只有连通图有贡献证明

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1 Proof

与随机过程中的情况相类比
moment expansion:

$$\left\langle e^{ikx}\right\rangle = \sum_{n=0}^{+\infty} \frac{(i'k)^n}{n!} \left\langle x^n \right\rangle \tag{1}$$

cumulant expansion:

$$\left\langle e^{ikx}\right\rangle = \exp\left[\frac{(ik)^l}{l!}\mu_l\right] \tag{2}$$

力学量平均值可表达为

$$\frac{\left\langle \hat{O} e^{-u}\right\rangle}{\left\langle e^{-u}\right\rangle} \tag{3}$$

其中, 分母部分

$$\begin{aligned} \left\langle e^{-u}\right\rangle_{\hat{O}} &= \exp\left[\sum_{l=0}^{+\infty} \frac{(-1)^l}{l!}\mu_l\right] \\ &= \exp\left[\sum_{l=0}^{+\infty} \frac{(-1)^l}{l!}\left\langle u^l\right\rangle_{\hat{O}}^C\right] \end{aligned} \tag{4}$$

即

$$\frac{1}{\left\langle e^{-u}\right\rangle} = \exp\left[-\sum_{l=0}^{+\infty} \frac{(-1)^l}{l!}\left\langle u\right\rangle_{\hat{O}}^C\right] \tag{5}$$

而分子部分

$$\left\langle \hat{O} e^{-u}\right\rangle = \sum_{n=0}^{+\infty} \left\langle \hat{O} u^n\right\rangle \frac{(-1)^n}{n!} \tag{6}$$

对于其中第 n 项, $\left\langle \hat{O} u^n\right\rangle$ 可分解为

$$\left\langle \hat{O} u^n\right\rangle \propto \left\langle \hat{O} u^l\right\rangle_{\hat{O}}^C \left\langle u^1\right\rangle_{\hat{O}}^{n_1} \left\langle u^2\right\rangle_{\hat{O}}^{n_2} \dots \tag{7}$$

$$l + n_1 + 2n_2 + 3n_3 + \dots = n \tag{8}$$

下面计算权重因子
$2n_2$ 个形成 $\left\langle u^2\right\rangle_0^{n_2}$ 配对, $\frac{C_{2n_2}^2 C_{2n_2-2}^2 \cdots}{n_2!} = \frac{(2n_2)!}{(2!)^{n_2} n_2!}$
$3n_3$ 个形成 $\left\langle u^3\right\rangle_0^{n_3}$ 配对, $\frac{C_{3n_3}^3 C_{3n_3-3}^3 \cdots}{n_3!} = \frac{(3n_3)!}{(3!)^{n_3} n_3!}$
以此类推

$$\begin{aligned} & C_n^l C_{n-l}^{n_1} C_{n-l-n_1}^{2n_2} \frac{1}{(2!)^{n_2} n_2!} \cdots \\ =& \frac{(-1)^{l+n_1+2n_2+\dots}}{n!} \frac{n!}{l!(n-l)!} \frac{(n-l)!}{(2n_2)!(n-l-2n_2)!} \cdots \\ =& \frac{(-1)^l}{l!} \frac{(-1)^{n_1}}{n_1!} \frac{\left[(-1)^2\right]^{l_2}}{(2!)^{n_2} n_2!} \cdots \end{aligned} \tag{9}$$

将会与分母部分抵消.
故
$$\frac{\left\langle \hat{O}e^{-u} \right\rangle}{\langle e^{-u} \rangle} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \left\langle \hat{O}u^n \right\rangle_{\hat{O}}^C \tag{10}$$