  are a pair of angular variables that are canonically conjugate to  $J$  and  $Q$  respectively; that is, the only nonvanishing commutation relations between the operators on the right-hand sides of (22) are (14),  $[\phi_0, J] = i$  and  $[\theta_0, Q] = i$ . For future use it is also useful define

$$\varphi_R(x) \equiv \phi(x) + \theta(x) \quad , \quad \varphi_L(x) \equiv \phi(x) - \theta(x). \quad (23)$$

From (22) it is clear that  $\varphi_R$  and  $\varphi_L$  are ‘chiral’ fields, as they only involve operators associated only with the right- and left-moving fermions respectively.

Our objective in introducing these operators is to produce a number of simple and elegant results. First, using (22), and the commutators just noted, we have

$$[\phi(x), \nabla\theta(y)] = [\theta(x), \nabla\phi(y)] = -i\pi\delta(x-y), \quad (24)$$

implying that  $-\nabla\theta/\pi$  is canonically conjugate to  $\phi$ , and  $-\nabla\phi/\pi$  is canonically conjugate to  $\theta$ ; alternatively, we can write the unified form

$$[\phi(x), \theta(y)] = i\frac{\pi}{2}\text{sgn}(x-y). \quad (25)$$

In terms of the chiral fields, the non-zero commutation relations are

$$[\varphi_R(x), \varphi_R(y)] = i\pi\text{sgn}(x-y) \quad , \quad [\varphi_L(x), \varphi_L(y)] = -i\pi\text{sgn}(x-y), \quad (26)$$

while  $\varphi_R$  and  $\varphi_L$  commute with each other. For future applications, it is also useful to express these commutation relations in terms of exponentials of the fields

$$\begin{aligned} e^{i\alpha\phi(x)} e^{i\beta\theta(y)} &= e^{i\beta\theta(y)} e^{i\alpha\phi(x)} e^{-i\alpha\beta(\pi/2)\text{sgn}(x-y)} \\ e^{i\alpha\varphi_R(x)} e^{i\beta\varphi_R(y)} &= e^{i\beta\varphi_R(y)} e^{i\alpha\varphi_R(x)} e^{-i\alpha\beta\pi\text{sgn}(x-y)} \\ e^{i\alpha\varphi_L(x)} e^{i\beta\varphi_L(y)} &= e^{i\beta\varphi_L(y)} e^{i\alpha\varphi_L(x)} e^{i\alpha\beta\pi\text{sgn}(x-y)}. \end{aligned} \quad (27)$$

Second, (15) can now be written in the compact, local form

$$\tilde{H}_R'' + \tilde{H}_L'' = \frac{v_F}{2\pi} \int_0^L dx \left[ \frac{1}{K}(\nabla\phi)^2 + K(\nabla\theta)^2 \right], \quad (28)$$

where the dimensionless coupling  $K$  has been introduced for future convenience; in the present situation  $K=1$ , but we will see later that interactions will lead to other values of  $K$ . The expressions (28) and (24) can be taken as defining relations, and we could have derived all the properties of the  $\rho_{Rn}$ ,  $\rho_{Ln}$ ,  $\theta_0$ ,  $\phi_0$  as consequences of the mode expansions (22), which follow after imposition of the periodic boundary conditions

$$\phi(x+L) = \phi(x) + \pi Q, \quad \theta(x+L) = \theta(x) + \pi J. \quad (29)$$

These conditions show that  $\phi(x)$  and  $\theta(x)$  are to be interpreted as angular variables of period  $\pi$ . Our final version of the bosonic form of  $\tilde{H}_R + \tilde{H}_L$  in (5) is contained in Eqns. (24), (28), and (29), and the two formulations are logically exactly equivalent. The Hilbert space splits apart into sectors defined by the integers  $Q = Q_R + Q_L$ ,  $J = Q_R - Q_L$  which measure the total charge of the left- and right-moving fermions. Note that

$$(-1)^Q = (-1)^J \quad (30)$$

and so the periods of  $\phi$  and  $\theta$  are together even or odd multiples of  $\pi$ . In terms of the chiral fields, this conditions translates into  $\varphi_R$  and  $\varphi_L$  being angular variables with period  $2\pi$ . All fluctuations in each charge sector are defined by the fluctuations of the local angular bosonic fields  $\phi(x)$  and  $\theta(x)$ , or equivalently by the fermionic fields  $\Psi_R(x)$  and  $\Psi_L(x)$ .

We close this subsection by giving the general form of the effective action for a Tomonaga-Luttinger liquid. The derivation above was limited to the case  $K = 1$ , but we will see later that the generalization to  $K \neq 1$  describes a wide class of interacting, compressible, quantum systems in one dimension. From the Hamiltonian (28) and the commutation relations (24) we can use the standard path-integral approach to write down the imaginary time action

$$\mathcal{S}_{TL} = \frac{v_F}{2\pi} \int dx d\tau \left[ \frac{(\nabla\phi)^2}{K} + K(\nabla\theta)^2 \right] - \frac{i}{\pi} \int dx d\tau \nabla\theta \partial_\tau \phi. \quad (31)$$

From this action, we can integrate out  $\theta$  to obtain an action for the  $\phi$  field alone

$$\mathcal{S}_{TL} = \frac{1}{2\pi K v_F} \int dx d\tau [(\partial_\tau \phi)^2 + v_F^2 (\nabla\phi)^2]. \quad (32)$$

This is just the action of a free, massless, relativistic scalar field. Conversely, we also have a “dual” formulation of  $\mathcal{S}_{TL}$  in which we integrate out  $\phi$ , and obtain the same action for  $\theta$  but with  $K \rightarrow 1/K$

$$\mathcal{S}_{TL} = \frac{K}{2\pi v_F} \int dx d\tau [(\partial_\tau \theta)^2 + v_F^2 (\nabla\theta)^2]. \quad (33)$$

Finally, it is useful to express (31) in terms of the chiral fields  $\varphi_R$  and  $\varphi_L$  using (23)

$$\begin{aligned} \mathcal{S}_{TL} = \frac{v_F}{8\pi} \int dx d\tau & \left[ \left( \frac{1}{K} + K \right) ((\nabla\varphi_R)^2 + (\nabla\varphi_L)^2) + 2 \left( \frac{1}{K} - K \right) \nabla\varphi_R \nabla\varphi_L \right] \\ & - \frac{i}{4\pi} \int dx d\tau [\nabla\varphi_R \partial_\tau \varphi_R - \nabla\varphi_L \partial_\tau \varphi_L]. \end{aligned} \quad (34)$$

The last kinematic “Berry phase” term reflects the commutation relations in (26). Note that the left- and right-movers decouple only at  $K = 1$ , and that is the only case with conformal invariance.

## B. Operator mappings

We are going to make extensive use of the fields  $\phi(x)$ ,  $\theta(x)$  in the following, and so their physical interpretation would be useful. The meaning of  $\phi$  follows from the derivative of (22), which with (12) gives

$$\nabla\phi(x) = \pi\rho(x) \equiv \pi(\rho_R(x) + \rho_L(x)). \quad (35)$$

So the gradient of  $\phi$  measures the total density of particles, and  $\phi(x)$  increases by  $\pi$  each time  $x$  passes through a particle. The expression (35) also shows that we can interpret  $\phi(x)$  as the displacement of the particle at position  $x$  from a reference state in which the particles are equally



spaced as in a crystal; that is,  $\phi(x)$  is something like a phonon displacement operator whose divergence is equal to the local change in density. Turning to  $\theta(x)$ , one interpretation follows from (24), which shows that  $\Pi_\phi(x) \equiv -\nabla\theta(x)/\pi$  is the canonically conjugate momentum variable to the field  $\phi(x)$ . So  $\Pi_\phi^2$  in the Hamiltonian is the kinetic energy associated with the “phonon” displacement  $\phi(x)$ .

A physical interpretation of  $\theta$  is obtained by taking the gradient of (22), and we obtain the analog of (35):

$$\nabla\theta(x) = \pi(\rho_R(x) - \rho_L(x)); \quad (36)$$

hence gradients of  $\theta$  measure the difference in density of right- and left-moving particles *i.e.* the current. Of course, we can combine (35) and (36) to obtain expressions for the chiral fields separately:

$$\nabla\varphi_R(x) = 2\pi\rho_R(x) \quad , \quad \nabla\varphi_L(x) = 2\pi\rho_L(x). \quad (37)$$

Finally, to complete the connection between the fermionic and bosonic theories, we need expressions for the single fermion annihilation and creation operators in terms of the bosons. Here, the precise expressions are dependent upon the short-distance regularization, but these fortunately only effect overall renormalization factors. With the limited aim of neglecting these non-universal renormalizations, the basic result can be obtained by some simple general arguments. First, note that if we annihilate a particle at the position  $x$ , from (35) the value of  $\phi(y)$  at all  $y < x$  has to be shifted by  $\pi$ . Such a shift is produced by the exponential of the canonically conjugate momentum operator  $\Pi_\phi$ :

$$\exp\left(-i\pi\int_{-\infty}^x \Pi_\phi(y)dy\right) = \exp(i\theta(x)). \quad (38)$$

However, it is not sufficient to merely create a particle. We are creating a fermion, and the fermionic antisymmetry of the wavefunction can be accounted for if we pick up a minus sign for every particle to the left of  $x$ , that is, with a Jordan–Wigner–like factor

$$\exp\left(im\pi\int_{-\infty}^x \Psi_F^\dagger(y)\Psi_F(y)dy\right) = \exp(imk_Fx + im\phi(x)), \quad (39)$$

where  $m$  is any odd integer, and  $\Psi_F^\dagger\Psi_F$  measures the *total* density of fermions (see (1)), including the contributions well away from the Fermi points. In the second expression in (39), the term proportional to  $k_F$  represents the density in the ground state, while  $\phi(x)$  is the integral of the density fluctuation above that. Combining the arguments leading to (38) and (39) we can assert the basic operator correspondence

$$\Psi_F(x) = \sum_{m \text{ odd}} A_m e^{imk_Fx + im\phi(x) + i\theta(x)}, \quad (40)$$

where the  $A_m$  are a series of unknown constants, which depend upon microscopic details. We will see shortly that the leading contribution to (40) comes from the terms with  $m = \pm 1$ , and the

remaining terms are subdominant at long distances. Comparison with (1) shows clearly that we may make the operator identifications for the right- and left- moving continuum fermion fields

$$\Psi_R \sim e^{i\theta+i\phi}, \quad \Psi_L \sim e^{i\theta-i\phi}. \quad (41)$$

The other terms in (40) arise when these basic fermionic excitations are combined with particle-hole excitations at wavevectors that are integer multiples of  $2k_F$ .

In terms of the chiral fields, the operator correspondences separate simply into left- and right-moving sectors, as they must:

$$\Psi_R \sim e^{i\varphi_R}, \quad \Psi_L \sim e^{-i\varphi_L}. \quad (42)$$

As alternative to the above derivation, we can also obtain (42) by using the commutation relations

$$[\rho_R(x), \Psi_R(y)] = -\delta(x-y)\Psi_R(y) \quad , \quad [\rho_L(x), \Psi_L(y)] = -\delta(x-y)\Psi_L(y) \quad (43)$$

It can now be verified that (37) and (42), combined with the commutation relations (26), are consistent with (43).

Actually, (42) is not precisely correct, but this will not be an issue in our subsequent discussion. From the commutation relations in (27) we can verify that  $\Psi_R(x)$  and  $\Psi_R(x')$  anti-commute with each other for  $x \neq x'$ , which is precisely the relationship expected for fermion operators (and similarly for  $\Psi_L$ ). However, upon using (42) with (27) we find that  $\Psi_R(x)$  commutes with  $\Psi_L(x')$ . This problem can be addressed by introducing the so-called Klein factors

$$\Psi_R \sim F_1 e^{i\varphi_R}, \quad \Psi_L \sim F_2 e^{-i\varphi_L}. \quad (44)$$

which obey the anti-commutation relations  $F_i F_j = -F_j F_i$  for  $i \neq j$ .

## II. INTERACTING FERMIONS

We now add two-body interactions between the  $\Psi_F$  fermions. For generic values of the wavevector  $k_F$ , the only momentum conserving interaction for spinless fermions near the Fermi points is

$$H_U = \frac{U}{2} \int dx [(\rho_R(x) + \rho_L(x))(\rho_R(x) + \rho_L(x))]. \quad (45)$$

For special commensurate densities, there can be additional ‘umklapp’ terms, but we defer consideration of such terms to the following section. Using the bosonization formula (35), we can write  $H_U$  as

$$H_U = \frac{U}{2\pi^2} \int dx (\nabla\phi)^2. \quad (46)$$

This can be easily absorbed into the bosonized version of  $H_{FL}$  in (28) by a redefinition of  $v_F$  and  $K$ . In this way we have shown that the Hamiltonian  $H_{FL} + H_{12}$  is equivalent to (28) but with the parameters

$$\begin{aligned} v_F &\rightarrow v_F \left[ 1 + \frac{Uv_F}{\pi} \right]^{1/2}, \\ K &= \left[ 1 + \frac{Uv_F}{\pi} \right]^{-1/2}. \end{aligned} \quad (47)$$

The values of the parameters only hold for small  $U$ ; however, the general result of a renormalization of  $v_F$  and  $K$ , but with no other change, is expected to hold more generally. Notice that now  $K \neq 1$ , as promised earlier.

We can now evaluate the correlators of the interacting fermion field using the operator mapping in Eq. (42). These can be obtained by use of the basic identity

$$\langle e^{i\mathcal{O}} \rangle = e^{-\langle \mathcal{O}^2 \rangle / 2}, \quad (48)$$

where  $\mathcal{O}$  is an arbitrary linear combination of  $\phi$  and  $\theta$  fields at different spacetime points; this identity is a simple consequence of the free-field (Gaussian) nature of (28). In particular, all results can be reconstructed by combining (48) with repeated application of some elementary correlators. The first of these is the two-point correlator of  $\phi$ :

$$\begin{aligned} \frac{1}{2} \langle (\phi(x, \tau) - \phi(0, 0))^2 \rangle &= \pi v_F K \int \frac{dk}{2\pi} T \sum_{\omega_n} \frac{1 - e^{i(kx - \omega_n \tau)}}{\omega_n^2 + v_F k^2} \\ &= \frac{K}{4} \ln \left[ \frac{\cosh(2\pi T x / v_F) - \cos(2\pi T \tau)}{(2\pi T / v_F \Lambda)^2} \right], \end{aligned} \quad (49)$$

where  $\Lambda$  is a large-momentum cutoff. Similarly, we have for  $\theta$ , the correlator

$$\frac{1}{2} \langle (\theta(x, \tau) - \theta(0, 0))^2 \rangle = \frac{1}{4K} \ln \left[ \frac{\cosh(2\pi T x / v_F) - \cos(2\pi T \tau)}{(2\pi T / v_F \Lambda)^2} \right]. \quad (50)$$

To obtain the  $\theta, \phi$  correlator we use the relation  $\Pi_\phi = -\nabla\theta/\pi$  and the equation of motion  $i\Pi_\phi = \partial_\tau\phi/(\pi v_F K)$  that follows from the Hamiltonian (28); then by an integration and differentiation of (49) we can obtain

$$\langle \theta(x, \tau) \phi(0, 0) \rangle = -\frac{i}{2} \arctan \left[ \frac{\tan(\pi T \tau)}{\tanh(\pi T x / v_F)} \right]. \quad (51)$$

This expression can also be obtained directly from (31). Finally, we can combine these expressions to obtain fermion correlator (in imaginary time)

$$\begin{aligned} \langle \Psi_R^\dagger(x, \tau) \Psi_R(0, 0) \rangle &\sim \exp \left[ -\frac{1}{4} (K + 1/K) \ln \left[ \frac{\cosh(2\pi T x / v_F) - \cos(2\pi T \tau)}{(2\pi T / v_F \Lambda)^2} \right] \right. \\ &\quad \left. - i \arctan \left[ \frac{\tan(\pi T \tau)}{\tanh(\pi T x / v_F)} \right] \right] \end{aligned} \quad (52)$$

In general, this is a complicated function, but it does have some useful limiting values. At  $K = 1$  it takes the simple form

$$\langle \Psi_R^\dagger(x, \tau) \Psi_R(0, 0) \rangle \sim \frac{1}{\sin(\pi T(v_F \tau - ix))} \quad (53)$$

expected for free fermions. Taking the Fourier transform of (52) for general  $K$ , and analytically continuing the resulting expressions to real frequencies is, in general, a complicated mathematical challenge; details can be obtained from Refs. 1 and 2. We quote some important results in the limit of  $T = 0$ . The fermion spectral function has the following singularity at small frequencies near  $\omega = v_F k$

$$-\text{Im}G_R^R(k, \omega) \sim \theta(\omega - v_F k) (\omega - v_F k)^{(K+1/K)/2-2} \quad , \quad \omega > 0, k > 0. \quad (54)$$

At  $K = 1$ , the spectrum function is a delta function  $\sim \delta(\omega - v_F k)$  and that is indicative of the presence of quasiparticles in the free fermion model. However, a key observation is that for  $K \neq 1$  the delta function transforms into a branch-cut in the frequency complex plane, and this indicates the absence of fermionic quasiparticles. We can obtain the equal-time fermion Green's function of the original fermion field  $\Psi_F$  in (1) directly from (52)

$$\langle \Psi_F^\dagger(x) \Psi_F(0) \rangle \sim \frac{\sin(k_F |x|)}{|x|^{(K+1/K)/2}}. \quad (55)$$

Taking the Fourier transform of this, we conclude that momentum distribution function of the fermions,  $n(k)$ , does indeed have a singularity at the Fermi wavevector  $k = k_F$ , but that this singularity is not generally a step discontinuity (as it is in Fermi liquids):

$$n(k) \sim -\text{sgn}(k - k_F) |k - k_F|^{(K+1/K)/2-1}. \quad (56)$$

Thus, interacting fermions in one dimension realize a new non-Fermi liquid phase, the Tomonaga-Luttinger liquid, whose momentum distribution function has a singularity at the Fermi surface, but the singularity is not a step discontinuity, and is instead given by (56).

### III. COMMENSURABLE DENSITIES

There are conditions under which the Luttinger liquid state is unstable to a gapped insulator: this requires that the fermion density,  $\rho_0$  is a rational number. The simplest example is when the spinless Fermi gas of Section I is at half-filling. Then  $\rho_0 = 1/2$  and  $k_F = \pi/2$ . This special value of  $k_F$  allows an *umklapp* process, when two right-moving fermions scatter to become two left-moving fermions: the total momentum transfer is  $2\pi$ , and this is allowed by the unit-periodicity of the underlying lattice. In the continuum formulation, this term is

$$H_u = v \int dx [\Psi_R^\dagger \nabla \Psi_R^\dagger \Psi_L \nabla \Psi_L + \Psi_L^\dagger \nabla \Psi_L^\dagger \Psi_R \nabla \Psi_R]. \quad (57)$$

We can now bosonize this using (41), and we again obtain the sine-Gordon theory for the Luttinger liquid in the presence of periodic potential:

$$\mathcal{S}_{sG} = \mathcal{S}_{TL} - \lambda \int dx d\tau \cos(4\phi) \quad (58)$$

From the chapter on the XY model, we deduce the RG equation

$$\frac{d\lambda}{d\ell} = (2 - 4K)\lambda. \quad (59)$$

Now the critical point is at  $K = 1/2$ , and for  $K < 1/2$  there is a flow towards large  $|\lambda|$ , and we have an instability to a strongly-coupled phase. This strong-coupling state is expected to be an insulator, but the insulator breaks translational symmetry (so strictly speaking, it is not a Mott insulator). The breaking of translational symmetry can be understood from the fact that  $\cos(2\phi)$  and  $\sin(2\phi)$  are observables which break translational symmetry. This follows from (41),

$$\Psi_R^\dagger \Psi_L \sim e^{-2i\phi}, \quad (60)$$

and the fact that

$$\Psi_R^\dagger \Psi_L \rightarrow (-1)^n \Psi_R^\dagger \Psi_L \quad (61)$$

under translation by  $n$  lattice spacings, for  $k_F = \pi/2$ . When  $\lambda$  flows to  $+\infty$  (say), then the values of  $\phi$  will be pinned at  $\pi p/2$ , where  $p$  is an integer. Consequently  $\cos(2\phi)$  takes the two values  $(-1)^p$ , and this implies a two-fold breaking of translational symmetry. A possible state is a charge density wave of fermions with period 2. Similarly, when  $\lambda$  flow to  $-\infty$ , there are two possible values of  $\sin(2\phi)$ , and this corresponds to a ‘valence bond solid’, or a dimerization of the lattice with period 2.

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- [1] J. Voit, “Charge-spin separation and the spectral properties of Luttinger liquids,” [Journal of Physics: Condensed Matter](#) **5**, 8305 (1993).
- [2] J. Voit, “One-dimensional Fermi liquids,” [Reports on Progress in Physics](#) **58**, 977 (1995), [cond-mat/9510014](#).