Bosonization

3-Dimensional Fermi liquid theory is mostly related to a picture of quasi-particles when we adiabatically switch on interactions, to obtain particle-hole excitations. These quasi-particles are directly related to the original fermions. Of course they also obey the Fermi-Dirac Statis- tics. Based on the free Fermi gas picture, the interaction term: (i) it renormalizes the free Hamiltonians of the quasi-particles such as the effective mass, and the thermodynamic properties; (ii) it introduces new collective modes. The existence of quasi-particles results in a fi- nite jump of the momentum distribution function n(k) at the Fermi surface, corresponding to a finite residue of the quasi-particle pole in the electrons Green function. 1-Dimensional Fermi liquids are very special because they keep a Fermi surface (by definition of the points where the momentum distribution or its derivatives have singularities) enclosing the same k-space vol- ume as that of free fermions, in agreement with Luttingers theorem.

1-Dimensional electrons spontaneously open a gap at the Fermi surface when they are coupled adiabatically to phonons with wave vector 2kF . The mean-field theory tells us that there is a charge or spin density wave instability at some finite temperature for repulsive interactions implying that there can be no Fermi liquid in 1-Dimension. There are no fermionic quasi-particles, and their elementary excitations are rather bosonic collective charge and spin fluctuations dispersing with different velocities. An incoming electron decays into such charge and spin excitations which then spatially separate with time (charge-spin separation). The correlations between these excitations are anoma- lous and show up as interaction-dependent nonuniversal power-laws in many physical quantities where those of ordinary metals are char- acterized by universal (interaction independent) powers.

1 Luttinger Liquid

Before looking at the Luttinger Liquid, we want to have a look at 1-D free Fermion system first. The lattice Hamiltonian for a non- interacting hopping Fermion system is written as:

$$H_f = -t \sum_{n\sigma} (\psi^+_{n\sigma}\psi_{n+1\sigma} + h.c.) + \mu \sum_{n\sigma} \psi^+_{n\sigma}\psi_{n\sigma} \tag{1}$$

where t is the hopping constant between nearest lattice sites, and μ is the chemical potential, and σ is the spin degree of freedom. Using the Fourier Transformation into the momentum space, we can find the eigenvalue for this Hamiltonian, is

$$\epsilon_k = -2t\cos ka + \mu \tag{2}$$

where $\mu = 0$ for $k_F = \pi/2a$, or, we can simply absorb the chemical potential μ when expanding around the chemical potential.

The Luttinger Liquid can be derived from the 1-D Hubbard model when the above free Fermion system includes an interaction term,

$$H = H_f + H_U = H_f + U \sum_n \rho_{n\uparrow}\rho_{n\downarrow} \tag{3}$$

for temperature much lower than the Fermi energy, we want to expand the energy around the Fermi points. By writing the site amplitudes for right and left moving components, where $k_F = \pi/2a$

$$\psi_{n\sigma} = e^{ik_F na}\psi_{n\sigma_+} + e^{-ik_F na}\psi_{n\sigma_-} = R_\sigma(n) + L_\sigma(n) \tag{4}$$

the assumption here for left and right moving Fermions is, they are slow varing in space, and have mean momemtum value of $\pm k_F$. Therefore, we can expand the Fermion field for position $(n+1)a$ at the center of the position na:

$$\psi_{n+1\sigma_\pm} = \psi_{n\sigma_\pm} + a\partial_x \psi_{n\sigma_\pm} + \cdots \tag{5}$$

plug this term in back into Eq. (4), retaining only the first order of a. Since

$$\begin{aligned} exp(ik_F a) + exp(-ik_F a) &= 0 \\ exp(ik_F a) - exp(-ik_F a) &= 2i \\ \sum_n exp(ik_F a(2n+1)) &= 0 \end{aligned} \tag{6}$$

the third equation vanishes because only $x = 2n\pi$ survives in the summation of $\sum_m exp(imx)$. Taking the notation that $a\sum_n \to \int_{-L/2}^{L/2} dx$, $\psi_{n\sigma\pm} = \sqrt{a}\psi_{\sigma_\pm}(x)$, now the 1-D free Fermi Hamiltonian is simplified into

$$H_F = -\hbar v_F \sum_\sigma \int_{-L/2}^{L/2} [\Psi^\dagger_{\sigma^+}(x)(i\partial_x)\Psi_{\sigma^+}(x)dx + \Psi^\dagger_{\sigma^-}(x)(-i\partial_x)\Psi_{\sigma^-}(x)dx], \tag{7}$$

where we define $\hbar v_F = 2at$ here.

The resultant dispersion relationship is explicitly of the form $E_\pm(k) = \pm\hbar v_F k$, where the energy of left and right movers is distinct. Inclusion of negative-energy states is a key difference between the Tomonaga and Luttinger models.

The negative-energy states introduce, however, a subtle complexity into the Luttinger model. In the ground state, the negative-momentum branch of the right-moving states and the positive-momentum branch of the left-moving states are occupied. This is illustrated in Fig.(b) and can be thought of as the filling of the negative-energy Dirac sea. The analogy here is between the positron

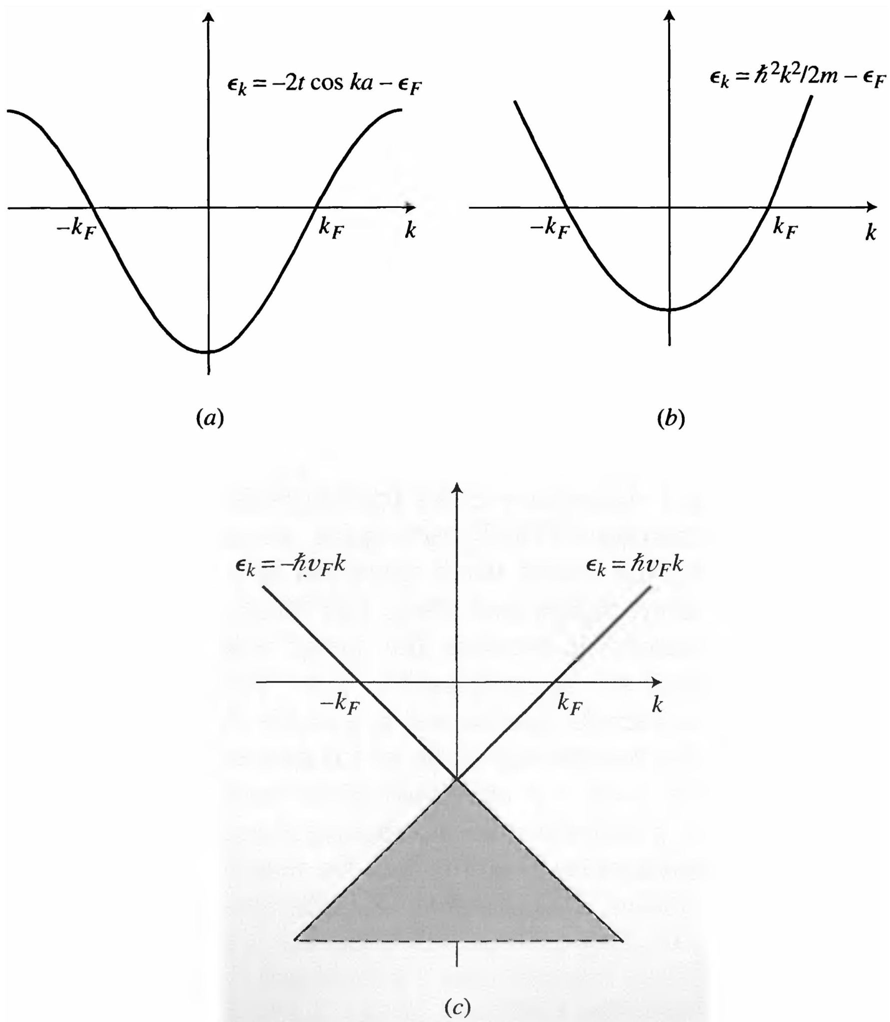

Figure 1: (a) Energy bands in the lattice, and (b) free-space models for noninteracting electrons in 1d. (c) Linearized energy band in the vicinity of the Fermi level. The shaded region represents the filling of the negative energy states, the Dirac sea.

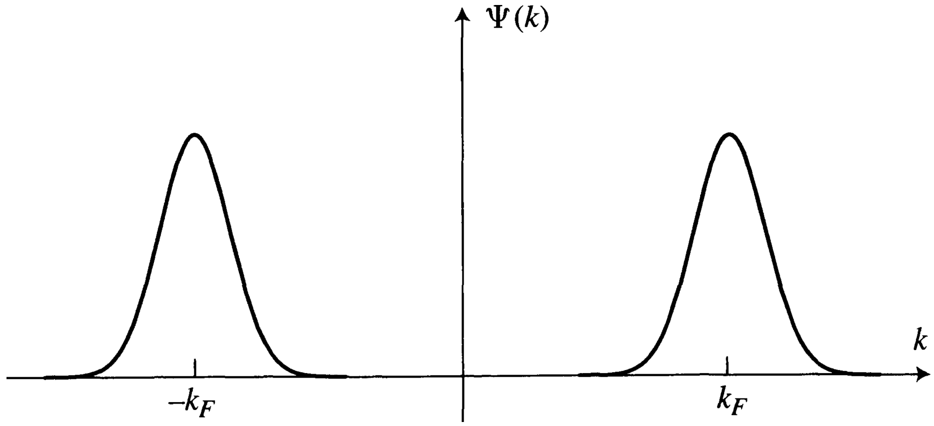

Figure 2: Fourier components of the fermion fields. In the bosonization procedure, we retain only the momentum components of the fermion fields at $\pm k_F$.

states in Dirac theory and the negative-energy single-particle states in the Luttinger model. However, the Hamiltonian, Eq.(7), is expressed in terms of creation operators for right-and left-moving fields, $\Psi_{\pm}^{\dagger}$, that act on the space a ground state energy. Luttinger alleviated this problem by performing a canonical transformation of the form

$$\Psi_{\sigma^{+}}(k) \rightarrow \begin{cases} b_{k\sigma} & k \geq 0 \\ c_{k\sigma}^{\dagger} & k<0 \end{cases} \tag{8}$$

$$\Psi_{\sigma^{-}}(k) \rightarrow \begin{cases} b_{k\sigma} & k<0 \\ c_{k\sigma}^{\dagger} & k \geq 0 \end{cases} \tag{9}$$

to the left-and right-moving fields, which effectively filled the negative-energy Dirac sea. The $b_{k\sigma}$'s and $c_{k\sigma}$'s obey the usual anti commutation relations. A consequence of this transformation is that the elementary excitations in momentum state k are pair excitations consisting of a particle and a hole. Further, it is linear combinations of these that now describe the collective or plasma excitations of the electron gas. Lutttinger, however, was not aware that this canonical transformation affected also the value of commutators involving the electron density. This is the profound change that the canonical transformation introduces into Luttinger model. Mattis and Lieb were the first to point this out and correctly solve the Luttinger model.

To illustrate the changes this switch of basis introduces, we rewrite the Hamiltonian in terms of the new particle-hole operators. The transformed free Hamiltonian

$$H_F = \hbar v_F \int_{-\infty}^{\infty} |p|\, dp[b_p^{\dagger} b_p + c_p^{\dagger} c_p] + W \tag{10}$$

contains the constant

$$W = -\hbar v_F \int_0^{\infty} p dp + \hbar v_F \int_{-\infty}^{0} p dp, \tag{11}$$

which represents the infinite energy of the filled Dirac sea. This result could have been obtained by normal ordering the operators in the original Hamiltonian. Consider an arbitrary reference state,$|\Omega\rangle$, and an operator, $\widehat{A}$, which can be written as a product of creation and annihilation operators. Normal ordering th creation and annihilation operators in $\widehat{A}$ means moving all the creation operators to the left and all the annihilation operators to the right. As a consequence, infinities arising from commutators are removed. We denote normal ordering by the symbol: $\widehat{A}$:. Consequently,: $\widehat{A}$: $|\Omega\rangle$ =0 and any average of: $\widehat{A}$:with respect to the ground state,$|\Omega\rangle$, identically vanishes, $\left\langle : \widehat{A} : \middle| \Omega \middle| : \widehat{A} : \right\rangle$ = 0. Hence, the transformation in Eq.(8) is equivalent to normal ordering the left-right moving-field operators with respect to the filled Dirac sea. In so doing, one can extract the infinite energy of the filled negative-energy states.

Let us look also at the momentum-space representation of the density of right and left movers:

$$\begin{aligned} n_{\sigma^{\pm}(x)} &= \Psi^{\dagger}_{\sigma^{\pm}}(x)\Psi_{\sigma^{\pm}}(x) \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{dq}{2\pi}\Psi^{\dagger}_{\sigma^{\pm}}(p)\Psi_{\sigma^{\pm}}(q)e^{i(q-p)x} \end{aligned} \tag{12}$$

with

$$n_{\sigma^{\pm}}(q) = \int_{-\infty}^{\infty} dx e^{iqx} n_{\sigma^{\pm}}(x). \tag{13}$$

Substituting the transformed form for the left and right movers into Eq.(12) and taking the Fourier transform (according to Eq. 13), we find that the form of $n_{\sigma^{\pm}}(k)$,

$n_{\sigma^{+}}(k \geq 0) = \int_0^{\infty} dq[b^{\dagger}_q b_{k+q} + c_{-k-q}c^{\dagger}_{-q} + c_{-q}b_{k-q}\theta(k-q)]$

$n_{\sigma^{+}}(k \leq 0) = \int_0^{\infty} dq[b^{\dagger}_{k-q} b_{-q} + c_{-q}c^{\dagger}_{k-q} + b^{\dagger}_q c^{\dagger}_{k+q}\theta(-k-q)]$

$n_{\sigma^{-}}(k \geq 0) = \int_0^{\infty} dq[b^{\dagger}_{-k-q} b_{-q} + c_q c^{\dagger}_{k+q} + b^{\dagger}_{-q} c^{\dagger}_{k-q}\theta(k-q)]$

$n_{\sigma^{-}}(k \leq 0) = \int_0^{\infty} dq[b^{\dagger}_{-q} b_{k-q} + c_{q-k}c^{\dagger}_{q} + c_q b_{k+q}\theta(-k-q)]$

depends on the sign of the momentum. The form derived here for the momentum components of the density is equivalent to the form given by Mattis and Liebb. The difference arises in the definition of the Fourier transform of the density defined in Eq. (13). The dependence of the density on the sign of the momentum arises from the pair nature of the fundamental excitations.

It is also from this complexity that differences arise when certain commutators are calculated in the original left-right and particle-hole bases. Consider the current operators for the total electron density

$$j_0^\sigma = \Psi^\dagger_{\sigma^+}(x)\Psi_{\sigma^+}(x) + \Psi^\dagger_{\sigma^-}(x)\Psi_{\sigma^-}(x) \tag{14}$$

and the difference

$$j_1^\sigma = \hbar v_F(\Psi^\dagger_{\sigma^+}(x)\Psi_{\sigma^+}(x) - \Psi^\dagger_{\sigma^-}(x)\Psi_{\sigma^-}(x)) \tag{15}$$

of the electron densities. In the original left-right basis, commutators involving j_0^σ and j_1^σ identically vanish. However, if the new particle-hole basis is used, Eq.(8), we find that the equal-time commutation relations between the components of the current,

$$[j_0^\sigma(x), j_1^\sigma(y)] = -\frac{i\hbar v_F}{\pi}\delta_x\delta(x-y) \tag{16}$$

$$[j_0^\sigma(x), j_0^\sigma(y)] = [j_1^\sigma(x), j_1^\sigma(y)] = 0, \tag{17}$$

obey Bose statistics. The right-hand side of the first commutator is a c-number. This discrepancy is a bit surprising but not totally unexpected if we write the density in first-quantized form,

$$n_{\sigma^\pm}(k) = \int_{-L/2}^{L/2} dx e^{-ikx}. \tag{18}$$

These components commute trivially. However, this is not the physically relevant form for the density operators when the Dirac sea is filled. Schwinger anticipated this apparent paradox that arises when the very existence of a ground state forces certain commutators to be nonzero, which would vanish trivially otherwise in the first-quantized language. The full consequences for the Luttinger model were worked out by Mattis and Lieb.

We turn now to be construction of an equivalent boson theory of the free part of the Luttinger model. To this end, we focus on the Heisenberg equations of motion,

$$-i\partial_t j_0^\sigma(x) = [H_F, j_0^\sigma(x)]. \tag{19}$$

It turns out that the value of the commutator in Eq.(19) is independent of the filling in the Dirac sea. That is, this commutator, unlike those involving the components of the density, is invariant to normal ordering. The commutators of the densities for left and right movers with the free Hamiltonian,

$$\begin{aligned}[H_F, n_{\sigma^+}(\pm k)] &= \mp k n_{\sigma^+}(\pm k) \\ [H_F, n_{\sigma^-}(\pm k)] &= \pm k n_{\sigma^-}(\pm k)\end{aligned} \tag{20}$$

depends on the sign of the momentum. Note the sign change relative to that of Mattis and Lieb. Here again, this difference arises from our definition of the Fourier transform, Eq.(13). Consequently,

$$\begin{aligned}\left[H_F, \int_{-\infty}^{\infty} dq e^{iqx} j_0^\sigma(q)\right] &= \hbar v_F \int_0^\infty dq e^{iqx} q(n_\sigma - (q) - n_{\sigma^+}(q)) \\ &+ \hbar v_F \int_0^\infty dq e^{-iqx} q(n_{\sigma^+}(-q) - n_{\sigma^-}(-q)) \\ &= \hbar v_F \int_{-\infty}^{\infty} dq e^{iqx} q(n_{\sigma^-}(q) - n_{\sigma^+}(q)) \\ &= i\partial_x j_1^\sigma,\end{aligned} \tag{21}$$

or equivalently,

$$\partial_t j_0^\sigma = -\partial_x j_1^\sigma. \tag{22}$$

If we introduce the notation, $t \to c_0$ and $x \to x_1$ with $x \equiv (xx_0, x_1)$, we can rewrite the Heisenberg equation of motion as a conservation equation:

$$\partial_\mu j_\mu^\sigma = 0. \tag{23}$$

Hence, the total fermion current, normal ordered with respect to the full Dirac sea, is conserved.

Conservation of the fermion current suggests that if we define a boson field $\Phi_\sigma(x)$ with conjugate momentum $\Pi_\sigma(x)$ such that

$$j_0^\sigma(x) = \frac{1}{\sqrt{\pi}} \partial_x \Phi_\sigma(x) \tag{24}$$

and

$$j_1^\sigma(x) = -\frac{1}{\sqrt{\pi}}\partial_t\Phi_\sigma(x) = -\frac{1}{\sqrt{\pi}}\Pi_\sigma(x), \tag{25}$$

The Heinsenberg equation of motion,

$$\partial_t[\partial_x\Phi_\sigma(x)] = \partial_x\Pi_\sigma(x), \tag{26}$$

will resemble an equation of motion for two conjuagate fields. To satisfy the Bose statistics of the currents,

$$[\Phi_\sigma(x), \Pi_{\sigma'}(y)] = i\delta_{\sigma\sigma'}\delta(x-y). \tag{27}$$

In the Bose basis, the free Hamiltonian density,

$$H_F = \frac{\hbar v_F}{2}\sum_\sigma \int dx[\Pi_\sigma^2(x) + (\partial_x\Phi_\sigma(x))^2], \tag{28}$$

yields an equation of motion that is equivalent to the Heisenberg evolution equation for the densities in the fermion basis. Hence, we have reformulated our fermion theory in terms of an equivalent boson theory. We emphasize that the boson theory has been constructed by an analogy based on the equations of motion. Hence, Eq.(28) is not a unique choice for the equivalent boson Hamiltonian density. By the construction of an equivalent boson theory, we imply no more than an equivalence between the equations of motion in the two accounts.

We can take the bosonization procedure a step further and construct explicitly the mapping between the fermion fields $\Psi_{\sigma^\pm}(x)$ and the new boson field, $\Phi_\sigma(x)$. Such a mapping is problematic because the $\Psi_{\sigma^\pm}(x)$'s are not the operators associated with the physical states of the electron gas. Recall that the $\psi_{\sigma^\pm}$'s describe particle-hole excitations. Hence, processes in which the electron number is changed are forbidden, For the moment, we set this problem aside and develop a mapping involving the boson fields that preserves the anti-commutation relations of the fermion fields. The key idea in bosonization is to associate (not equate) the left-and right-moving fermion fields with boson fields of the form,

$$\begin{aligned}\Psi_{\sigma^\pm}(x) &= \frac{1}{\sqrt{2\pi a}}e^{-\imath\sqrt{\pi}[\int_{-\infty}^{x}\Pi_\sigma(x')dx' \mp \Phi_\sigma(x)]} \\ &= \frac{1}{\sqrt{2\pi a}}e^{\pm\imath\sqrt{\pi}\Phi_{\sigma^\pm}(x)}\end{aligned} \tag{29}$$

where

$$\Phi_{\sigma^{\pm}}(x) = \Phi_{\sigma}(x) \mp \int_{-\infty}^{x} \Pi_{\sigma}(x')dx'. \tag{30}$$

It is crucial that the operator equivalence, Eq.(29), be understood strictly in the normal-ordered sense. For example, if Eq.(29) were naively substituted into the continuum version of the Hamiltonian, Eq.(7), the resultant boson Hamiltonian would correspond to twice Eq.(28). Such an equation would yield the incorrect equations of motion. This should underscore our disclaimer that the fermion fields should be associated, not equated, with the boson field, Eq.(29). True equalities arise in the two theories when average values of physical observables are computed. The physical motivation for Eq.(29) is as follows. The amplitude that a particle is at x is $\Psi_{\sigma^{\pm}}(x)$. Classically, the evolution of a particle from x to $x+a$ is brought about by the displacement operator,$e^{a\partial_x}$. Quantum mechanically, this quantity becomes the exponential of the momentum operator. Hence, $\Psi_{\sigma^{\pm}}(x)$ must be proportional to the exponential of the momentum operator. We call this quantity $\Pi_{\sigma}(x)$. However, if this is the only dependence, then $\Psi_{\sigma^{\pm}}(x)$ would commute with itself. The simplest form that ensures the fermion anti-commutation relations is Eq.(29).

To solve the problem that the bosonized form of the fermion fields cannot be used, in their current form, to connect electronic states that differ in particle number, we introduce ladder operators, which change the particle number by integer values. Such ladder operators (traditionally known as Klein *factors*) must lie outside the space of the bosonic operators because bosons are necessarily composites of even numbers of fermions. No combination of bosonic operators can ever create a single electron. Let N_{σ} represent the deviation of the electron occupation number from the ground state value. Following the notation of Kotliar and Si, we introduce the operator $F_{\sigma}(F_{\sigma}^{\dagger})$, which lowers (raises) N_{σ} by one. The F_{σ}'s commute with all the boson fields in $\psi_{\sigma^{\pm}}$ for q $\neq 0$ modes are particle-hole excitations. The commutation relations of the F_{σ}'s are

$$\begin{aligned} F_{\sigma}^{\dagger}F_{\sigma} &= F_{\sigma}F_{\sigma}^{\dagger} = 1 \\ F_{\sigma}^{\dagger}F_{\sigma'} &= -F_{\sigma'}F_{\sigma}^{\dagger} \\ F_{\sigma}F_{\sigma'} &= -F_{\sigma'}F_{\sigma} \end{aligned} \tag{31}$$

The physical states of the electron are now captured by

$$\widetilde{\psi}_{\sigma^{\pm}}^{\dagger}(x) = F_{\sigma}^{\dagger} e^{\frac{2\pi\imath x N_{\sigma}}{L}} \psi_{\sigma^{\pm}}^{\dagger}. \tag{32}$$

For processes that do not conserve particle number, Klein factors must be included. However, as the problems on which we focus conserve particle number, we will omit the Klein factors, as they cannot change the physics. Nonetheless, they are part of the complete story of bosonization.

We now turn to the bosonization of the interaction terms. In the continuum limit, the Hubbard interaction can be written as

$$H_U = aU\int_{-L/2}^{L/2} dx(R^\dagger_\uparrow(x) + L^\dagger_\uparrow(x))(R_\uparrow(x) + L_\uparrow(x)) \times (R^\dagger_\downarrow(x) + L^\dagger_\downarrow(x))(R_\downarrow(x) + L_\downarrow(x)), \tag{33}$$

where $R_\sigma(x)(L_\sigma(x)) = \Psi_{n\sigma^\pm} e^{\pm\imath k_F x}/\sqrt{a}$. To implement the bosonization scheme, we must normal order the operators in the Hubbard interaction. As in the free system, normal ordering of the operators in the Hubbard interaction,

$$H_U =: H_U : +Q, \tag{34}$$

will generate a constant term (Q) that is again infinite. In this case, the infinite term is associated with the infinite charge in the negative energy states. The important point here is that the infinite term is constant. Hence, it can be ignored.
If we now expand Eq.(33), keeping only the non-oscillatory terms, we find that the Hubbard interaction reduces to

$$H_U \to H_{int} = aU\int_{-L/2}^{L/2} dx[(: \Psi^\dagger_{\downarrow^+}\Psi_{\uparrow^+} : + : \Psi^\dagger_{\uparrow^-}\Psi_{\uparrow^-} :)\times(: \Psi^\dagger_{\downarrow^+}\Psi_{\downarrow^+} : + : \Psi^\dagger_{\downarrow^-}\Psi_{\downarrow^-} :) + aU[: \Psi^\dagger_{\uparrow^+}\Psi_{\uparrow^-} :: \Psi^\dagger_{\downarrow^-}\Psi_{\downarrow^+} : +h.c.]]. \tag{35}$$

Diagrams illustrating the two types of scattering processes retained in H_{int} are shown in Fig.(9.3). In Fig. 9.3a, the net momentum transfer across each vertex is zero. Hence, the first term in Eq.(35) corresponds to forward scattering of the electrons. In the second term (see Fig. 9.3b), particles are converted from left(right) into right(left) movers. Hence, this term corresponds to a back scattering process. We have dropped the terms which have a pre-factor of $e^{\pm\imath 4k_F}$. At half-filling, $k_F = \pi/2$, and the exponential pre-factor reduces to unity. The $4k_F$ term is referred to as an *Umklapp process* in which two right(left) movers are destroyed and two left(right) movers are created. We will come back to these terms later.

To complete the bosonization scheme, we need a rule for writing products of fermion operators in Eq.(35) in terms of the Bose fields. To proceed, we use the Baker-Hausdorff identity,

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}. \tag{36}$$

Using the commutation relation, Eq.(27), we reduce the operator product,

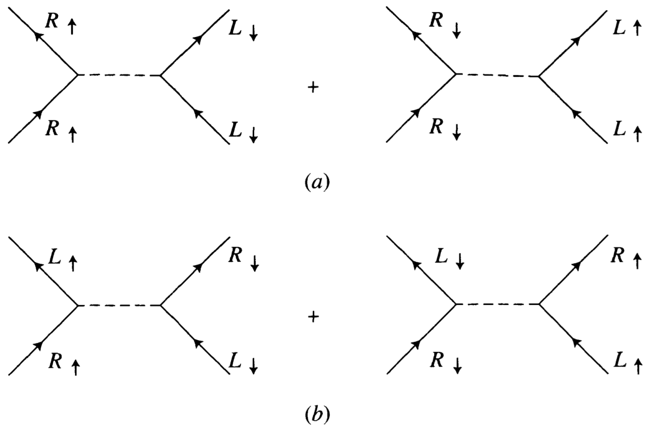

Figure 3: (a) Forward-scattering process corresponding to the firlst terms in equation (35). $R(L)$ represent right- and left-moving fields. At each vertex of a forward-scattering term, the momentum change is zero. The wavy line indicates the Coulomb interaction aU. (b) Backscattering corresponds to the second term in eq(35). The magnitude of the momentum change at each vertex is $|2k_F|$.

$$\begin{aligned}\Psi^{\dagger}_{\sigma^+}\Psi_{\sigma^-} &= \frac{1}{2\pi a}e^{-i\sqrt{4\pi}\phi_\sigma(x)}e^{-\pi i/2} \\ &= -i\frac{1}{2\pi a}e^{-i\sqrt{4\pi}\phi_\sigma(x)},\end{aligned} \tag{37}$$

in the back scattering terms to a simple exponential of the Bose field. The factor of $-i$ arises from the commutator,

$$\begin{aligned}[\Phi^+_\sigma(x), \Phi^-_\sigma(y)] &= \left[\Phi_\sigma(x), \int_{-\infty}^{y}\Pi(x')dx'\right] + \left[\Phi_\sigma(y), \int_{-\infty}^{x}\Pi_\sigma(x')dx'\right] \\ &= i\int_{-\infty}^{y}dx'\delta(x-x') + i\int_{-\infty}^{x}dx'\delta(y-x') \\ &= i.\end{aligned} \tag{38}$$

Further simplification requires additional identities. The fermion field $\psi_{\sigma^\pm}$ is an exponential of two fields that obey Bose statistics. In analogy with the harmonic oscillator, we partition the exponential into a sum of creation and annihilation operators parts. Consequently, the $\psi_{\sigma^\pm}$ fields are of the form

$$\psi_{\sigma^\pm} = e^A = e^{A^+ + A^-} \tag{39}$$

where $A^+(A^-)$ represents the creation(annihilation) operator part of A. Using the Baker-Hausdorff operator identity, Eq.(36), we find that

$$e^A = e^{A^+} e^{A^-} e^{-\frac{1}{2}[A^+,A^-]}. \tag{40}$$

Because the normal ordering places all the creation operators to the left and the annihilation operators to the right,

$$: e^A :\equiv e^{A^+} e^{A^-} = e^{\frac{1}{2}[A^+,A^-]} e^A \tag{41}$$

is the corresponding normal-ordered form of the $\psi_{\sigma^\pm}$ fields. This expression is of the utmost utility in the bosonization procedure. For example, it is common in the literature to write the left-moving and right-moving fermion fields in the normal-ordered form. The normal-ordered form differs from Eq.(29) by the phase factor in Eq.(41). In the context of the back-scattering terms, we can use Eq.(41) to evaluate a product of two normal-ordered operators. Upon using Eq.(41) twice, we find that

$$: e^A :: e^B := e^{[A^-,B^+]} : e^{A+B} :, \tag{42}$$

where $[A^\pm, B^\pm] = 0$. In general $[A^\pm, B^\pm]$ is a c-number. For the case that $\Phi_{\sigma^+} = e^A$ amd $\Phi_{\sigma^-} = e^B$, the commutator in the exponential factor of Eq.(42) has been evaluated by Mattis [M1974] and Mandelstam. Its value is unity. As a consequence,

$$: \Psi^\dagger_{\uparrow^+} \Psi^\dagger_{\downarrow^-} :: \Psi^\dagger_{\downarrow^-} \Psi_{\downarrow^+} : + h.c. = \frac{1}{2\pi^2 a^2} : \cos\Big(\sqrt{4\pi} (\Phi_\uparrow(x) - \Phi_\downarrow(x)) \Big) : . \tag{43}$$

The forward-scattering terms are slightly more difficult because operator products of the form $\Psi^\dagger_{\uparrow^+}(x+\epsilon)\Psi_{\uparrow^+}(x-\epsilon)$ are not well defined in the limit that $\epsilon \to 0$. Naively, one might expect this quantity to equal unity. This is not so. We can evaluate this product by considering the fermion correlator,

$$\begin{aligned} \langle \Psi^\dagger_{\sigma^+}(x) \Psi_{\sigma^+}(0) \rangle &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{-\imath q x} \langle \Psi^\dagger_{\sigma^+}(q) \Psi_{\sigma^+}(k) \rangle \\ &= \int_0^\infty \frac{dk}{2\pi} e^{\imath k x}. \end{aligned} \tag{44}$$

The double integrals contract to $2\pi\delta(k-q)$, with $k \leq 0$. The later constraint arises because right movers exist only in the negative-momentum states in the ground state. Eq. (44) is not well behaved for large momentum, but it can be regularized by introducing the exponential factor,e^{-ak}, where the lattice constant acts as a short-distance cutoff. We find then that

$$\langle \Psi^\dagger_{\sigma^+}(x)\Psi_{\sigma^+}(0)\rangle = \frac{1}{2\pi}\frac{1}{a-ix}. \tag{45}$$

For left movers, the corresponding correlator is the complex conjugate of Eq.(45).

If use the bosonized form of these field operators to evaluate the density correlator, we must obtain the same result. To proceed, we simplify a product of the form $e^A e^B$ using Eqs.(36) and (42),

$$e^A e^B =: e^{A+B} : e^{[A^-,B^+]+\frac{[A^-,A^+]+[B^-,B^+]}{2}} \tag{46}$$

In general, the commutators appearing in Eq.(46) are c-numbers. Further, because $\langle (A^\pm)^2\rangle = \langle A^+A^-\rangle = 0$, it follows immediately that $\langle [A^-, A^+]\rangle = \langle a^2\rangle$. Consequently,

$$\langle e^A e^B\rangle =: e^{A+B} : e^{\langle AB+\frac{A^2+B^2}{2}}. \tag{47}$$

The average value of the exponential of any normal-ordered operator is unity, $\langle : e^A :\rangle = 1$. Hence, we obtain the all-important rule for evaluating a correlation function,

$$\langle e^A e^B\rangle = e^{\langle AB+\frac{A^2+B^2}{2}\rangle}, \tag{48}$$

from which it follows that

$$\langle \Psi^\dagger_{\sigma^+}(x)\Psi_{\sigma^+}(0)\rangle = \frac{1}{2\pi a}e^{\pi\langle \Phi_{\sigma^+}(x)\Phi_{\sigma^+}(0)-\Phi^2_{\sigma^+}(0)\rangle} \tag{49}$$

By analogy with Eq.(9.44), we find that

$$\begin{aligned}\langle \Psi^\dagger_{\sigma^+}(x)\Psi_{\sigma^+}(0)\rangle &= \frac{1}{2\pi a}e^{\pi G_+(x)} \\ &= \frac{1}{2\pi a}\frac{a}{a-ix}\end{aligned} \tag{50}$$

or, equivalently,

$$\langle e^{\imath\eta\Phi_{\sigma^\pm}} e^{\imath\eta\Phi_{\sigma^\pm}}\rangle = \left(\frac{a}{a\mp ix}\right)^{\eta^2/\pi} \equiv e^{\eta^2 G_\pm}. \tag{51}$$

We now have the tools to evaluate the original fermion product:

$$
\begin{aligned}
:\Phi^{\dagger}_{\sigma^{\pm}}(x)\Phi_{\sigma^{\pm}}(x): &= \lim_{\epsilon\to 0}\frac{1}{2\pi a} : e^{\mp \imath\sqrt{\pi}\Phi_{\sigma^{\pm}}(x+\epsilon)\pm\Phi_{\sigma^{\pm}}(x-\epsilon)}: e^{\pi G_{\pm}} \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi a} : 1 \mp \sqrt{\pi}\partial_x\Phi_{\sigma^{\pm}}\epsilon + \cdots : \left(\pm\frac{ia}{\epsilon \pm ia}\right) \\
&= \lim_{\epsilon\to 0}\frac{\pm i}{2\pi(\epsilon + ia)} + : \frac{1}{\sqrt{\pi}}\partial_x\Phi_{\sigma^{\pm}} + O(\epsilon) : .
\end{aligned}
\tag{52}
$$

For arbitrarily small a, the first term in this expression is infinite, a reflection of the infinite number of right and left movers in the negative energy states. Normal ordering the operators in the charge density results in the removal of the divergent charge density in the ground state. As a consequence, the normal-ordered charge density,

$$
:\Psi^{\dagger}_{\sigma^{\pm}}(x)\Psi_{\sigma^{\pm}}(x) := \frac{1}{\sqrt{\pi}}\partial_x\Phi_{\sigma^{\pm}}(x), \tag{53}
$$

has a well-defined interpretation in the boson basis. Noting that

$$
:\Psi^{\dagger}_{\sigma^{+}}(x)\Psi_{\sigma^{+}}(x): + :\Psi^{\dagger}_{\sigma^{-}}\Psi_{\sigma^{-}}(x) := \frac{1}{\sqrt{\pi}} : \partial_x\Phi_{\sigma}(x) : \tag{54}
$$

and using Eq. (43), we simplify the interaction terms to

$$
H_{int} = aU\int dx \sum_{\sigma}\left[\frac{:\partial_x\Phi_{\uparrow}\partial_x\Phi_{\downarrow}:}{\pi} + \frac{1}{2\pi^2 a^2} : \cos\Big(\sqrt{4\pi}(\Phi_{upward}(x) - \Phi_{\downarrow}(x))\Big) : \right]. \tag{55}
$$

The utility of this expression and the bosonization procedure is made clear by introducing the charge,Φ_c, and spin, Φ_s, boson fields,

$$
\Phi_c = \frac{\Phi_{\uparrow} + \Phi_{\downarrow}}{\sqrt{2}}, \Phi_s = \frac{\Phi_{\uparrow} - \Phi_{\downarrow}}{\sqrt{2}}. \tag{56}
$$

The new charge and spin fields obey the usual Bose commutation relations, as they are simply sums and differences of Bose fields. If we substitute these expressions into the bosonized pieces of the

Hamiltonian, we obtain a Hamiltonian, $H_B = H_c + H_s$, in which the charge,

$$H_c = \frac{\hbar v_F}{2} \int dx[\Pi_c^2 + g_c^2(\partial_x \Phi_c)^2], \tag{57}$$

and spin degrees of freedom,

$$H_s = \frac{\hbar v_F}{2} \int dx[\Pi_s^2 + g_s^2(\partial_x \Phi_s)^2] + \frac{U}{2\pi^2 a} \int dx : \cos \sqrt{8\pi}\Phi_s :, \tag{58}$$

are completely decoupled. The coupling constants,

$$g_c^2 = 1 + \frac{aU}{2\pi\hbar v_F} \tag{59}$$

and

$$g_s^2 = 1 - \frac{aU}{2\pi\hbar v_F}, \tag{60}$$

which are now a function of the interactions, can be used to define new velocities for the spin and charge degrees of freedom. Let $v_F^c = v_F g_c$ and $v_F^2 = v_F g_s$. To see that the spin and charge sectors are now moving with different velocities, we consider the transformation $\Phi_\gamma \sqrt{g_\gamma} \to \tilde{\Phi}_\gamma$ and $\Pi_\gamma 1/\sqrt{g_\gamma} \to \tilde{\Pi}_\gamma$, where $\gamma = c, s$. This transformation leaves intact the Bose commutation relations, Eq.(38). In terms of the re-scaled fields, the charge

$$H_c = \frac{\hbar v_F^c}{2} \int dx[\tilde{\Pi}_c^2 + (\partial_x \tilde{\Phi}_c)^2] \tag{61}$$

degrees of freedom resemble a collection of non-interacting fermions but with a new Fermi velocity,v_F^c, that increases as the strength of the on-site repulsions increases. In the spin sector,

$$H_s = \frac{\hbar v_F^s}{2} \int dx[\tilde{\Pi}_s^2 + (\partial_x \tilde{\Phi}_s)^2] + \frac{U}{2\pi^2 a} \int dx : \cos \sqrt{8\pi/g_s}\tilde{\Phi}_s : \tag{62}$$

and the new velocity, v_F^s, of the spins lags behind that of the charge degrees of freedom for repulsive interactions between the electrons. Hence, the spin and charge degrees of freedom move with completely different velocities. Another difference between the spin and charge sectors is the $: \cos \tilde{\Phi}_s :$ term. In the bosonized language, this factor is equivalent to a mass term in the original fermion Hamiltonian. Physically, the mass term represents an energy gap. However, only in the case of attractive Coulomb interactions, in which case electrons form bound antiferromagnetic pairs, does a spin gap arise, thereby heightening the difference between the spin and charge sectors in 1d. As a result, for 1d interacting electron liquids, the electron falls apart into distinct spin and charge quasi particles. We can think of an electron then as a composite particle made out of two distinct entities. We will refer to the entity that carries the charge as the *holon* (or eon) and the spin part as the *spinon*. Holons (or eons) and spinons obey Bose statistics. The emergence of holon and spinon excitations in 1d represents a marked departure from Fermi-liquid behavior. In Fermi-liquid theory, there is a one-to-one correspondence between the excited states of the interacting and non-interacting systems. Spin and charge separation in 1d fundamentally destroy this correspondence because now the electron gives rise to two excitations rather than the single excitation indicative of Fermi-liquid theory.

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