8

Regularization of Feynman Integrals

For dimensions close to $D = 4$, the Feynman integrals in momentum space appearing in Chapter 4 do not converge since their integrands fall off too slowly at large momenta. Divergences arising from this short-wavelength region of the integrals are called *ultraviolet (UV)-divergences.* For massive fields, these are the only divergences of the integrals. Since this is ordinarily the case, we may hereafter omit specifying their UV character. In the zero-mass limit relevant for critical phenomena, there exists further divergences at small momenta and long wavelength. These so-called *infrared (IR)-divergences* will be discussed in Chapter 12.

In this chapter we shall consider only UV-divergences. They can be dealt with by various mathematical methods whose advantages and disadvantages will be pointed out, and from which we shall select the best method for our purposes.

In principle, all masses and coupling constants occurring here ougth to carry a subscript B indicating that the perturbative calculations are done starting from the *bare energy functional* $E_B[\phi_B]$ with *bare mass* m_B and *bare field* ϕ_B, introduced earlier in Section 7.3.1:

$$E_B[\phi_B] = \int d^D x \left[\frac{1}{2}(\partial\phi_B)^2 + \frac{m_B^2}{2}\phi_B^2\right], \tag{8.1}$$

and perturbing it with the bare interaction

$$E_B^{\rm int}[\phi_B] \equiv \int d^D x \frac{\lambda_B}{4!}\phi_B^4(\mathbf{x}). \tag{8.2}$$

However, in Chapter 9 we shall see that the renormalized quantities can eventually be calculated from the same Feynman integrals with the experimentally observable mass m and coupling constant λ. For this reason, the subscripts B will be omitted in all integrals.

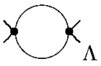
8.1 Regularization

In four dimensions, the integrals of two- and four-point functions diverge. With the help of so-called *regularization procedures* they can be made finite. A regularization parameter is introduced, so that all divergences of the integrals appear as singularities in this parameter. There are various possible regularization procedures:

(a) Momentum *cutoff* Λ regularization

In field descriptions of condensed matter systems, Feynman diagrams are regularized naturally at length scales a, where the field description breaks down. All momentum integrals are limited naturally to a region $|\mathbf{p}| < \Lambda = \pi/a$, so that no UV-divergences can occur. Examples of Feynman integrals in a four-dimensional ϕ^4-theory with cutoff regularization are

$$\frac{1}{2}\,(\text{diagram})_\Lambda = -\frac{\lambda}{2}\int_\Lambda \frac{d^4p}{(2\pi)^4}\frac{1}{\mathbf{p}^2+m^2} = -\frac{gm^2}{32\pi^2}\left[\frac{\Lambda^2}{m^2}-\log\frac{\Lambda^2}{m^2}\right]+\mathcal{O}\left[\left(\Lambda^{-1}\right)^0\right],$$



$$\frac{3}{2}\,(\text{diagram})_\Lambda = -\frac{3\lambda}{2}\int_\Lambda \frac{d^4p}{(2\pi)^2}\frac{1}{[(\mathbf{p}-\mathbf{k})^2+m^2](\mathbf{p}^2+m^2)} = \frac{3\lambda^2}{32\pi^2}\left[\log\frac{\Lambda^2}{m^2}\right]+\mathcal{O}\left[\left(\Lambda^{-1}\right)^0\right].$$

The subscript Λ of the diagrams emphasizes that the momentum integrals are carried out only up to $\mathbf{p}^2=\Lambda^2$. Both integrals are divergent for $\Lambda \longrightarrow \infty$. The first behaves for large Λ like Λ^2, and is called *quadratically divergent.* The second behaves like $\log\Lambda$ and is called *logarithmically divergent.*

Phase transitions do not depend on the properties of the system at short distances and should therefore not depend on the cutoff. If a field theory is to give a correct description of the phase transition, it must be possible to go to the limit $\Lambda\to\infty$ at the end without changing the critical behavior.

The cutoff regularization has an undesirable feature of destroying the translational invariance. Methods that do not suffer from this are

(b) Pauli-Villars regularization [1]

In this case, convergence is enforced by changing the propagator in such a way that it decreases for $|\mathbf{p}|\to\infty$ faster than before. This is done by the replacement

$$(\mathbf{p}^2+m^2)^{-1}\to(\mathbf{p}^2+m^2)^{-1}-(\mathbf{p}^2+M^2)^{-1}. \tag{8.3}$$

in which M^2 plays the role of a cutoff.

(c) Analytic regularization [2]

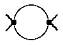
The propagator is substituted by $(\mathbf{p}^2+m^2)^{-z}$ where z is a complex number with $\mathrm{Re}(z)$ large enough to make the integrals convergent. The result is then continued analytically to a region around the physical value $z=1$. All divergences manifest themselves as simple poles for $z=1$. Finite physical quantities for $z=1$ can be defined by subtracting the pole terms.

(d) Dimensional regularization [3, 4, 5]

Instead of changing the power of the propagator, the measure of momentum integration is changed by allowing the dimension D in the integrals to be an arbitrary complex number. This regularization will be introduced in detail in the next section and used in all our calculations. It was invented by 't Hooft and Veltmanto regularize nonabelian gauge theories where all previous cutoff methods failed. There are several attractive features of dimensional regularization. First, it preserves all symmetries of the theory, in particular gauge symmetry. Second, it allows an easy identification of the divergences. Third, it suggests in a natural way a *minimal subtraction scheme* (MS scheme), that greatly simplifies the calculations. Fourth, it regularizes at the same time IR-divergences in massless theories, as will be discussed in Section 12.3.

A difficulty with dimensional regularization is the treatment of certain tensors whose definition does not permit an analytic extrapolation to an arbitrary complex number of spatial dimensions, the most prominent example being the completely antisymmetric tensor $\varepsilon_{\alpha\beta\gamma\delta}$. Fortunately, this tensor does not appear in the theories to be discussed in this text, so that dimensional regularization can be applied without problem.

8.2 Dimensional Regularization

The integrand in the diagram  behaves for large loop momenta $\mathbf{p}$ like $|\mathbf{p}|^{-2}$, and the integrand in the diagram  like $|\mathbf{p}|^{-4}$. The momentum integrals are therefore defined only for dimensions $D < 2$ and $D < 4$ respectively. The idea is to calculate a Feynman integral for a continuous-valued number of dimensions D for which convergence is assured. For this the Feynman integrals for integer D must be extrapolated analytically to complex D.

The concept of a continuous dimension was introduced by Wilson and Fisher [6], who first calculated physical quantities in $D = 4 - \varepsilon$ dimensions with $\text{Re}\,\varepsilon > 0$, and expanded them in powers of the deviation ε from the dimension $D = 4$. This concept was subsequently incorporated into quantum field theory [7], giving rise to many applications in statistical physics. The dimensional regularization by 't Hooft and Veltman was the appropriate mathematical tool for such expansions.

We shall first derive all formulas needed for the upcoming calculations. The analytic extrapolation to noninteger dimension will be based on the extrapolation of the factorial of integer numbers to real numbers by the Gamma function. In Subsection 8.2.2 we shall describe another approach in which D-dimensional Gaussian integrals are used to arrive at the same formulas.

For completeness, the original procedure of 't Hooft and Veltman will be reviewed in Subsection 8.2.4, to be followed in Subsection 8.2.5 by a slightly different method of Collins [8], who introduced integrals in continuous dimensions D via a certain subtraction method.

8.2.1 Calculation in Dimensional Regularization

To explain the method of dimensional regularization, consider first the simplest Feynman integral

$$I(D) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2}. \tag{8.4}$$

It is UV-divergent for $D \geq 2$, and IR-divergent for $D \leq 0$. After introducing polar coordinates as explained in Appendix 8A, it can be rewritten as

$$I(D) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2} = \frac{2\pi}{(2\pi)^D} \prod_{k=1}^{D-2} \int_0^\pi \sin^k \vartheta_k \, d\vartheta_k \int_0^\infty dp \, p^{D-1} \frac{1}{p^2 + m^2} \tag{8.5}$$

$$= \frac{S_D}{(2\pi)^D} \int_0^\infty dp \, p^{D-1} \frac{1}{p^2 + m^2}, \tag{8.6}$$

where

$$S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{8.7}$$

is the surface of a unit sphere in D dimensions [recall (1.98)]. The resulting one-dimension integral can, after the substitution $p^2/m^2 y$, be cast into the form of an integral for the Beta function

$$B(\alpha, \gamma) \equiv \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha + \gamma)} = \int_0^\infty dy \, y^{\alpha - 1} (1 + y)^{-\alpha - \gamma}, \tag{8.8}$$

where $\Gamma(z)$ is the Gamma function with the integral representation

$$\Gamma(z) = \int_0^\infty dt \, t^{z-1} e^{-t}. \tag{8.9}$$

We then find

$$\begin{aligned}I(D) &= \frac{S_D}{(2\pi)^D}\int_0^\infty dp\, p^{D-1}\,\frac{1}{p^2+m^2} = \frac{S_D}{2(2\pi)^D}(m^2)^{D/2-1}\int_0^\infty dy\, y^{D/2-1}(1-y)^{-1} \\ &= \frac{1}{(4\pi)^{D/2}\Gamma(D/2)}(m^2)^{D/2-1}\frac{\Gamma(D/2)\Gamma(1-D/2)}{\Gamma(1)} = \frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}}\Gamma(1-D/2). \qquad (8.10)\end{aligned}$$

The Gamma function provides us with an analytical extrapolation of the integrals in integer dimensions D to any complex D.

In general, Feynman integrals contain more general denominators than just $\mathbf{p}^2+m^2$. The simplest integral of this type is

$$I(D;\mathbf{q}) = \int \frac{d^Dp}{(2\pi)^D}\frac{1}{\mathbf{p}^2+2\mathbf{pq}+m^2}. \qquad (8.11)$$

It can be reduced to the previous integral (8.4) by completing the squares in the denominator, yielding

$$I(D;\mathbf{q}) = \int \frac{d^Dp}{(2\pi)^D}\frac{1}{\mathbf{p}^2+2\mathbf{pq}+m^2} = \int \frac{d^Dp}{(2\pi)^D}\frac{1}{\mathbf{p}^2+m^2-\mathbf{q}^2} = \frac{1}{(4\pi)^{D/2}}(m^2-\mathbf{q}^2)^{D/2-1}\Gamma(1-D/2). \qquad (8.12)$$

By differentiating this with respect to the mass, we get a formula for arbitrary integer powers of such propagators:

$$I(D,a;\mathbf{q}) = \int \frac{d^D\mathbf{p}}{(2\pi)^D}\frac{1}{(\mathbf{p}^2+2\mathbf{pq}+m^2)^a} = \frac{1}{(4\pi)^{D/2}}\frac{\Gamma(a-D/2)}{\Gamma(a)}(m^2-\mathbf{q}^2)^{D/2-a}, \qquad (8.13)$$

which can be extended analytically to arbitrary complex powers a.

We may differentiate this equation with respect to the external momentum $\mathbf{q}$, and obtain a further formula

$$I^\mu(D,a;\mathbf{q}) \equiv \int \frac{d^Dp}{(2\pi)^D}\frac{p_\mu}{(\mathbf{p}^2+2\mathbf{pq}+m^2)^a} = \frac{1}{(4\pi)^{D/2}}\frac{\Gamma(a-D/2)}{\Gamma(a)}\frac{q_\mu}{(m^2-\mathbf{q}^2)^{a-D/2}}. \qquad (8.14)$$

More differentiations with respect to $\mathbf{q}$ yield formulas with higher tensors $q^{\mu_1}\cdots q^{\mu_n}$ in the integrand.

For products of different propagators, the integrals are reduced to the above form with the help of *Feynman's parametric integral formula*:

$$\frac{1}{A^a\,B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 dx\frac{x^{a-1}\,(1-x)^{b-1}}{[Ax+B(1-x)]^{a+b}}, \qquad (8.15)$$

which is a straightforward generalization of the obvious identity

$$\frac{1}{AB} = \frac{1}{B-A}\left(\frac{1}{A}-\frac{1}{B}\right) = \frac{1}{B-A}\int_A^B dz\frac{1}{z^2} = \int_0^1 dx\frac{1}{[Ax+B(1-x)]^2}. \qquad (8.16)$$

Differentiation with respect to A and B yields (8.15) for integer values of the powers a and b, and the resulting equation can be extrapolated analytically to arbitrary complex powers. More generally, Feynman's formula reads:

$$\frac{1}{A_1\cdots A_n} = \Gamma(n)\int_0^1 dx_1\cdots\int_0^1 dx_n\,\frac{\delta(1-x_1+\cdots+x_n)}{(x_1A_1+\ldots+x_nA_n)^n}, \qquad (8.17)$$

as can easily be proved by induction [10]. By differentiating both sides a_i times with respect to A_i, one finds

$$\frac{1}{A_1^{a_1}\cdots A_n^{a_n}}=\frac{\Gamma\left(a_1+\ldots+a_n\right)}{\Gamma\left(a_1\right)\cdots\Gamma(a_n)}\int_0^1 dx_1\cdots\int_0^1 dx_n\frac{\delta(1-x_1+\cdots+x_n)x_1^{a_1-1}\cdot x_n^{a_n-1}}{(x_1A_1+\ldots+x_nA_n)^{a_1+\ldots+a_n}},\tag{8.18}$$

valid for integer $a_1,\ldots,a_n$. By analytic extrapolation, this formula remains valid for complex values of a_i. When introducing Feynman parameter one has to avoid divergences in integrals over x_i.

As an example, take the integral containing two propagators of arbitrary power, which becomes

$$\begin{aligned}\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(\mathbf{p}^2+m^2)^a[(\mathbf{p}-\mathbf{k})^2+m^2]^b}&=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 dx\int\frac{d^Dp}{(2\pi)^D}\frac{(1-x)^{a-1}\,x^{b-1}}{[\mathbf{p}^2+m^2-2\mathbf{pk}\,x+\mathbf{k}^2\,x]^{a+b}}\\&=\frac{1}{(4\pi)^{D/2}}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 dx\frac{(1-x)^{a-1}\,x^{b-1}}{[m^2+\mathbf{k}^2\,x\,(1-x)]^{a+b-D/2}}.\end{aligned}\tag{8.19}$$

8.2.2 Dimensional Regularization via Proper Time Representation

Schwinger observed that all propagators may be rewritten as Gaussian integrals by using a so-called *proper time representation* (see Appendix 8C) of Feynman integrals, also referred to as *parametric representation.* This permits definition of a momentum integration in D complex dimensions via a generalization of the Gaussian integral to D complex dimensions [4, 11, 12, 13], which involves again the analytic continuation of Gamma functions. This method is closely related to what is called *analytic regularization*, and many properties are common to the two approaches.The results are the same in the three approaches.

In Schwinger's proper time representation, each scalar propagator is rewritten as an integral

$$\frac{1}{\mathbf{p}^2+2\mathbf{pq}+m^2}=\int_0^\infty d\tau\; e^{-\tau(\mathbf{p}^2+2\mathbf{pq}+m^2)}.\tag{8.20}$$

The variable τ is called *proper time* for reasons which are irrelevant to the present development. With the integral representation of the Gamma function (8.9), this can be generalized to

$$\frac{1}{(\mathbf{p}^2+2\mathbf{pq}+m^2)^a}=\frac{1}{\Gamma(a)}\int_0^\infty d\tau\;\tau^{a-1}e^{-\tau(\mathbf{p}^2+2\mathbf{pq}+m^2)},\tag{8.21}$$

valid for $a>0$. We now assume that in a typical D-dimensional momentum integral associated with a Feynman diagram

$$\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(\mathbf{p}^2+2\mathbf{pq}+m^2)^a}=\frac{1}{\Gamma(a)}\int\frac{d^Dp}{(2\pi)^D}\int_0^\infty d\tau\;\tau^{a-1}e^{-\tau(\mathbf{p}^2+2\mathbf{pq}+m^2)},\tag{8.22}$$

the integral over the proper time can be exchanged with the momentum integral, which therefore becomes Gaussian:

$$\frac{1}{\Gamma(a)}\int_0^\infty d\tau\;\tau^{a-1}\int\frac{d^Dp}{(2\pi)^D}e^{-\tau(\mathbf{p}^2+2\mathbf{pq}+m^2)}.\tag{8.23}$$

Now, a D-dimensional Gaussian integral

$$\int\frac{d^Dp}{(2\pi)^D}\,e^{-\tau\mathbf{p}^2}=\left[\int\frac{dp}{2\pi}\,e^{-\tau\,p^2}\right]^D=\left(\frac{1}{4\pi\,\tau}\right)^{D/2}\tag{8.24}$$

can be easily generalized from integer values of D to any complex number D of dimensions. While the first equation in (8.24) makes sense only if D is integer valued, the right-hand side exists for any complex D, so that Eq. (8.24) can be used as a definition of the Gaussian integral on the left-hand side for complex dimensions D. This analytic interpolation is completely equivalent to those obtained by the methods in the previous two subsections [see the proof at the end of Appendix 8A]. In the Feynman integral the order of integration of the proper-time integral and the momentum integral is interchanged and the momentum integral is carried out with Eq. (8.24):

$$\int \frac{d^D p}{(2\pi)^D} e^{-\tau(\mathbf{p}^2+2\mathbf{pq}+m^2)} = \left(\frac{1}{4\pi\tau}\right)^{D/2} e^{-\tau(m^2-\mathbf{q}^2)}. \tag{8.25}$$

We can now perform the proper time integral using the integral representation (8.9) of the Gamma function and find

$$\int_0^\infty d\tau\, \tau^{a-D/2-1} e^{-\tau(m^2-\mathbf{q}^2)} = \Gamma(a-D/2)\frac{1}{(m^2-\mathbf{q}^2)^{a-D/2}}. \tag{8.26}$$

The left-hand side is defined only for $D < 2a$, but the Gamma function possesses a unique analytic continuation to larger D. Using (8.25) and (8.26), we obtain for the Feynman integral (8.22)

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{(\mathbf{p}^2+2\mathbf{pq}+m^2)^a} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(a-D/2)}{\Gamma(a)} \frac{1}{(m^2-\mathbf{q}^2)^{a-D/2}}. \tag{8.27}$$

By expanding this integral in powers of a up to order $\mathcal{O}(a)$ and comparing the coefficients of a we find a further important integral which will be needed for the calculation of vacuum energies in Eqs. (8.117) and (10.131):

$$\int \frac{d^D p}{(2\pi)^D} \log(\mathbf{k}^2+2\mathbf{kq}+m^2) \;=\; \frac{1}{(4\pi)^{D/2}} \frac{2}{D} \Gamma\left(1-D/2\right)\, (m^2-\mathbf{q}^2)^{D/2}, \tag{8.28}$$

Strictly speaking, the logarithm on the left-hand side does not make sense since its argument has the dimension of a square mass. It should therefore always be written as $\log[(\mathbf{k}^2+2\mathbf{kq}+m^2)/\mu^2]$ with some auxiliary mass μ. If $m \neq 0$, the auxiliary mass μ can, of course, be taken to be m itself. In dimensional regularization, however, this proper way of writing the logartihm does not change the integral at all, as we shall soon demonstrate. The reason is that it merely adds an integral over a constant, and this vanishes by the so-called Veltman formula, to be derived in Eq. (8.33).

A proper-time representation exists for any Feynman integral. Each propagator is replaced by an integral over τ_i. As a result, all momentum integrals are of the type (8.27), (8.14), and their straightforward generalizations. The UV-divergences arise then from the integration region $\tau_i \approx 0$, which is regulated by the analytic continuation of the Gamma function.

Massless Tadpole Integrals

Let us now derive a result which will turn out to be important for the evaluation of massless Feynman integrals associated with so-called *tadpole diagrams*. Tadpole diagrams are quadratically divergent diagrams which have only one external vertex. They contain therefore no external momenta. In the ϕ^4-theory, the simplest tadpole diagram is  . The detailed discussion of these diagrams will take place in Section 11.4, in particular their role in the renormalization process.

Consider the following Feynman integral:

$$\int \frac{d^D p}{(2\pi)^D} \frac{m^2}{\mathbf{p}^2(\mathbf{p}^2+m^2)} = \int \frac{d^D p}{(2\pi)^D} \left[\frac{1}{\mathbf{p}^2} - \frac{1}{(\mathbf{p}^2+m^2)} \right]. \tag{8.29}$$

The left hand side is calculated with the help of Schwinger's proper-time integral as

$$\begin{aligned} m^2 \int \frac{d^D p}{(2\pi)^D} \int_0^\infty d\tau_1\, d\tau_2\, e^{-(\tau_1+\tau_2)\mathbf{p}^2-\tau_2 m^2} &= \frac{m^2}{(4\pi)^{D/2}} \int_0^\infty d\tau_1\, d\tau_2\, (\tau_1+\tau_2)^{-D/2} e^{-\tau_2 m^2} \\ = \frac{m^2}{(4\pi)^{D/2}} \int_0^\infty d\tau_2 \int_{\tau_2}^\infty d\tau_{12}\, \tau_{12}^{-D/2} e^{-\tau_2 m^2} &= \frac{m^2}{(4\pi)^{D/2}} \int_0^\infty d\tau_2 \frac{-\tau_2^{-D/2+1}}{1-D/2} e^{-\tau_2 m^2} \\ &= -\frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}} \Gamma(1-D/2). \end{aligned} \tag{8.30}$$

The same result is obtained by applying Eq. (8.27) to the second term on the right-hand side of (8.29):

$$-\int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2+m^2} = -\frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}} \Gamma(1-D/2). \tag{8.31}$$

This implies that for any complex D, the first term on the right-hand side of (8.29) must vanish:

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2} = 0. \tag{8.32}$$

The vanishing of this integral was extended to what is known as *Veltman's formula* [9]:

$$\int \frac{d^D p}{(2\pi)^D} (\mathbf{p}^2)^\kappa = 0 \quad \text{for } \kappa,\ D \text{ complex}. \tag{8.33}$$

Within dimensional regularization, this formula can be derived from Eq. (8.27) by taking the limit $a \to 0$, which is well defined for all $D \neq 2, 4, \ldots$:

$$\int d^D k = 0. \tag{8.34}$$

It can now be argued that this equation holds for all D by analytic interpolation. Then, after a polar decomposition of the measure of integration as in Eq. (8.6), we obtain

$$S_D \int_0^\infty dk\, k^{D-1} = 0 \tag{8.35}$$

for any complex D, thus implying (8.33).

The consistency of (8.33) has been discussed in various ways [9, 8]. It is nontrivial owing to the absence of any regime of convergence as a function of D, so that no simple analytic continuation can be invoked. Leibbrandt introduced an extended Gaussian integral containing an auxiliary mass term which permits taking simultaneously the limits $m^2 \to 0$ and $D \to 4$. In this text we shall prove the property (8.33) once more with Collins' subtraction method in Subsections 8.2.4 and 8.2.5.

The vanishing of the integrals over simple powers of the momentum has the pleasant consequence that in the course of renormalization via the so-called minimal subtraction procedure it will be superfluous to calculate diagrams which contain a tadpole, as will be explained in Sections 11.4 and 11.7–11.8 (see also on page 205).

8.2.3 Tensor Structures

The generalization of tensors in integer space dimension D to complex values proceeds by replacing four-vectors by D-vectors. The integrations are performed and the results interpolated analytically. Considering for instance the integral

$$\int d^D p \, p_\mu e^{-\tau \mathbf{p}^2} = 0,$$

which vanishes in integer dimension. It is taken to be zero for all D. As another example, we evaluate

$$\begin{aligned} &\int \frac{d^D p}{(2\pi)^D} \frac{p_\mu}{(\mathbf{p}^2+m^2)^a} \frac{1}{[(\mathbf{p}-\mathbf{k})^2+m^2]^b} \\ &= \frac{k_\mu}{(4\pi)^{D/2}} \frac{\Gamma(a+b-D/2)}{\Gamma(a)\Gamma(b)} \int_0^1 d\tau \, \tau^b (1-\tau)^{a-1} \left[\mathbf{k}^2 \tau(1-\tau)+m^2\right]^{D/2-a-b}. \end{aligned} \tag{8.36}$$

When dealing with Feynman integrals of second or higher order we frequently encounter the unit tensor. For consistency, its trace which is defined initially only for integer values of D must be assumed to have a continuous value D for arbitrary complex D:

$$\sum_\mu \delta_{\mu\mu} = D. \tag{8.37}$$

This relation can be used to simplify integrals over tensors in momentum space:

$$\int \frac{d^D p}{(2\pi)^D} p_\mu p_\nu \, f(\mathbf{p}^2) = \frac{1}{D} \int \frac{d^D p}{(2\pi)^D} \, \delta_{\mu\nu} \mathbf{p}^2 \, f(\mathbf{p}^2). \tag{8.38}$$

The correctness of this can be verified by contraction with $\delta_{\mu\nu}$ and using the previous integration rules in a complex number D of dimensions.

8.2.4 Dimensional Regularization by 't Hooft and Veltman

We are now going to review the construction of an analytical continuation in the number of dimensions originally used by 't Hooft and Veltman [3, 4, 5]. This regularization procedure may be exemplified using the integral (8.4), which is obviously UV-divergent for $D \geq 2$ and IR-divergent for $D \leq 0$. It can be used to *define* the integration in a continuous number of dimensions D in the region $D < 2$. In order to extrapolate the integral to a larger domain, 't Hooft and Veltman introduced a procedure called "*partial p*", which is based on inserting into the integrand the unit differential operator

$$\frac{1}{D} \frac{\partial p}{\partial p} = 1, \tag{8.39}$$

and carrying out a partial integration in the region $0 < D < 2$. The surface term is explicitly zero for $0 < D < 2$ and we remain with:

$$I(D) = -\frac{1}{D} \int d^D p p_i \frac{\partial}{\partial p_i} \frac{1}{\mathbf{p}^2+m^2} \tag{8.40}$$

$$= -\frac{S_D}{D} \int dp p^{D-1} \frac{2p^2}{(p^2+m^2)^2}. \tag{8.41}$$

Inserting $m^2 - m^2$ and re-expressing the right-hand side by $I(D)$ leads to

$$I(D) = \frac{2\,S_D\,m^2}{D-2}\int dp p^{D-1}\frac{1}{(p^2+m^2)^2}. \tag{8.42}$$

The region of convergence of the integral is now extended to $0 < D < 4$. There is a pole at $D = 2$ as a consequence of the UV-divergence. This expression provides us with the desired analytic extrapolation of (8.6). The procedure can be repeated to yield a convergent integral in the region $0 < D < 6$. The momentum integrations can all be performed yielding results expressed in terms of Beta functions (8.8). The momentum integration in (8.42) yields

$$\int_0^\infty dp\,p^{D-1}\frac{1}{(p^2+m^2)^2} = \frac{(m^2)^{D/2-2}}{2}\frac{\Gamma(D/2)\Gamma(2-D/2)}{\Gamma(2)}. \tag{8.43}$$

The UV-divergence of the integration is reflected by the pole of the Gamma function at $D = 4$. After including the prefactor of (8.42) and the definition of S_D, we obtain

$$I(D) = \int d^D p\,\frac{1}{\mathbf{p}^2+m^2} = \pi^{D/2}(m^2)^{D/2-1}\Gamma(1-D/2). \tag{8.44}$$

The Gamma function is analytic in the entire complex-D plane, except for isolated poles at $D = 2, 4, \ldots$.

We now observe that the same analytic expression was obtained by a naive evaluation of (8.6) in (8.10), without an explicit consideration of ranges of convergence of the momentum integrals. The considerations of 't Hooft and Veltman serve to give a justification for the simple direct result.

The above analytic continuation in the dimension of a specific class of integrals can now easily be extended to more general Feynman integrals. There the integrand will not only depend on the magnitude of the momentum $\mathbf{p}$ but also on its direction, which may point in some n-dimensional subspace of the D-dimensional space, where n is an integer number $n \leq 4$. Let this space be spanned by unit vectors $\mathbf{q}_i$, $(i = 1, \ldots, n)$. The integrand is then some function $f(\mathbf{p}^2, \mathbf{p}\cdot\mathbf{q}_1, \mathbf{p}\cdot\mathbf{q}_2, \ldots, \mathbf{p}\cdot\mathbf{q}_n)$. Now 't Hooft and Veltman split the measure of integration into an n-dimensional part $d^n p_\parallel$ and a remainder $d^{D-n}p_\perp$, and write the integral as

$$\begin{aligned}&\int d^D p\,f(\mathbf{p}^2, \mathbf{p}\cdot\mathbf{q}_1, \mathbf{p}\cdot\mathbf{q}_2, \ldots, \mathbf{p}\cdot\mathbf{q}_n) \\ &\quad= \int_{-\infty}^{\infty} dp_1\cdots dp_n \int d^{D-n}p_\perp f(\mathbf{p}_\parallel^2, \mathbf{p}_\perp^2, \mathbf{p}\cdot\mathbf{q}_1, \mathbf{p}\cdot\mathbf{q}_2, \ldots, \mathbf{p}\cdot\mathbf{q}_n).\end{aligned} \tag{8.45}$$

The integration over the n-dimensional subspace is an ordinary integration in integer dimensions. The remaining $D - n$ -dimensional integration is integration in continuous dimensions. The function f is independent of the direction of $\mathbf{p}_\perp$ because the scalar products $\mathbf{p}\cdot\mathbf{q}_1, \ldots, \mathbf{p}\cdot\mathbf{q}_n$ depend only on $\mathbf{p}_1, \ldots, \mathbf{p}_n$, such that $\mathbf{p}_\perp$ appears only in the argument $\mathbf{p}^2 = p_1^2 + \ldots + p_n^2 + p_\perp^2$. The $D-n$ -dimensional integration over $\mathbf{p}_\perp$ is therefore rotationally invariant. Now the angular integration can be carried out in arbitrary dimensions $D - n$, leaving only a one-dimensional radial integral:

$$\begin{aligned}&\int d^{D-n}p_\perp f(\mathbf{p}_\parallel^2, \mathbf{p}_\perp^2, \mathbf{p}\cdot\mathbf{q}_1, \mathbf{p}\cdot\mathbf{q}_2, \ldots, \mathbf{p}\cdot\mathbf{q}_n) \\ &\quad= S_{D-n}\int_0^\infty dp_\perp\,p_\perp^{D-n-1} f(\mathbf{p}_\parallel^2, \mathbf{p}_\perp^2, \mathbf{p}\cdot\mathbf{q}_1, \mathbf{p}\cdot\mathbf{q}_2, \ldots, \mathbf{p}\cdot\mathbf{q}_n),\end{aligned} \tag{8.46}$$

where S_{D-n} is the surface of a unit sphere in $D - n$ dimensions [see (8.7)].

Together with the remaining integrations in (8.45) we have

$$\int d^D p\, f(\mathbf{p}^2, \mathbf{p}\cdot\mathbf{q}_1, \ldots, \mathbf{p}\cdot\mathbf{q}_n) = \frac{2\pi^{(D-n)/2}}{\Gamma((D-n)/2)} \int_{-\infty}^{\infty} dp_1 \ldots dp_n \times \int dp_\perp p_\perp^{D-n-1} f(\mathbf{p}_\parallel^2, \mathbf{p}_\perp^2, \mathbf{p}\cdot\mathbf{q}_1, \ldots, \mathbf{p}\cdot\mathbf{q}_n). \tag{8.47}$$

The splitting of the measure produces artificial IR-divergences in the remaining radial integral via the factor $p_\perp^{D-n-1}$. In 't Hooft and Veltman approach they are eliminated by partial integration, throwing away the surface terms. This procedure is shown in Appendix 8B for the simple integral of Eq. (8.4) in which the splitting of the measure also generates artificial IR-divergences. Only after generating a finite domain of convergence can the procedure "partial p" be used to go to higher D. The results show again that the Gamma function provides us with a universal analytic continuation in D.

Note that, for brevity, we have omitted here the typical factor $1/(2\pi)^D$ in the measure of all momentum integrations since they are irrelevant to the above arguments. This factor will again be present in all subsequent calculations.

8.2.5 Subtraction Method

Another procedure for the analytic extrapolation of Feynman integrals in D dimensions was presented by Collins [8]. He differs from 't Hooft and Veltman by giving an explicit procedure to subtract the artificial IR-divergences which disappear in the previous approach by discarding the surface terms.

For simplicity, consider only a rotationally invariant integrand $f(\mathbf{p}^2)$, in which case we may take $n = 0$ in Eq. (8.45). The integrand is supposed to fall off sufficiently fast for large $\mathbf{p}$ to give the one-dimensional integral in Eq. (8.46) a finite region of convergence $0 < D < D'$. For the analytic extrapolation to smaller $D < 0$, the subtraction procedure consists in adding and subtracting the leading orders of an expansion of the integrand around $\mathbf{p}^2 = 0$, which contain the IR-divergences. The continuation to $-2 < \operatorname{Re} D < D'$ is given by

$$\int \frac{d^D p}{(2\pi)^D} f(\mathbf{p}^2) = \frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \left\{ \int_C^\infty dp\, p^{D-1} f(p^2) + \int_0^C dp\, p^{D-1} \left[f(p^2) - f(0) \right] + f(0) \frac{C^D}{D} \right\}. \tag{8.48}$$

The integral over $f(p^2) - f(0) \sim p^2 f^{(1)}(0) + \mathcal{O}(p^4)$ obviously converges at the origin for $-2 < \operatorname{Re} D < 0$. This formula holds initially only for $0 < D < D'$. We now derive the left-hand side by extrapolating the right-hand side analytically to $D < 0$. There, the limit $C \to \infty$ can be taken since $f(0)C^D/D \to 0$, leaving

$$\int \frac{d^D p}{(2\pi)^D} f(\mathbf{p}^2) = \frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_0^\infty dp\, p^{D-1} \left[f(p^2) - f(0) \right], \tag{8.49}$$

which is a convergent definition for the left-hand side valid for $-2 < \operatorname{Re} D < 0$.

Repeated application of the subtraction procedure leads to a continuation formula for $-2l - 2 < \operatorname{Re} D < -2l$ $(l = 0, 1, \ldots)$:

$$\int d^D p\, f(\mathbf{p}^2) = \frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_0^\infty dp\, p^{D-1} \Big[f(p^2) - f(0) - p^2 f'(0) - \cdots - p^{2l} \frac{f^{(l)}(0)}{l!} \Big]. \tag{8.50}$$

After a sufficient number of steps, the one-dimensional integral will be convergent and can be evaluated with the help of the integral representation of the Beta function (8.8) and the Gamma function (8.9). This, in turn, can be extrapolated analytically to higher D, thereby exhibiting poles at isolated values of D. This continuation to higher D is found to be consistent with the subtraction formula as all subtraction levels lead to the same Gamma function. We demonstrate this by treating the one-loop integral (8.4):

$$I(D) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{(\mathbf{p}^2 + m^2)}, \tag{8.51}$$

which is UV-divergent for $D \geq 2$ and becomes IR-divergent for $D \leq 0$ upon splitting the measure of integration in Eq. (8.6). Expressing the integral directly in terms of Gamma functions as in (8.10) gives

$$I(D) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2} = \frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}} \Gamma(1 - D/2). \tag{8.52}$$

To find a definition for the integral for $-2 < \operatorname{Re} D < 0$, we use the subtraction formula (8.49). We derive

$$\begin{aligned} I(D) &= \int \frac{d^D p}{(2\pi)^D} \frac{1}{(\mathbf{p}^2 + m^2)} = \frac{S_D}{(2\pi)^D} \int dp\, p^{D-1} \left(\frac{1}{p^2 + m^2} - \frac{1}{m^2} \right) \\ &= \frac{S_D}{(2\pi)^D} \int dp\, p^{D-1} \frac{-p^2}{m^2(p^2 + m^2)} = \frac{-(m^2)^{D/2-1}}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty dy\, y^{D/2} (1+y)^{-1} \\ &= \frac{-(m^2)^{D/2-1}}{(4\pi)^{D/2}\Gamma(D/2)} B(D/2+1, -D/2) = \frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}} \frac{\Gamma(1 - D/2)}{\Gamma(1)}. \end{aligned} \tag{8.53}$$

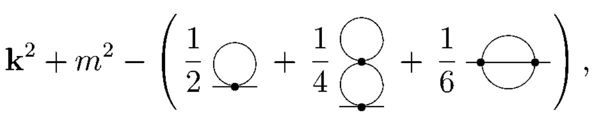
This is the same result as the one obtained for the unsubtracted formula (8.52). The same is true for higher subtractions like $(\mathbf{p}^2/m^2)^2$, implying that the integration over any pure power of $\mathbf{p}^2$ gives zero in this regularization.

These manipulations show again that naive integration in Subsection 8.2.1 performed without any subtractions provides us with the desired analytic extrapolation to any complex dimension D.

8.3 Calculation of One-Particle-Irreducible Diagrams up to Two Loops in Dimensional Regularization

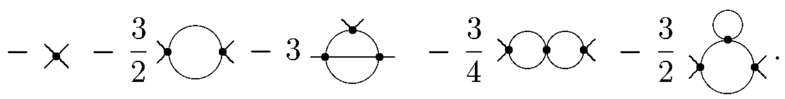
In order to illustrate dimensional regularization, we shall now calculate explicitly the diagrams in $\Gamma^{(2)}(\mathbf{k})$ and $\Gamma^{(4)}(\mathbf{k}_i)$, and $\Gamma^{(0)}$ up to two loops:

$\Gamma^{(2)}(\mathbf{k}) =$



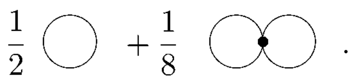
(8.54)

$\Gamma^{(4)}(\mathbf{k}_i) =$



(8.55)

$\Gamma^{(0)} =$



(8.56)

In Subsection 8.3.4, we shall also do the calculation for the vacuum diagrams.

8.3.1 One-Loop Diagrams

At the one-loop level, there are only two divergent diagrams. The Feynman integral associated with the diagram  is divergent for $D \geq 2$.

$$= -\lambda \int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2}. \tag{8.57}$$

Using Eq. (8.28), we find

$$= -\lambda \frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}} \Gamma(1 - D/2). \tag{8.58}$$

The Feynman integral is UV-divergent in two, four, six, ... dimensions, which is reflected by poles in the Gamma function at $D = 2, 4, 6, \ldots$. The poles can be subtracted in various ways, parametrized by an arbitrary mass parameter μ. Introducing the *dimensionless coupling constant* g [similar to the dimensionless $\hat{\lambda} = \lambda m^{D-4}$ of Eq. (7.28)]

$$g \equiv \lambda \mu^{D-4} = \lambda \mu^{-\varepsilon}, \tag{8.59}$$

the integral reads in terms of g and ε

$$= -m^2 \frac{g}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2} \right)^{\varepsilon/2} \Gamma(\varepsilon/2 - 1). \tag{8.60}$$

The arbitrary mass parameter μ appears in a dimensionless ratio with the mass. It is this kind of terms which contains IR-divergences in the limit $m^2 \to 0$. These are expanded in powers of ε:

$$\left(\frac{4\pi\mu^2}{m^2} \right)^{\varepsilon/2} = 1 + \frac{\varepsilon}{2} \log \left(\frac{4\pi\mu^2}{m^2} \right) + \mathcal{O}(\varepsilon^2). \tag{8.61}$$

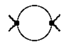
The ε-expansion of the Gamma function in (8.60) reads (see Appendix 8D):

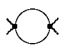
$$\Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!} \left\{ \frac{1}{\varepsilon} + \psi(n+1) + \frac{\varepsilon}{2} \left[\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1) \right] + \mathcal{O}(\varepsilon^2) \right\}, \tag{8.62}$$

where $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ is the *Euler Digamma function.* Inserting these into (8.60) we find the Laurent expansion in ε:

$$= m^2 \frac{g}{(4\pi)^2} \left[\frac{2}{\varepsilon} + \psi(2) + \log \left(\frac{4\pi\mu^2}{m^2} \right) + \mathcal{O}(\varepsilon) \right]. \tag{8.63}$$

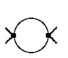
The residue of the pole is proportional to m^2 and *independent* of μ.

The integration over two propagators in  is convergent for $D < 4$:



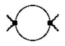
$$= \lambda^2 \int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2} \frac{1}{(\mathbf{p} + \mathbf{k})^2 + m^2}, \tag{8.64}$$

where the external momentum $\mathbf{k}$ is the sum of the incoming momenta: $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$. Using formula (8.16), we can rewrite the Feynman integral (8.64) with Feynman parameters as



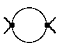
$$\begin{aligned}
&= \lambda^2 \int_0^1 dx \int \frac{d^D p}{(2\pi)^D} \frac{1}{\{(\mathbf{p}^2 + m^2)(1-x) + [(\mathbf{p}+\mathbf{k})^2 + m^2]\, x\}^2} && (8.65)\\
&= \lambda^2 \int_0^1 dx \int \frac{d^D p}{(2\pi)^D} \frac{1}{(\mathbf{p}^2 + 2\mathbf{p}\mathbf{k}x + \mathbf{k}^2 x + m^2)^2} && (8.66)\\
&= \frac{\lambda^2}{(4\pi)^{D/2}} \frac{\Gamma(2 - D/2)}{\Gamma(2)} \int_0^1 dx \frac{1}{[\mathbf{k}^2 x(1-x) + m^2]^{2-D/2}}. && (8.67)
\end{aligned}$$

The divergence for $D = 4$ is contained in the Gamma function which possesses poles at $D = 4, 6, \ldots$. The remaining parameter integral is finite for any D as long as $m^2 \neq 0$. In terms of ε and g, the expression for the simple loops reads



$$= g\mu^{\varepsilon} \frac{g}{(4\pi)^2} \Gamma(\varepsilon/2) \int_0^1 dx \left[\frac{4\pi\mu^2}{\mathbf{k}^2 x(1-x) + m^2} \right]^{\varepsilon/2} . \tag{8.68}$$

In order to separate the pole terms, we expand each term in powers of ε. The Gamma function is expanded with (8.62) to yield



$$\begin{aligned} &= g\mu^{\varepsilon} \frac{g}{(4\pi)^2} \left[\frac{2}{\varepsilon} + \psi(1) + \mathcal{O}(\varepsilon) \right] \left\{ 1 + \frac{\varepsilon}{2} \int_0^1 dx \log \left[\frac{4\pi\mu^2}{\mathbf{k}^2 x(1-x) + m^2} \right] + \mathcal{O}(\varepsilon^2) \right\} \\ &= g\mu^{\varepsilon} \frac{g}{(4\pi)^2} \left\{ \frac{2}{\varepsilon} + \psi(1) + \int_0^1 dx \log \left[\frac{4\pi\mu^2}{\mathbf{k}^2 x(1-x) + m^2} \right] + \mathcal{O}(\varepsilon) \right\} . \end{aligned} \tag{8.69}$$

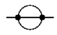
The prefactor $g\mu^{\varepsilon}$ is, in fact, the coupling constant λ which is to be renormalized, and which is therefore not expanded in ε. Only the expression behind it contributes to the renormalization constant Z_g. The pole term of this contribution is then independent of the arbitrary mass parameter μ. This mass parameter appears only in the finite part, exhibiting a degree of freedom in the renormalization procedure.

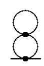
The result (8.67) can be generalized to Feynman integrals of the type (8.64), in which the denominators appear with an arbitrary power. Using Eqs. (8.15) and (8.27), we find

$$\begin{aligned} &\int \frac{d^D p}{(2\pi)^D} \frac{1}{(\mathbf{p}^2 + m^2)^a} \frac{1}{[(\mathbf{p} - \mathbf{k})^2 + m^2]^b} \\ &\quad = \frac{\Gamma(a + b - D/2)}{(4\pi)^{D/2} \Gamma(a) \Gamma(b)} \int_0^1 dx\, x^{b-1} (1-x)^{a-1} \left[\mathbf{k}^2 x(1-x) + m^2 \right]^{D/2 - a - b} . \end{aligned} \tag{8.70}$$

The parameter integral on the right-hand side develops IR-divergences for $m^2 = 0$ and $D = 4$ if a or $b \geq 2$.

8.3.2 Two-Loop Self-Energy Diagrams

At the two-loop level, there are two diagrams  and  contributing to the self-energy. The Feynman integral associated with the first one factorizes into two independent momentum integrals:

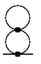


$$= \lambda^2 \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{1}{\mathbf{p}_1^2 + m^2} \frac{1}{(\mathbf{p}_2^2 + m^2)^2} . \tag{8.71}$$

The first integral coincides with the previous integral (8.57), which is expanded in powers of ε in Eq. (8.63). The second integral is calculated using formula (8.27):

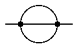
$$\begin{aligned} \lambda \int \frac{d^D p_2}{(2\pi)^D} \frac{1}{(\mathbf{p}_2^2 + m^2)^2} &= \frac{g\mu^{\varepsilon}}{(4\pi)^{D/2}} \frac{\Gamma(2 - D/2)}{\Gamma(2)} \frac{1}{(m^2)^{2-D/2}} \\ &= \frac{g}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2} \right)^{\varepsilon/2} \Gamma(\varepsilon/2) \\ &= \frac{g}{(4\pi)^2} \left[\frac{2}{\varepsilon} + \psi(1) + \log \frac{4\pi\mu^2}{m^2} + \mathcal{O}(\varepsilon) \right] , \end{aligned} \tag{8.72}$$

having expanded the Gamma function according to formula (8.62). This result can also be found directly in (8.69) for $\mathbf{k}^2 = 0$. The result of the integration over $\mathbf{p}_1$ is given in (8.63). The product of the integrations over $\mathbf{p}_1$ and $\mathbf{p}_2$ gives

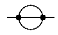


$$= -\frac{m^2 g^2}{(4\pi)^4}\left[\frac{4}{\varepsilon^2} + 2\frac{\psi(1)+\psi(2)}{\varepsilon} - \frac{4}{\varepsilon}\log\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\varepsilon^0)\right]. \tag{8.73}$$

We come now to the calculation of the so-called *sunset diagram*:

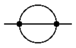


$$= \lambda^2 \int \frac{d^D p_1}{(2\pi)^D}\frac{d^D p_2}{(2\pi)^D}\frac{1}{\mathbf{p}_1^2 + m^2}\frac{1}{\mathbf{p}_2^2 + m^2}\frac{1}{(\mathbf{q}+\mathbf{p}_1+\mathbf{p}_2)^2 + m^2}, \tag{8.74}$$

in which $\mathbf{q}$ is the incoming momentum. The calculation of  is rather difficult, because a naive introduction of the parameter integrals results in divergences of the parameter integral. The problem is solved by lowering the degree of divergence via partial integration. Using the trivial identity $\partial p^\mu/\partial p^\nu = \delta^{\mu\nu}$ and the trace property (8.38), we see that

$$1 = \frac{1}{2D}\left(\frac{\partial p_1^\mu}{\partial p_1^\mu} + \frac{\partial p_2^\mu}{\partial p_2^\mu}\right). \tag{8.75}$$

Inserting this identity into (8.74), and performing a partial integration in which the surface term is discarded, we obtain a sum of two integrals:



$$\begin{aligned} &= -\frac{\lambda^2}{D-3}\int \frac{d^D p_1}{(2\pi)^D}\frac{d^D p_2}{(2\pi)^D}\frac{3m^2 + \mathbf{q}(\mathbf{q}+\mathbf{p}_1+\mathbf{p}_2)}{(\mathbf{p}_1^2+m^2)(\mathbf{p}_2^2+m^2)[(\mathbf{q}+\mathbf{p}_1+\mathbf{p}_2)^2+m^2]^2} \\ &= -\frac{\lambda^2}{D-3}\left[3m^2 A(\mathbf{q}) + B(\mathbf{q})\right]. \end{aligned} \tag{8.76}$$

In this way, the original integral which was initially quadratically divergent is decomposed into a logarithmically divergent integral $A(\mathbf{q})$ and a linearly divergent integral $B(\mathbf{q})$. Let us first evaluate the integral for $A(\mathbf{q})$:

$$A(\mathbf{q}) = \int \frac{d^D p_1}{(2\pi)^D}\frac{d^D p_2}{(2\pi)^D}\frac{1}{(\mathbf{p}_1^2+m^2)(\mathbf{p}_2^2+m^2)[(\mathbf{q}+\mathbf{p}_1+\mathbf{p}_2)^2+m^2]^2}. \tag{8.77}$$

Replacing the momentum $\mathbf{p}_2$ by $-\mathbf{p}-\mathbf{q}-\mathbf{p}_1$, the integral over $\mathbf{p}_1$ coincides with that in the diagram  in Eq. (8.64). It can be performed in the same way, and we obtain

$$A(\mathbf{q}) = \frac{1}{(4\pi)^{D/2}}\frac{\Gamma(2-D/2)}{\Gamma(2)}\int_0^1 dx \int \frac{d^D p}{(2\pi)^D}\frac{1}{[(\mathbf{q}+\mathbf{p})^2 x(1-x) + m^2]^{2-D/2}(\mathbf{p}^2+m^2)^2}. \tag{8.78}$$

Applying the Feynman formula (8.15), this becomes

$$A(\mathbf{q}) = \frac{1}{(4\pi)^{D/2}}\Gamma(4-D/2)\int_0^1 dx[x(1-x)]^{D/2-2}\int_0^1 dy \int \frac{d^D p}{(2\pi)^D}\frac{y(1-y)^{1-D/2}}{[f(\mathbf{q},\mathbf{p},x,y)]^{4-D/2}}. \tag{8.79}$$

with

$$\begin{aligned} f(\mathbf{q},\mathbf{p},x,y) &= (\mathbf{p}^2+m^2)y + \left[(\mathbf{p}+\mathbf{q})^2 + \frac{m^2}{x(1-x)}\right](1-y) \\ &= \mathbf{p}^2 + 2\mathbf{p}\mathbf{q}(1-y) + \mathbf{q}^2(1-y) + m^2\left[y + \frac{1-y}{x(1-x)}\right]. \end{aligned} \tag{8.80}$$

The momentum integral in (8.79) is now carried out with the help of formula (8.27), leading to

$$A(\mathbf{q}) = \frac{\Gamma(4-D)}{(4\pi)^D} \int_0^1 dx[x(1-x)]^{D/2-2} \int_0^1 dy \frac{y(1-y)^{1-D/2}}{\left[\mathbf{q}^2 y(1-y) + m^2 \left(y + \frac{1-y}{x(1-x)}\right)\right]^{4-D}}. \tag{8.81}$$

Inserting $D = 4 - \varepsilon$ and expanding the denominator in ε, we find

$$\begin{aligned} A(\mathbf{q}) &= \frac{\Gamma(\varepsilon)}{(4\pi)^4} \left(\frac{4\pi}{m^2}\right)^\varepsilon \int_0^1 dx[x(1-x)]^{-\varepsilon/2} \int_0^1 dy\, y(1-y)^{\varepsilon/2-1} \\ &\times \left\{1 - \varepsilon \log\left[\frac{\mathbf{q}^2}{m^2} y(1-y) + \left(y + \frac{1-y}{x(1-x)}\right)\right] + \mathcal{O}(\varepsilon^2)\right\}. \end{aligned} \tag{8.82}$$

Considering only the first term in the brackets, the parameter integrals can be evaluated with the help of the integral formula [note the difference with respect to (8.8)]:

$$B(\alpha, \beta) \equiv \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 dy\, y^{\alpha-1}(1-y)^{\beta-1}, \tag{8.83}$$

and the expansion (see Appendix 8D):

$$\Gamma(n+1+\varepsilon) = n! \left\{1 + \varepsilon\, \psi(n+1) + \frac{\varepsilon^2}{2}\left[\psi'(n+1) + \psi(n+1)^2\right] + \mathcal{O}(\varepsilon^3)\right\}, \tag{8.84}$$

yielding

$$\int_0^1 dx\, x^{-\varepsilon/2}(1-x)^{-\varepsilon/2} = \frac{\Gamma(1-\varepsilon/2)\Gamma(1-\varepsilon/2)}{(1-\varepsilon)\Gamma(1-\varepsilon)} = 1 + \varepsilon + \mathcal{O}(\varepsilon^2), \tag{8.85}$$

$$\int_0^1 dy\, y(1-y)^{\varepsilon/2-1} = \frac{\Gamma(2)\Gamma(\varepsilon/2)}{\Gamma(2+\varepsilon/2)} = \frac{2}{\varepsilon(1+\varepsilon/2)} = \frac{2}{\varepsilon} - 1 + \mathcal{O}(\varepsilon). \tag{8.86}$$

Only the y-integration is singular for $\varepsilon \to 0$. The singularity comes from the endpoint at $y = 1$. Using this fact we find for $A(\mathbf{q})$ from the first term of the bracket:

$$A(\mathbf{q}) = \frac{1}{(4\pi)^4} \left(\frac{4\pi}{m^2}\right)^\varepsilon \frac{\Gamma(1+\varepsilon)}{\varepsilon} \left[\frac{2}{\varepsilon} - 1 + \mathcal{O}(\varepsilon)\right] \left[1 + \varepsilon + \mathcal{O}(\varepsilon^2)\right]. \tag{8.87}$$

The second term in the bracket is of order ε. This ε is canceled against the $1/\varepsilon$ coming from the factor $\Gamma(\varepsilon)$. Since the logarithm itself is convergent, it contributes a pole term only for $y \to 1$ where the y-integration (8.86) generates a pole. But for $y \to 1$, the logarithm goes to zero as $\log y$. Therefore, the second term in the bracket makes no divergent contribution, and the result for $A(\mathbf{q})$ is the one in (8.87). Expanding $\Gamma(1+\varepsilon) = \Gamma(1) + \psi(1)\varepsilon + \mathcal{O}(\varepsilon^2)$ according to formula (8.84), and $(4\pi/m^2)^\varepsilon = 1 + \varepsilon \log(4\pi/m^2) + \mathcal{O}(\varepsilon^2)$, we obtain

$$A(\mathbf{q}) = \frac{1}{(4\pi)^4} \left\{\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left[1 + 2\,\psi(1) + 2\log\frac{4\pi}{m^2}\right] + \mathcal{O}(\varepsilon^0)\right\}. \tag{8.88}$$

There are many ways of calculating $B(\mathbf{q})$, most of them involving cumbersome expressions. The easiest way uses the fact that the integrand of $B(\mathbf{q})$ can be rewritten as:

$$\begin{aligned} &\frac{q^\mu(q+p_1+p_2)^\mu}{(\mathbf{p}_1^2+m^2)(\mathbf{p}_2^2+m^2)[(\mathbf{q}+\mathbf{p}_1+\mathbf{p}_2)^2+m^2]^2} \\ &\quad = -\frac{q^\mu}{2}\frac{\partial}{\partial q^\mu} \frac{1}{(\mathbf{p}_1^2+m^2)(\mathbf{p}_2^2+m^2)[(\mathbf{q}+\mathbf{p}_1+\mathbf{p}_2)^2+m^2]^2}. \end{aligned} \tag{8.89}$$

Then the integral for $B(\mathbf{q})$ becomes

$$B(\mathbf{q}) = -\frac{q^\mu}{2}\frac{\partial}{\partial q^\mu}\int \frac{d^Dp_1}{(2\pi)^D}\frac{d^Dp_2}{(2\pi)^D}\frac{1}{(\mathbf{p}_1^2+m^2)(\mathbf{p}_2^2+m^2)[(\mathbf{q}+\mathbf{p}_1+\mathbf{p}_2)^2+m^2]}. \tag{8.90}$$

Introducing Feynman parameters as before, we find

$$\begin{aligned} B(\mathbf{q}) &= -\frac{q^\mu}{2}\frac{\Gamma(3-D)}{(4\pi)^D}\frac{\partial}{\partial q^\mu}\int_0^1 dx[x(1-x)]^{D/2-2} \\ &\quad \times \int_0^1 dy\, y^{1-D/2}\left\{\mathbf{q}^2 y(1-y) + m^2\left[1-y+\frac{y}{x(1-x)}\right]\right\}^{D-3} \\ &= \mathbf{q}^2\frac{(3-D)\Gamma(3-D)}{(4\pi)^D}\int_0^1 dx[x(1-x)]^{D/2-2} \\ &\quad \times \int_0^1 dy\, y^{2-D/2}(1-y)\left\{\mathbf{q}^2 y(1-y) + m^2\left[1-y+\frac{y}{x(1-x)}\right]\right\}^{D-4}. \end{aligned} \tag{8.91}$$

In terms of $D = 4-\varepsilon$, this reads

$$\begin{aligned} B(\mathbf{q}) &= \mathbf{q}^2\frac{\Gamma(\varepsilon)}{(4\pi)^4}\left(\frac{4\pi}{m^2}\right)^\varepsilon \int_0^1 dx[x(1-x)]^{-\varepsilon/2} \\ &\times \int_0^1 dy\, y^{\varepsilon/2}(1-y)\left\{1-\varepsilon\log\left[\frac{\mathbf{q}^2}{m^2}y(1-y)+\left(1-y+\frac{y}{x(1-x)}\right)\right]+\mathcal{O}(\varepsilon^2)\right\}. \end{aligned} \tag{8.92}$$

The parameter integrals without the brackets give no pole in ε:

$$\int_0^1 dx\,[x(1-x)]^{-\varepsilon/2} = \frac{\Gamma(1-\varepsilon/2)\Gamma(1-\varepsilon/2)}{(1-\varepsilon)\Gamma(1-\varepsilon)} = 1+\varepsilon+\mathcal{O}(\varepsilon^2), \tag{8.93}$$

$$\begin{aligned} \int_0^1 dy\,(1-y)\,y^{\varepsilon/2} &= \frac{\Gamma(2)\Gamma(1+\varepsilon/2)}{\Gamma(3+\varepsilon/2)} = \frac{\Gamma(1+\varepsilon/2)}{(2+\varepsilon/2)(1+\varepsilon/2)\Gamma(1+\varepsilon/2)} \\ &= \frac{1}{2}\left[1-\frac{3}{4}\varepsilon+\mathcal{O}(\varepsilon^2)\right]. \end{aligned} \tag{8.94}$$

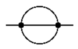
The only pole in ε comes from the prefactor $\Gamma(\varepsilon)$ in (8.92). Since the second term in the brackets of (8.92) carries a factor ε, it does not contribute to the pole term of $B(\mathbf{q})$, and we have

$$\begin{aligned} B(\mathbf{q}) &= \frac{\mathbf{q}^2}{(4\pi)^4}\left(\frac{4\pi}{m^2}\right)^\varepsilon \frac{\Gamma(1+\varepsilon)}{\varepsilon}\left[1+\varepsilon+\mathcal{O}(\varepsilon^2)\right]\frac{1}{2}\left[1-\frac{3}{4}\varepsilon+\mathcal{O}(\varepsilon^2)\right] \\ &= \frac{\mathbf{q}^2}{(4\pi)^4}\left(\frac{4\pi}{m^2}\right)^\varepsilon \frac{1}{2\,\varepsilon}+\mathcal{O}(\varepsilon^0). \end{aligned} \tag{8.95}$$

Expanding $(4\pi/m^2)^\varepsilon$ as before, the final expression for the pole term of $B(\mathbf{q})$ is

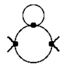
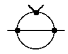
$$B(\mathbf{q}) = \frac{\mathbf{q}^2}{(4\pi)^4}\frac{1}{2\,\varepsilon}+\mathcal{O}(\varepsilon^0). \tag{8.96}$$

Together with the result for $A(\mathbf{q})$ in (8.88), we find for the sunset diagram in (8.76) with $\lambda = g\mu^\varepsilon$:



$$= -g^2\frac{m^2}{(4\pi)^4}\left\{\frac{6}{\varepsilon^2}+\frac{6}{\varepsilon}\left[\frac{3}{2}+\psi(1)+\log\frac{4\pi\mu^2}{m^2}\right]+\frac{\mathbf{q}^2}{2m^2\,\varepsilon}+\mathcal{O}(\varepsilon^0)\right\}. \tag{8.97}$$

8.3.3 Two-Loop Diagram of Four-Point Function

There are three two-loop diagrams contributing to the four-point function: , , . The first is a product of two independent integrals:

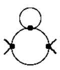

$$= -\lambda^3 \int \frac{d^D p}{(2\pi)^D} \frac{1}{[(\mathbf{p}-\mathbf{k})^2+m^2](\mathbf{p}^2+m^2)} \int \frac{d^D q}{(2\pi)^D} \frac{1}{[(\mathbf{q}-\mathbf{k})^2+m^2](\mathbf{q}^2+m^2)}, \tag{8.98}$$

where $\mathbf{k}$ denotes either of the three different momentum combinations $\mathbf{k}_1+\mathbf{k}_2$, $\mathbf{k}_1+\mathbf{k}_3$, and $\mathbf{k}_1+\mathbf{k}_4$. The pole term in (8.98) is easily calculated with Eq. (8.69). Setting $\lambda = g\mu^\varepsilon$, we find

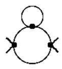

$$= -g\mu^\varepsilon \frac{g^2}{(4\pi)^4} \left\{ \frac{2}{\varepsilon} + \psi(1) + \int_0^1 dx \log\left[\frac{4\pi\mu^2}{\mathbf{k}^2 x(1-x)+m^2}\right] + \mathcal{O}(\varepsilon) \right\}^2 \tag{8.99}$$

$$= -g\mu^\varepsilon \frac{g^2}{(4\pi)^4} \left\{ \frac{4}{\varepsilon^2} + \frac{4}{\varepsilon}\psi(1) + \frac{4}{\varepsilon} \int_0^1 dx \log\left[\frac{4\pi\mu^2}{\mathbf{k}^2 x(1-x)+m^2}\right] + \mathcal{O}(\varepsilon^0) \right\}. \tag{8.100}$$

The second diagram is associated with the following integral


$$= -\lambda^3 \int \frac{d^D p}{(2\pi)^D} \frac{1}{[(\mathbf{p}-\mathbf{k})^2+m^2](\mathbf{p}^2+m^2)^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(\mathbf{q}^2+m^2)}. \tag{8.101}$$

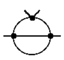
It is calculated using Eqs. (8.70) and (8.63), replacing gain λ by $g\mu^\varepsilon$:


$$= -m^2 \frac{g}{(4\pi)^2} \left[\frac{2}{\varepsilon} + \psi(2) + \log\left(\frac{4\pi\mu^2}{m^2}\right) + \mathcal{O}(\varepsilon) \right] \tag{8.102}$$

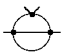
$$\times g\mu^\varepsilon \frac{g}{(4\pi)^2} \Gamma(1+\varepsilon/2) \left[1 + \frac{\varepsilon}{2} \log 4\pi\mu^2 + \mathcal{O}(\varepsilon^2) \right] \left[\int_0^1 dx \frac{1-x}{\mathbf{k}^2 x(1-x)+m^2} - \mathcal{O}(\varepsilon) \right] \tag{8.103}$$

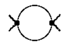
$$= -g\mu^\varepsilon \frac{g^2}{(4\pi)^4} \frac{2}{\varepsilon} \left[\int_0^1 dx \frac{m^2(1-x)}{\mathbf{k}^2 x(1-x)+m^2} + \mathcal{O}(\varepsilon) \right]. \tag{8.104}$$

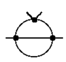
The third Feynman integral,


$$= -\lambda^3 \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{(\mathbf{p}^2+m^2)\left[(\mathbf{k}_1+\mathbf{k}_2-\mathbf{p})^2+m^2\right]} \frac{1}{(\mathbf{q}^2+m^2)\left[(\mathbf{p}-\mathbf{q}+\mathbf{k}_3)^2+m^2\right]}, \tag{8.105}$$

can be written as


$$= -\lambda^3 \int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2+m^2} \frac{1}{(\mathbf{k}_1+\mathbf{k}_2-\mathbf{p})^2+m^2} I(\mathbf{p}+\mathbf{k}_3), \tag{8.106}$$

where $I(\mathbf{k})$ is the same integral (8.64) which occurred in the one-loop diagram . Using the result of that integral in (8.68), we obtain


$$= (g\mu^\varepsilon)^2 \frac{g}{(4\pi)^2} \Gamma(\varepsilon/2) \int_0^1 dx \tag{8.107}$$

$$\times \int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2+m^2} \frac{1}{(\mathbf{k}_1+\mathbf{k}_2-\mathbf{p})^2+m^2} \left[\frac{4\pi\mu^2}{(\mathbf{k}_3+\mathbf{p})^2 x(1-x)+m^2}\right]^{\varepsilon/2}.$$

The denominators are combined via (8.17), and we get

$$g\mu^{\varepsilon}\frac{g^2}{(4\pi)^4}(4\pi\mu^2)^{\varepsilon}\Gamma(2+\varepsilon/2)\frac{\Gamma(\varepsilon)}{\Gamma(2+\varepsilon/2)}\int_0^1 dx\,[x(1-x)]^{-\varepsilon/2}\int_0^1 dy(1-y)^{\varepsilon/2-1}y$$
$$\times\int_0^1 dz\Big\{yz(1-yz)(\mathbf{k}_1+\mathbf{k}_2)^2+y(1-y)\mathbf{k}_3^2$$
$$-2yz(1-y)\mathbf{k}_3(\mathbf{k}_1+\mathbf{k}_2)+m^2\left[y+\frac{1-y}{x(1-x)}\right]\Big\}^{-\varepsilon}. \tag{8.108}$$

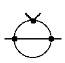
The only pole term comes from the end-point singularity in the integral at $y=1$. For $y=1$, the curly bracket can be expanded as

$$\{\dots\}=\left[(\mathbf{k}_1+\mathbf{k}_2)^2\right]^{-\varepsilon}\left\{1-\varepsilon\log\left[z(1-z)+\frac{m^2}{(\mathbf{k}_1+\mathbf{k}_2)^2}\right]+\mathcal{O}\left(\varepsilon^2\right)\right\}. \tag{8.109}$$

Using formula (8D.24), the prefactor of (8.108) is seen to have an expansion

$$\frac{1}{\varepsilon}[1-\varepsilon\psi(1)]+\mathcal{O}(\varepsilon),$$

so that we have



$$= g\mu^{\varepsilon}\frac{g^2}{(4\pi)^4}(4\pi\mu^2)^{\varepsilon}\frac{1}{\varepsilon}[1+\varepsilon\psi(1)]\int_0^1 dx\,[x(1-x)]^{-\varepsilon/2}\int_0^1 dy(1-y)^{\frac{\varepsilon}{2}-1}y$$
$$\times\left[(\mathbf{k}_1+\mathbf{k}_2)^2\right]^{-\varepsilon}\int_0^1 dz\left\{1-\varepsilon\log\left[z(1-z)+\frac{m^2}{(\mathbf{k}_1+\mathbf{k}_2)^2}\right]\right\}+\mathcal{O}(\varepsilon). \tag{8.110}$$

The first two terms in the integral are independent of x and y, and give

$$\frac{\Gamma^2(1-\varepsilon/2)}{\Gamma(2-\varepsilon)}\frac{\Gamma(\varepsilon/2)\Gamma(2)}{\Gamma(2+\varepsilon/2)}\left\{1-\varepsilon\int_0^1 dx\log\left[x(1-x)+\frac{m^2}{(\mathbf{k}_1+\mathbf{k}_2)^2}\right]\right\}. \tag{8.111}$$

The Gamma functions are combined using the following formula:

$$\frac{\prod_n\Gamma(1+a_n\varepsilon)}{\prod_m\Gamma(1+a'_m\varepsilon)}=1+\mathcal{O}\left(\varepsilon^2\right), \tag{8.112}$$

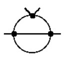
which holds if

$$\sum_n a_n-\sum_m a'_m=0. \tag{8.113}$$

We then find for the prefactor in (8.111):

$$\frac{\Gamma^2(1-\varepsilon/2)}{\Gamma(2-\varepsilon)}\frac{\Gamma(\varepsilon/2)\Gamma(2)}{\Gamma(2+\varepsilon/2)}=\frac{2}{\varepsilon}\frac{1}{(1-\varepsilon)(1+\varepsilon/2)}[1+\mathcal{O}(\varepsilon^2)]. \tag{8.114}$$

Up to this point, the singular part of (8.107) is therefore

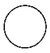
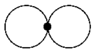


$$= g\mu^{\varepsilon}\frac{g^2}{(4\pi)^4}\frac{2}{\varepsilon^2}\left\{1+\frac{\varepsilon}{2}+\varepsilon\psi(1)-\varepsilon\int_0^1 dx\log\left[\frac{(\mathbf{k}_1+\mathbf{k}_2)^2x(1-x)+m^2}{4\pi\mu^2}\right]\right\}+\mathcal{O}(\varepsilon^0), \tag{8.115}$$

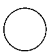
where we have used the mass parameter μ to make the logarithm dimensionless. We now look at the effect of the last logarithm in (8.109). It is accompanied by a factor ε and vanishes at $y=1$, such that it fails to lead to an end-point singularity at $y=1$, $\varepsilon=0$. Therefore, it contributes to order $\mathcal{O}(\varepsilon^0)$ and can be neglected as far as the singular parts are concerned, leaving Eq. (8.111) unchanged.

8.3.4 Two-Loop Vacuum Diagrams

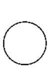
For the discussion of the effective potential in Section 10.6 we shall also need the vacuum diagrams in dimensional regularization. Up to two loops, they are

$$\Gamma^{(0)} = \frac{1}{2}$$  $$+\frac{1}{8}$$  . (8.116)

The analytic expression for the one-loop diagram is, according to Eq. (5.45),

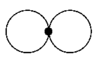
$$\frac{1}{2}$$  $$= \frac{N}{2}\int \frac{d^D p}{(2\pi)^D} \log(\mathbf{p}^2 + m^2). \tag{8.117}$$

Recalling formula Eq. (8.28), and expanding the result of the integration in powers of ε, we obtain

$$\frac{1}{2}$$ 
$$\begin{aligned} &= \frac{N}{2}\frac{2}{D}\frac{(m^2)^{D/2}}{(4\pi)^{D/2}}\Gamma(1-D/2) \\ &= \frac{N}{2}\frac{m^4}{\mu^\varepsilon}\frac{1}{(4\pi)^2}\left[-\frac{1}{\varepsilon}+\frac{1}{2}\log\frac{m^2}{4\pi\mu^2}-\frac{1}{4}-\frac{1}{2}\psi(2)\right]+\mathcal{O}(\varepsilon), \end{aligned} \tag{8.118}$$

where the value of the Digamma function $\psi(2)$ can be taken from Eq. (8D.12). The one-loop vacuum diagram is the only diagram with no vertex. To go to the second line with a dimensionless argument of the logarithm, we have inserted the mass parameter μ by inserting the identity $1 = \mu^\varepsilon/\mu^\varepsilon$ and expanded $(m^2/\mu^2)^{\varepsilon/2}$ in powers of ε. Note that the parameter μ in the prefactor $1/\mu^\varepsilon$ cannot be absorbed into the coupling constant.

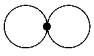
The two-loop diagram for $N = 1$ corresponds to the Feynman integral

$$\frac{1}{8}$$  $$= -\frac{\lambda}{8}\left(\int \frac{d^D p}{(2\pi)^D}\frac{1}{\mathbf{p}^2+m^2}\right)^2. \tag{8.119}$$

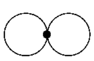
For an O(N)-symmetric theory, this is multiplied by the contracted tensor (6.25):

$$T^{(1)}_{\alpha\alpha\beta\beta} = \frac{N(N+2)}{3}. \tag{8.120}$$

Comparison of (8.119) with (8.57) and (8.58) yields

$$\frac{1}{8}$$  $$= -\frac{N(N+2)}{3}\frac{\lambda}{8}\frac{(m^2)^{D-2}}{(4\pi)^D}\Gamma^2(1-D/2). \tag{8.121}$$

This has the ε-expansion in terms of $g = \lambda\mu^{-\varepsilon}$:

$$\frac{1}{8}$$ 
$$\begin{aligned} &= -\frac{N(N+2)}{3}\frac{1}{2}\frac{m^4}{\mu^\varepsilon}\frac{g}{(4\pi)^4}\left\{\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\left(\log\frac{m^2}{4\pi\mu^2}-\psi(2)\right)\right. \\ &\left. -\psi(2)\log\frac{m^2}{4\pi\mu^2}+\frac{1}{2}\left(\log\frac{m^2}{4\pi\mu^2}\right)^2+\frac{1}{4}\left[\frac{\pi^2}{3}+2\psi^2(2)-\psi'(2)\right]+\mathcal{O}(\varepsilon)\right\}. \end{aligned}$$

Appendix 8A Polar Coordinates and Surface of a Sphere in D Dimensions

The polar coordinates in D dimensions are:

$$(p_1.\dots,p_D) \;\to\; (p,\varphi,\vartheta_1,\dots.\vartheta_{D-2})\,, \quad p^2 = p_\mu p_\mu\,, \tag{8A.1}$$

$$d^Dp \;=\; p^{D-1}dp\,d\varphi\,\sin\vartheta_1\,d\vartheta_1\,\sin^2\vartheta_2\,d\vartheta_2\dots\sin^{D-2}\vartheta_{D-2}d\vartheta_{D-2}\,, \tag{8A.2}$$

where $0<p<\infty,\ 0<\varphi<2\pi,\ 0<\vartheta_i<\pi,\ i=1,\dots,D-2$. We shall denote the directional integral as

$$\int d\hat{\mathbf{p}} \equiv \int d\varphi\,\sin\vartheta_1\,d\vartheta_1\,\sin^2\vartheta_2\,d\vartheta_2\dots\sin^{D-2}\vartheta_{D-2}d\vartheta_{D-2}. \tag{8A.3}$$

The result of the directional integral is the surface of a sphere of unit radius in D dimensions:

$$S_D = \int d\hat{\mathbf{p}}. \tag{8A.4}$$

The $D-1$ angular integrations can be carried out using the integral formula

$$\int_0^{\pi/2}(\sin t)^{2x-1}(\cos t)^{2y-1}\,dt = \frac{1}{2}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,, \qquad \mathrm{Re}\,x,\,\mathrm{Re}\,y>0\,. \tag{8A.5}$$

Inserting here $y=\frac{1}{2}$ and $x=\frac{k+1}{2}$, we obtain

$$\int_0^{\pi}dt\,(\sin t)^k = 2\int_0^{\pi/2}dt\,(\sin t)^k = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{k+2}{2})} = \sqrt{\pi}\,\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})}\,, \tag{8A.6}$$

and thus:

$$S_D = 2\pi\prod_{k=1}^{D-2}\int_0^{\pi}\sin^k\vartheta_k\,d\vartheta_k = 2\pi^{\frac{D}{2}}\,\frac{\Gamma(1)\prod_{k=2}^{D-2}\Gamma(\frac{k+1}{2})}{\Gamma(\frac{D}{2})\prod_{k=1}^{D-3}\Gamma(\frac{k+2}{2})} = 2\pi^{\frac{D}{2}}\,\frac{\Gamma(1)}{\Gamma(\frac{D}{2})}\,. \tag{8A.7}$$

The surface of the unit sphere in D dimensions is therefore

$$S_D = \frac{2\,\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}\,. \tag{8A.8}$$

Integration of a rotationally invariant integrand gives

$$\begin{aligned} I(D) \;&=\; \int d^Dp\,f(\mathbf{p}^2) = 2\pi\prod_{k=1}^{D-2}\int_0^{\pi}\sin^k\vartheta_k\,d\vartheta_k\int_0^\infty dp\,p^{D-1}\,f(p^2) \\ &=\; S_D\int_0^\infty dp\,p^{D-1}\,f(p^2)\,. \end{aligned} \tag{8A.9}$$

These calculations hold initially for integer values of the dimension D. By an analytic continuation of S_D in D, the final results make sense also for continuous values of D. As an example, we extrapolate analytically a D-dimensional Gaussian integral to any complex D. Using Eq. (8A.9) we find

$$\int\frac{d^Dp}{(2\pi)^D}e^{-\mathbf{p}^2} \;=\; \frac{S_D}{(2\pi)^D}\int_0^\infty dp\,p^{D-1}e^{-p^2} \;=\; \frac{S_D}{2(2\pi)^D}\int_0^\infty dx\,x^{D/2-1}e^{-x} \tag{8A.10}$$

$$=\; \frac{S_D}{2(2\pi)^D}\,\Gamma(D/2) \;=\; \left(\frac{1}{4\pi}\right)^{D/2}. \tag{8A.11}$$

Appendix 8B Dimensional Regularization of 't Hooft and Veltman by Splitting the Integration Measure

The splitting of the integration measure may be exemplified using the integral

$$I(D) = \int d^D p \frac{1}{\mathbf{p}^2 + m^2} = S_D \int_0^\infty dp\, p^{D-1} \frac{1}{p^2 + m^2}, \tag{8B.1}$$

where S_D is the surface of a unit sphere in D dimensions calculated in Eq. (8A.8). The integral in (8B.1) is obviously UV-divergent for $D \geq 2$ and IR-divergent for $D \leq 0$.

The idea of 't Hooft and Veltman is to define Feynman integrals in *noninteger dimensions* by splitting the measure of integration into a physical four- and an additional artificial $D-4$-dimensional one. Let the four-dimensional momentum variable part be $\mathbf{p}_{\|}$, the remaining $D-4$-dimensional one $\mathbf{p}_{\perp}$, so that

$$I(D) = \int d^4 p_{\|} \int d^{D-4} p_{\perp} \frac{1}{p_{\|}^2 + p_{\perp}^2 + m^2}. \tag{8B.2}$$

The $D-4$-dimensional integrals can be reduced to a one-dimensional integral by observing that the integrand depends only on the length of the $D-4$-dimensional integration variable. Proceeding as in Appendix 8A, we perform the angular integration in noninteger dimension which yields the surface of a unit sphere in $D-4$ dimensions:

$$S_{D-4} = \frac{2\pi^{(D-4)/2}}{\Gamma((D-4)/2)}. \tag{8B.3}$$

The resulting expression for $I(D)$ contains only ordinary integrals:

$$I(D) = S_{D-4} \int d^4 p_{\|} \int_0^\infty dp_{\perp} \frac{p_{\perp}^{D-4-1}}{p_{\|}^2 + p_{\perp}^2 + m^2}. \tag{8B.4}$$

This integral does not converge for any D. For $D \geq 2$ it is UV-divergent, whereas for $D \leq 4$ it is IR-divergent. The reason is the artificial IR-divergence generated by splitting the measure of integration. An integral with the original region of convergence is constructed by *partial integration*. For this we rewrite

$$dp_{\perp} p_{\perp}^{D-4-1} = \frac{1}{2} dp_{\perp}^2 p_{\perp}^{D-6} = dp_{\perp}^2 \frac{1}{D-4} \frac{d}{dp_{\perp}^2} (p_{\perp}^2)^{D/2-2} \tag{8B.5}$$

and integrate (8B.4) partially over $p_{\perp}^2$, yielding

$$I(D) = \frac{S_{D-4}}{D-4} \int d^4 p_{\|} \left[\frac{(p_{\perp}^2)^{D/2-2}}{p_{\|}^2 + p_{\perp}^2 + m^2} \bigg|_0^\infty + \int_0^\infty dp_{\perp}^2 \frac{(p_{\perp}^2)^{D/2-2}}{(p_{\|}^2 + p_{\perp}^2 + m^2)^2} \right]. \tag{8B.6}$$

The surface term is UV-divergent for $D \geq 2$, and IR-divergent for $D \leq 4$, and is discarded. The remaining integral is UV-divergent for $D \geq 2$ and IR-divergent for $D \leq 2$. Hence there is still no region of convergence. We must repeat the procedure, and arrive with $\Gamma(D/2-2)(D/2-2)(D/2-1) = \Gamma(D/2)$ at an expression

$$I(D) = \frac{2\pi^{D/2-2}}{\Gamma(D/2)} \int d^4 p_{\|} \int_0^\infty dp_{\perp}^2 \frac{(p_{\perp}^2)^{D/2-1}}{(p_{\|}^2 + p_{\perp}^2 + m^2)^3}. \tag{8B.7}$$

This integral is now convergent for $0 < D < 2$. It is the defining integral for the D-dimensional integral in Eq. (8.4) in this interval and can be used as a starting point for an analytic extrapolation to higher D.

For this purpose, 't Hooft and Veltman used their procedure "*partial p*", inserting into the integrand the unit differential operator

$$\frac{1}{D}\left(\frac{\partial p_{\|i}}{\partial p_{\|i}} + \frac{\partial p_\perp}{\partial p_\perp}\right) = 1, \tag{8B.8}$$

and carrying out a partial integration in the region $0 < D < 2$. This time, the surface term is explicitly zero for $0 < D < 2$. The region of convergence of the remaining integral is extended to $0 < D < 4$, except for a pole at $D = 2$ as a consequence of the UV-divergence. After re-expressing the right-hand side in terms of I, we find

$$I(D) = -3m^2 \frac{4\,\pi^{D/2-2}}{(D/2-1)\Gamma(D/2)} \int d^4p_\| \int_0^\infty dp_\perp \frac{p_\perp^{D-1}}{(p_\|^2 + p_\perp^2 + m^2)^4}. \tag{8B.9}$$

This expression provides us with the desired analytic extrapolation of (8B.7).

The procedure can be repeated to yield a convergent integral in the region $0 < D < 6$:

$$I(D) = 3\cdot 4\,m^4 \frac{4\,\pi^{D/2-2}}{(D/2-2)(D/2-1)\Gamma(D/2)} \int d^4p_\| \int_0^\infty dp_\perp \frac{p_\perp^{D-1}}{(p_\|^2 + p_\perp^2 + m^2)^5}. \tag{8B.10}$$

The only traces of the original UV-divergence are the poles for $D = 2$ and $D = 4$. The integrations can be performed and the result expressed in terms of Beta functions (8.8). The four-dimensional momentum integration in (8B.10) yields

$$\int d^4p_\| \int_0^\infty dp_\perp \frac{p_\perp^{D-1}}{(\mathbf{p}_\|^2 + p_\perp^2 + m^2)^5} = \pi^2 \frac{\Gamma(3)}{\Gamma(5)} \int_0^\infty dp_\perp \frac{p_\perp^{D-1}}{(p_\perp^2 + m^2)^3}, \tag{8B.11}$$

and the remaining one-dimensional integration gives

$$\pi^2 \frac{(m^2)^{D/2-3}}{2} \frac{\Gamma(D/2)\Gamma(3-D/2)}{\Gamma(5)}. \tag{8B.12}$$

The UV-divergence of the integrations is reflected by the pole of the Gamma function at $D = 6$. After including the prefactor in (8B.10), we obtain

$$I(D) = \int d^Dp\, \frac{1}{\mathbf{p}^2 + m^2} = \pi^{D/2}(m^2)^{D/2-1}\Gamma(1 - D/2). \tag{8B.13}$$

This is the same result as in (8.44) which was reached without splitting the integration measure. But the splitting is useful for integrals that depend on external momentum.

Appendix 8C Parametric Representation of Feynman Integrals

We follow Itzykson and Zuber [15] in the derivation of a proper-time representation of a Feynman integral with $L = I - V + 1$ loops, I internal lines, and V vertices. It is superficially UV-convergent for $\omega(G) = \mathrm{Re}(D)L - 2I < 0$.

An oriented diagram is defined with the help of a so-called *incidence matrix* $\epsilon_{v\,l}$ with $v = 1, \ldots, V$ and $l = 1, \ldots, I$:

$$\epsilon_{v\,l} = \begin{cases} +1 & \text{if } v \text{ is starting vertex of line } l \\ -1 & \text{if } v \text{ is ending vertex of line } l \\ 0 & \text{if line } l \text{ is not incident on vertex } v \end{cases} . \tag{8C.1}$$

The integral of an n-point diagram G containing no tadpole part is called $I'_G(\mathbf{k}_1, \ldots, \mathbf{k}_n)$. The sum of the external momenta entering the diagram at a vertex v is denoted by $\bar{\mathbf{k}}_v$. Momentum conservation at each vertex is expressed by $\delta^{(D)}(\bar{\mathbf{k}}_v - \sum_l \epsilon_{v\,l}\mathbf{p}_l)$, where $\mathbf{p}_l$ are the internal momenta. The integral has the following general form:

$$I'_G(\mathbf{k}_1, \ldots, \mathbf{k}_n) = (-\lambda)^V W_G \int \prod_{l=1}^{I} \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}_l^2 + m^2} \prod_{v=1}^{V} (2\pi)^D \delta^{(D)} \left(\bar{\mathbf{k}}_v - \sum_l \epsilon_{v\,l}\mathbf{p}_l \right), \tag{8C.2}$$

where W_G is the weight factor. Each propagator and the δ-distributions at each vertex are expressed by an integral representation:

$$\frac{1}{\mathbf{p}^2 + m^2} = \int_0^\infty d\tau \, e^{-\tau(\mathbf{p}^2 + m^2)}, \tag{8C.3}$$

$$(2\pi)^D \delta^{(D)}(\bar{\mathbf{k}}_v - \sum_l \epsilon_{v\,l}\mathbf{p}_l) = \int d^D y_v \, e^{-i\mathbf{y}_v \cdot (\bar{\mathbf{k}}_v - \sum_l \epsilon_{v\,l}\mathbf{p}_l)}. \tag{8C.4}$$

Insertion of these two formulas and interchange of the order of integration in Eq. (8C.2) leads to $\mathbf{p}$-integrations of the following form:

$$\begin{aligned} \int \frac{d^D p_l}{(2\pi)^D} e^{-\tau_l \left(\mathbf{p}_l^2 - \frac{i}{\tau_l} \sum_v \mathbf{y}_v \epsilon_{v\,l} \mathbf{p}_l \right)} &= \int \frac{d^D p_l}{(2\pi)^D} e^{-\tau_l \left(\mathbf{p}_l - \frac{i}{2\tau_l} \sum_v \mathbf{y}_v \epsilon_{v\,l} \right)^2 - \frac{1}{4\tau_l} \left(\sum_v \mathbf{y}_v \epsilon_{v\,l} \right)^2} \\ &= \frac{1}{(4\pi\tau_l)^{D/2}} e^{-\frac{1}{4\tau_l} \left(\sum_v \mathbf{y}_v \epsilon_{v\,l} \right)^2}. \end{aligned} \tag{8C.5}$$

The interchange is justified if $\mathrm{Re}(D)L < 2I$ for the entire diagram as well as for all subdiagrams, or if the integrals are regularized. The complete expression for I'_G becomes

$$I'_G(\mathbf{k}_i) = \frac{(-\lambda)^V W_G}{(4\pi)^{ID/2}} \int \prod_{v=1}^{V} d^D y_v \int_0^\infty \prod_{l=1}^{I} \left[d\tau_l \, \frac{e^{-\tau_l m^2 - \left(\sum_v \mathbf{y}_v \epsilon_{v\,l} \right)^2 / 4\tau_l}}{\tau_l^{D/2}} \right] e^{-i \sum_{v=1}^{V} \mathbf{y}_v \cdot \bar{\mathbf{k}}_v}. \tag{8C.6}$$

After a variable transformation

$$\mathbf{y}_1 = \mathbf{z}_1 + \mathbf{z}_V, \quad \mathbf{y}_2 = \mathbf{z}_2 + \mathbf{z}_V, \quad \ldots \quad , \quad \mathbf{y}_V = \mathbf{z}_V,$$

and using the fact that

$$\mathbf{z}_V \sum_{v=1}^{V} \epsilon_{v\,l} = 0,$$

the integration over $\mathbf{z}_V$ yields the factor

$$\int d^D z_V \exp\left(-i\mathbf{z}_V \sum_{v=1}^{V} \bar{\mathbf{k}}_v\right) = (2\pi)^D \delta^{(D)} \left(\sum_v \bar{\mathbf{k}}_v \right) = (2\pi)^D \delta^{(D)} \left(\sum_i^n \mathbf{k}_i \right).$$

It is useful to change the notation and remove this factor, since the integrals in $\Gamma^{(n)}$ are defined without it. Thus we introduce

$$I'_G = (2\pi)^D \delta^{(D)} \left(\sum_i^n \mathbf{k}_i \right) I_G, \tag{8C.7}$$

and the remaining integral is rewritten as

$$I_G(\mathbf{k}_i) = (-\lambda)^V W_G \int \prod_{v=1}^{V-1} d^D z_v \int_0^\infty \prod_{l=1}^{I} \left[d\tau_l \, \frac{e^{-\tau_l m^2 - \left(\sum_{v_1,v_2}^{V-1} \mathbf{z}_{v_1} \epsilon_{v_1 l} \frac{1}{\tau_l} \epsilon_{v_2 l} \mathbf{z}_{v_2} \right)/4}}{(4\pi\tau_l)^{D/2}} \right] e^{-i \sum_{v=1}^{V-1} \mathbf{z}_v \bar{\mathbf{k}}_v}.$$

The matrix

$$[d_G]_{v_1 v_2} = \sum_l \epsilon_{v_1 l} \frac{1}{\tau_l} \epsilon_{v_2 l}$$

is nonsingular and its determinant is found to be [16]

$$\Delta_G \equiv \det[d_G] = \sum_{\text{trees}\,\mathcal{T}} \prod_{l \in \mathcal{T}} \frac{1}{\tau_l}, \tag{8C.8}$$

where $\sum_{\text{trees}\,\mathcal{T}}$ denotes a sum over all trees diagrams contained in G. Recall the definition tree diagrams on page 72 and in Fig. 5.5 on page 72. They consist of a sum over all maximally connected subsets of lines in G which contain no loop but all vertices of G. As each tree has $V-1$ lines, Δ_G is a homogeneous polynomial in τ_l^{-1} of order $V-1$. Integrating over $\mathbf{z}_i$ gives

$$I_G(\mathbf{k}_i) = \frac{(-\lambda)^V W_G}{(4\pi)^{LD/2}} \int_0^\infty \prod_{l=1}^{I} d\tau_l \, \frac{e^{-\tau_l m^2 - \sum_{v_1,v_2}^{V-1} \bar{\mathbf{k}}_{v_1} [d^{-1}]_{v_1 v_2} \bar{\mathbf{k}}_{v_2}}}{\prod_{l=1}^{I} \tau_l^{D/2} \Delta_G(\tau)^{D/2}}.$$

The denominator in the integrand has the final form $[M_G(\tau)]^{D/2}$, where

$$M_G(\tau) = \tau_1, \ldots, \tau_I \times \sum_{\text{trees}\ \mathcal{T}} \prod_{l \in \mathcal{T}} \frac{1}{\tau_l} = \sum_{\text{trees}\ \mathcal{T}} \prod_{l \notin \mathcal{T}} \tau_l, \tag{8C.9}$$

which is a homogeneous polynomial of degree $I - (V-1) = L$.

Let us denote the quadratic form in the exponent of the integrand by $Q_G(\bar{\mathbf{k}}, \tau)$. It is given by a ratio of two homogeneous polynomials in τ of degree $L+1$ and L respectively:[0]

$$\sum_{v_1,v_2}^{V-1} \bar{\mathbf{k}}_{v_1} [d^{-1}]_{v_1 v_2} \bar{\mathbf{k}}_{v_2} = \frac{1}{M_G} \sum_{\text{cuts}\ C} \Big(\sum_{v \in G_1(C)} \bar{\mathbf{k}}_v \Big)^2 \prod_{l \in C} \tau_l \equiv Q_G(\bar{\mathbf{k}}, \tau). \tag{8C.10}$$

A cut is a subset of the lines of G which divide G in exactly two parts, say $G_1(C)$ and $G_2(C)$ if they are taken away. A tree becomes a cut if one of its lines is taken away. Therefore a cut has $I - (V-1) + 1 = L + 1$ lines. With these notations the final form reads

$$I_G(\mathbf{k}_i) = \frac{(-\lambda)^V W_G}{(4\pi)^{LD/2}} \int_0^\infty \prod_{l=1}^{I} d\tau_l \, \frac{e^{-\tau_l m^2 - Q_G(\bar{\mathbf{k}}, \tau)}}{M_G(\tau)^{D/2}}. \tag{8C.11}$$

The dependence on D emerges explicitly from the one dimensional τ-integrations. For this reason, formula (8C.11) has been used to define Feynman integrals in D dimension with complex D [12]. The UV-divergences are found in the $\tau \to 0$ limit.

The superficial divergence of the parametric integral is seen explicitly if the τ-integration is rescaled by a homogeneity parameter σ: $\tau_l \to \sigma\tau_l$, such that

$$Q_G(\bar{\mathbf{k}}, \sigma\tau) = \sigma Q_G(\bar{\mathbf{k}}, \tau), \tag{8C.12}$$

$$M_G(\sigma\tau) = \sigma^L M_G(\tau). \tag{8C.13}$$

Inserting the trivial identity

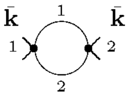
$$1 = \int_0^\infty d\sigma\, \delta\left(\sigma - \sum_l^I \tau_l\right) \tag{8C.14}$$

into (8C.11) gives then for I_G:

$$\begin{aligned} I_G(\mathbf{k}_i) &= (-\lambda)^V W_G \int_0^\infty \frac{d\sigma}{\sigma}\, \sigma^{I-DL/2} \int_0^1 \prod_{l=1}^I d\tau_l \, \frac{e^{-\sigma[Q_G(\bar{\mathbf{k}},\tau)+m^2\sum_l \tau_l]}}{[(4\pi)^L M_G(\tau)]^{D/2}}\, \delta(1-\sum_l \tau_l) \\ &= (-\lambda)^V W_G \Gamma(I-DL/2) \int_0^1 \prod_{l=1}^I d\tau_l \, \frac{[Q_G(\bar{\mathbf{k}},\tau)+m^2\sum_l \tau_l]^{\frac{DL}{2}-I}}{[(4\pi)^L M_G(\tau)]^{D/2}}\, \delta(1-\sum_l \tau_l). \end{aligned} \tag{8C.15}$$

The integration over σ produces the superficial divergence of G. It is convergent at $\sigma = 0$ only if $2I - \mathrm{Re}(D)L = -\omega(G) > 0$. The integral is analytically continued with the Gamma function $\Gamma((2I-DL)/2) = \Gamma(-\omega(G)/2)$, which gives rise to simple poles for $\omega(G) = 0, 2, \ldots$, where the divergences are logarithmic or quadratically. All possible subdivergences come from the remaining τ-integrations, which produce further Gamma functions [13].

We shall exemplify the above formula for a one-loop diagram which has $V = 2$, $I = 2$, $L = 1$ and therefore $I - DL/2 = 2 - D/2$. The momentum conservation is taken care of by a factor $(2\pi)^D \delta^{(D)}(\sum_i^n \mathbf{k}_i)$ to be omitted.



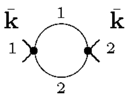
$$\begin{aligned} &= \lambda^2 \frac{3}{2} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(\mathbf{p}^2+m^2)[(\mathbf{p}-\bar{\mathbf{k}})^2+m^2]} \\ &= \lambda^2 \frac{3}{2} \int \frac{d\sigma}{\sigma} \sigma^{2-D/2} \int_0^1 d\tau_1 d\tau_2 \frac{\mathrm{e}^{-\sigma[Q_G(\bar{\mathbf{k}},\tau_1,\tau_2)+m^2(\tau_1+\tau_2)]}}{[4\pi\, M_G(\tau_1,\tau_2)]^{D/2}}\, \delta(1-\tau_1-\tau_2) \\ &= \lambda^2 \frac{3}{2}\, \Gamma(2-D/2) \int_0^1 d\tau_1 d\tau_2 \frac{[Q_G(\bar{\mathbf{k}},\tau_1,\tau_2)+m^2(\tau_1+\tau_2)]^{\frac{D}{2}-2}}{[4\pi\, M_G(\tau_1,\tau_2)]^{D/2}}\, \delta(1-\tau_1-\tau_2). \end{aligned} \tag{8C.16}$$

This simple diagram contains only two trees, $\mathcal{T}_1$ and $\mathcal{T}_2$. The first consists of line 1, the second of line 2. There is also one cut $\mathcal{C}$, which consists of line 1 and 2, such that $G_1(\mathcal{C})$ contains vertex 1 and $G_2(\mathcal{C})$ contains vertex 2. The relations for the number of lines of a tree and a cut are fulfilled: $I(\mathcal{T}) = V - 1 = 1$ and $I(\mathcal{C}) = L + 1 = 2$. We then find for M_G and Q_G:

$$M_G = \prod_{l \notin \mathcal{T}_1} \tau_l + \prod_{l \notin \mathcal{T}_2} \tau_l = \tau_2 + \tau_1, \tag{8C.17}$$

$$Q_G = \frac{1}{\tau_1+\tau_2} \bar{\mathbf{k}}^2\, \tau_1\tau_2, \tag{8C.18}$$

and the integral becomes



$$\begin{aligned} &= \lambda^2 \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}} \int_0^1 d\tau_1 d\tau_2 \frac{\left[\bar{\mathbf{k}}^2 \frac{\tau_1\tau_2}{\tau_1+\tau_2} + m^2(\tau_1+\tau_2)\right]^{D/2-2}}{(\tau_1+\tau_2)^{D/2}}\, \delta(1-\tau_1-\tau_2) \\ &= \lambda^2 \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}} \int_0^1 d\tau \left[\bar{\mathbf{k}}^2 \tau(1-\tau) + m^2\right]^{D/2-2}. \end{aligned} \tag{8C.19}$$

We have reproduced the result in (8.67), which we obtained with Feynman's parameter integral formula.

Appendix 8D Expansion of Gamma Function

The Gamma function is defined by the integral (8.9). This integral has poles for $z = 0$ and negative integers. Partial integration applied to the integral representation of $\Gamma(z+1)$ leads to the identity

$$\Gamma(z+1) = z\int_0^\infty dt\, t^{z-1}e^{-t} = z\,\Gamma(z), \tag{8D.1}$$

which shows that the Gamma function is the generalization of the factorial to arbitrary complex variables z.

In the context of dimensional regularization, we need a series expansion of the Gamma function near zero or negative integer values of z. Using (8D.1), we see that

$$\Gamma(2+\varepsilon) = (1+\varepsilon)\,\Gamma(1+\varepsilon) = (\varepsilon+1)\,\varepsilon\,(\varepsilon-1)\,\Gamma(-1+\varepsilon). \tag{8D.2}$$

This relates the expansion of $\Gamma(-1+\varepsilon)$ to the expansion of $\Gamma(2+\varepsilon)$ in powers of ε, which reads

$$\begin{aligned}\Gamma(2+\varepsilon) &= \Gamma(2) + \Gamma'(2)\,\varepsilon + \frac{1}{2}\Gamma''(2)\,\varepsilon^2 + \mathcal{O}(\varepsilon^3) &\text{(8D.3)}\\ &= 1 + \psi(2)\,\varepsilon + \frac{1}{2}\frac{\Gamma''(2)}{\Gamma(2)}\,\varepsilon^2 + \mathcal{O}(\varepsilon^3). &\text{(8D.4)}\end{aligned}$$

In the second line, we have inserted $\Gamma(2) = 1$ as well as the definition of the Euler Digamma function $\psi(z)$:

$$\psi(z) \equiv \Gamma'(z)/\Gamma(z). \tag{8D.5}$$

From this follows the recurrence relation:

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{d\,\log(z-1)}{dz} + \frac{\log\Gamma(z-1)}{dz} = \frac{1}{z-1} + \psi(z-1), \tag{8D.6}$$

which for integer values of z implies

$$\psi(n) = \psi(1) + \sum_{l=1}^{n-1}\frac{1}{l}. \tag{8D.7}$$

Analogously, we find for $\psi'(z)$:

$$\psi'(z) = \frac{-1}{(z-1)^2} + \psi'(z-1), \tag{8D.8}$$

and

$$\psi'(n) = \psi'(1) - \sum_{l=1}^{n-1}\frac{1}{l^2}. \tag{8D.9}$$

The value of $\psi(z)$ at $z = 1$ is equal to the negative of *Euler's constant* γ:

$$\psi(1) = -\gamma = -0.5772156649\ldots. \tag{8D.10}$$

The derivative of $\psi(z)$ has at $z = 1$ the value

$$\psi'(1) = \pi^2/6. \tag{8D.11}$$

Thus we find:

$$\psi(n) = -\gamma + \sum_{l=1}^{n-1} \frac{1}{l}, \tag{8D.12}$$

$$\psi'(n) = \frac{\pi^2}{6} - \sum_{l=1}^{n-1} \frac{1}{l^2},. \tag{8D.13}$$

Differentiating Eq. (8D.5), we obtain

$$\psi'(z) = \Gamma''(z)/\Gamma(z) - [\Gamma'(z)]^2/\Gamma(z)^2, \tag{8D.14}$$

and thus an expression for $\Gamma''(z)/\Gamma(z)$ which may be inserted into the expansion Eq. (8D.4) to find

$$\Gamma(2+\varepsilon) = 1 + \psi(2)\varepsilon + \frac{1}{2}\left[\psi'(2) + \psi(2)^2\right]\varepsilon^2 + \mathcal{O}(\varepsilon^3). \tag{8D.15}$$

For $\Gamma(-1+\varepsilon)$, we derive the expansion

$$\Gamma(-1+\varepsilon) = -\frac{\Gamma(2+\varepsilon)}{\varepsilon(1+\varepsilon)(1-\varepsilon)} = -\frac{1}{\varepsilon}\Gamma(2+\varepsilon)\left[1+\varepsilon^2+\mathcal{O}(\varepsilon^3)\right] \tag{8D.16}$$

$$= -\left\{\frac{1}{\varepsilon} + \psi(2) + \varepsilon\left[1 + \frac{1}{2}\psi'(2) + \frac{1}{2}\psi(2)^2\right]\right\} + \mathcal{O}(\varepsilon^2). \tag{8D.17}$$

A generalization of this formula to any integer n is obtained using repeatedly the identity (8D.1):

$$\Gamma(-n+\varepsilon) = \frac{\Gamma(n+1+\varepsilon)}{(n+\varepsilon)(n+\varepsilon-1)\dots(\varepsilon+1)\,\varepsilon\,(\varepsilon-1)\dots(\varepsilon-n)} \tag{8D.18}$$

$$= \frac{\Gamma(n+1+\varepsilon)}{(-1)^n\varepsilon\,(1+\varepsilon)(1-\varepsilon)(2+\varepsilon)(2-\varepsilon)\dots(n+\varepsilon)(n-\varepsilon)} \tag{8D.19}$$

$$= \frac{(-1)^n\,\Gamma(n+1+\varepsilon)}{n!^2\,\varepsilon\,(1-\varepsilon^2)\left[1-\left(\frac{\varepsilon}{2}\right)^2\right]\left[1-\left(\frac{\varepsilon}{3}\right)^2\right]\dots\left[1-\left(\frac{\varepsilon}{n}\right)^2\right]} \tag{8D.20}$$

$$= \frac{(-1)^n}{n!^2\,\varepsilon}\,\Gamma(n+1+\varepsilon)\left[1+\varepsilon^2\sum_{j=1}^{n}\frac{1}{j^2}+\mathcal{O}(\varepsilon^4)\right]. \tag{8D.21}$$

Together with Eq. (8D.13), this becomes

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!^2\,\varepsilon}\Gamma(n+1+\varepsilon)\left\{1+\varepsilon^2\left[\frac{\pi^2}{6}-\psi'(n+1)\right]+\mathcal{O}(\varepsilon^4)\right\}. \tag{8D.22}$$

We further need the Taylor expansion of $\Gamma(n+1+\varepsilon)$:

$$\Gamma(n+1+\varepsilon) = \Gamma(n+1)\left[1+\varepsilon\,\psi(n+1)+\frac{1}{2}\frac{\Gamma''(n+1)}{\Gamma(n+1)}\varepsilon^2+\mathcal{O}(\varepsilon^3)\right]$$

$$= n!\left\{1+\varepsilon\,\psi(n+1)+\frac{\varepsilon^2}{2}\left[\psi'(n+1)+\psi(n+1)^2\right]+\mathcal{O}(\varepsilon^3)\right\}. \tag{8D.23}$$

Using (8D.21) and (8D.23), the expansion for $\Gamma(-n+\varepsilon)$ takes the form:

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left\{\frac{1}{\varepsilon}+\psi(n+1)+\frac{\varepsilon}{2}\left[\frac{\pi^2}{3}+\psi(n+1)^2-\psi'(n+1)\right]+\mathcal{O}(\varepsilon^2)\right\}. \tag{8D.24}$$

Notes and References

For the dimensional regularization see also
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