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PERTURBATION THEORY WITHOUT GAUGE FIXING

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ABSTRACT

We propose to formulate the perturbative expansion for field theory starting from the Langevin equation which describes the approach to equilibrium. We show that this formulation can be applied to gauge theories to compute gauge invariant quantities without fixing the gauge. A very simple example is worked out in detail. We also discuss the speed of approaching to equilibrium of the solution of the Langevin equation in the framework of perturbation theory.

I. INTRODUCTION

In the standard perturbative approach to gauge theories, it is necessary to introduce a gauge fixing term. In the Abelian case this introduction does not give serious problems (if one uses a linear gauge condition), while in non-abelian gauge theories it is necessary to introduce the Faddeev-Popov ghost[1]. However, this procedure breaks down in a nonperturbative approach if one uses a covariant gauge: the gauge condition does not fix uniquely the gauge for large gauge potentials and this phenomenon goes under the name of Gribov ambiguity[2].

In lattice gauge theories the situation is very different: no gauge fixing is needed and owing to the compactness of the gauge group, all quantities which are not gauge invariant are zero[3].

The aim of our work is to construct a new perturbation theory for continuum gauge theories without introducing a gauge fixing. The quantization method we propose has the advantage of working also in the non-perturbative region independently of the Gribov ambiguity. Moreover, we think that it is better to respect as far as possible the symmetry of the problem at all stages in the computation. In our case we would still find that quantities which transform homogeneously under a gauge transformation (e.g. a charged field $\phi: \delta\phi = i\alpha\phi$) must have vanishing expectation values. Therefore, quantities such as the propagator for a charged field become zero when the group charge e is different from zero. This discontinuity in the behaviour of non-gauge-invariant quantities as a function of e will be reflected in the presence of divergences in their perturbative expansion. However, the perturbative expansion for gauge invariant quantities is obviously free of these divergences.

Our approach is based on the Langevin equation of non-equilibrium statistical mechanics[4]. As it will be clear later, the Langevin equation (a stochastic evolution equation) is strongly connected to the Monte Carlo procedure which is used to do computer simulation in gauge theories[5][1]. Now it is interesting to study in the framework of perturbation theory the speed of approaching to equilibrium of these random constructive procedures, as we shall do in this paper.

We first recall the general properties of the Langevin equation in Sec. II, and write down the diagrammatic rules in Sec. III. Then, in Sec. IV we show how to obtain the correct results in a case where standard perturbation theory cannot be used. In Sec. V we write the Langevin equation and the diagrammatical rules for gauge theories, and present a simple computation to show how the correct result is obtained. The arguments for the correctness of our new perturbative expansion at all orders are given in Sec. VI, and finally Sec. VII is devoted to the presentation of our conclusions.

II. Langevin Equation

Let us consider, for definiteness, an Euclidean scalar field theory. Usually we want to compute the correlation function, e.g., $\langle\phi(x)\phi(y)\rangle$, where the bracket denotes the statistical expectation value at a temperature T:

$$\langle\phi(x)\phi(y)\rangle = \frac{\int d[\phi]\phi(x)\phi(y)\exp\{-\beta V(\phi)\}}{\int d[\phi]\exp\{-\beta V(\phi)\}}, \tag{2.1}$$

where $\beta = 1/kT$.

It may be convenient to generalize the problem. We can consider that the field ϕ is also a function of a time $t(0 < t < \infty)$, and it is coupled with a heat reservoir at temperature T. It will reach the equilibrium distribution for large time t. If we know the evolution equation of the field $\phi(x,t)$, we can use it to compute the large time behaviour of $\phi(x,t)$, and consequently the equilibrium distribution and the correlation function Eq. (2.1). In other words, we have the freedom to assign to the field $\phi(x,t)$ any time evolution equation so as to reach equilibrium for large times.

The simplest equation we can write is the so-called Langevin equation[4],

$$\frac{\partial\phi(x,t)}{\partial t} = -\frac{\delta V}{\delta\phi(x,t)} + \eta(x,t), \tag{2.2}$$

where $\eta(x,t)$ are Gaussian random variables:

$$\langle\eta(x,t)\rangle = 0,$$

1) In certain cases the Monte Carlo procedure can be considered as a time-discretized Langevin equation, where the discretization is done in such a way as to preserve the same asymptotic limit for large times.

$$\langle \eta(x,t)\eta(y,t')\rangle = 2\beta^{-1}\delta(x-y)\delta(t-t')$$
$$= \frac{\int d[\eta]\eta(x,t)\eta(y,t')\exp\left\{-\frac{\beta}{2}\int d^D x dt \eta^2(x,t)\right\}}{\int d[\eta]\exp\left\{-\frac{\beta}{2}\int d^D x dt \eta^2(x,t)\right\}}, \tag{2.3}$$

$$\langle \eta_1\eta_2\eta_3\eta_4\rangle_c \equiv \langle \eta_1\eta_2\eta_3\eta_4\rangle - \langle \eta_1\eta_2\rangle\langle \eta_3\eta_4\rangle - \langle \eta_1\eta_3\rangle\langle \eta_2\eta_4\rangle$$
$$- \langle \eta_1\eta_4\rangle\langle \eta_2\eta_3\rangle = 0, \tag{2.4}$$

where $\eta_i = \eta(x,t_i)(i = 1,2,3,4)$. If we impose a boundary condition at $t = 0$ (in the following we will assume $\phi(x,0) = 0$ and $V(0) = 0$), the solution of Eq. (2.2) is uniquely given in terms of η; let us call it $\phi^\eta(x,t)$. The stochastic correlation function are defined by $\langle \phi^\eta(x,t)\phi^\eta(x',t')\rangle$, where the bracket indicates the mean value over η. (In the rest of the paper we shall write $\phi(x,t)$ at the place of $\phi^\eta(x,t)$.)[1] It is a well-known theorem of statistical mechanics[4] that when t goes to infinity,

$$\langle \phi(x,t)\phi(x',t)\rangle \to \langle \phi(x)\phi(x')\rangle, \tag{2.5}$$

i.e. the equal time non-equilibrium correlation functions tend to the equilibrium ones for large times.

We note that the probability distribution $P(\phi,t)$ satisfies the Fokker-Planck equation, as is proved in [4], (we set $\beta = 1$)

$$\frac{d}{dt}P(\phi,t) = \frac{\delta^2 P}{\delta\phi(x)^2} + \frac{\delta}{\delta\phi(x)}\left(P\frac{\delta V}{\delta\phi(x)}\right)$$
$$= -2\exp\left(-\frac{1}{2}V\right)\hat{H}\left[P(\phi,t)\exp\left(\frac{1}{2}V\right)\right], \tag{2.6}$$

where

$$\hat{H} = -\frac{1}{2}\frac{\delta^2}{\delta\phi(x)^2} + U, \quad U = \frac{1}{8}\left(\frac{\delta V}{\delta\phi}\right)^2 - \frac{1}{4}\frac{\delta^2 V}{\delta\phi^2}. \tag{2.7}$$

Eq. (2.5) can be proved in many ways. In the following we present a simple proof in the case in which ϕ is defined only on one point, i.e. it is a number q, not a function. In this case we have

$$\hat{H} = \frac{1}{2}P^2 + U(q). \tag{2.8}$$

If $V(q)$ increases fast at infinity, $\hat{H}$ has a discrete spectrum. Let us denote by $\phi_n(q)$ and λ_n its eigenvectors and eigenvalues:

$$\hat{H}\phi_n(q) = \lambda_n\phi_n(q), \quad (\lambda_{i+1} > \lambda_i). \tag{2.9}$$

We can write the correlation functions at equal times as

$$\langle q(t)^K\rangle = \sum_{n=0}^{\infty} c_n \exp(-2\lambda_n t)\int dq q^K \phi_n(q)\phi_0(q). \tag{2.10}$$

1) We write the Langevin equation using a simple, but not mathematically correct notation. The rigorous notation uses the Ito differential calculus[4].

It is very easy to verify that $\exp\left(-\frac{1}{2}V(q)\right)$ is an eigenvector of $\hat{H}$ with eigenvalue zero; $\exp\left(-\frac{1}{2}V(q)\right)$ is also the ground state of $\hat{H}$, because it is a function without zeros. Therefore, for large times we have

$$\langle q(t)^K\rangle = \frac{\int dqq^K \exp(-V(q))}{\int dq \exp(-V(q))} + O[\exp(-2\lambda_1 t)], \tag{2.11}$$

i.e. for large times we reach the equilibrium distribution $\exp(-V(q))$, and the corrections are exponentially small. It will be useful to note that the exponent λ_1 can be written as the eigenvalue of a Schrödinger operator. This implies that λ_1 must be a continuous function of V.

The same argument can be done for the general case. The Hamiltonian now is

$$H = \int d^D x \left\{\frac{1}{2}\pi(x)^2 + U(\phi(x))\right\}, \quad [\pi(x), \phi(y)] = -i\delta(x-y), \tag{2.12}$$

and

$$\phi_0[\phi] = \exp\left\{-\frac{1}{2}\int d^D x V(\phi(x))\right\} \tag{2.13}$$

is the solution of the following Schrödinger functional equation with $\lambda_0 = 0$:

$$\left\{-\frac{1}{2}\frac{\delta^2}{\delta\phi(x)^2} + U(\phi(x))\right\}\phi_0[\phi] = \lambda_0\phi_0[\phi]. \tag{2.14}$$

It is possible to consider a more general Langevin equation:

$$\dot{\phi}(x,t) = -\int d^D y M(x,y)\frac{\delta V}{\delta\phi(y)} + \eta(x,t). \tag{2.15}$$

If

$$\langle\eta(x,t)\eta(y,t')\rangle = 2M(x,y)\delta(t-t') \tag{2.16}$$

and M is a positive matrix, one finds the same conclusions of the previous case.

These results imply that the functional integral formulation of field theories can be replaced by a parabolic nonlinear stochastic equation[1]. The Langevin equation can be used both in perturbative theory or in a nonperturbative framework. In this paper we will show how to use the Langevin equation for constructing a perturbative expansion in cases where the standard perturbative approach must be modified.

III. Diagrams

Let us study the diagrammatic approach to the solution of the Langevin equation. We will consider for simplicity the case in which,

$$V(\phi) = \int d^D x \left\{\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{3}g\phi^3\right\}. \tag{3.1}$$

1) In Ref. [6] it is shown how to use an elliptic nonlinear stochastic equation instead of a parabolic one.

The Langevin equation with $M(x,y) = \delta(x-y)$ is

$$\left.\begin{aligned} &\dot{\phi} = \partial^2\phi - m^2\phi + g\phi^2 + \eta, \\ &\langle \eta(x,t)\eta(x',t')\rangle = 2\delta(x-x')\delta(t-t'). \end{aligned}\right\} \tag{3.2}$$

Let us first study the case $g = 0$. The solution of Eq. (3.2) can be written as

$$\phi(x,t) = \int_0^t d\tau \int d^D y G(x-y, t-\tau)\eta(y,\tau), \tag{3.3}$$

where $G(x,t)$ is the retarded Green function which satisfies

$$\left.\begin{aligned} &\frac{\partial}{\partial t} G(x,t) = (\partial^2 - m^2)G(x,t) + \delta(x)\delta(t), \\ &G(x,t) = 0. \quad (\text{for } t < 0) \end{aligned}\right\} \tag{3.4}$$

It is obvious that

$$G(x,t) = \int \frac{d^D k}{(2\pi)^D} \exp\{-t(k^2 + m^2) + ik\cdot x\}\theta(t). \tag{3.5}$$

Eq. (3.3) implies that $\phi(x,t)$ is a Gaussian stochastic variable, being the linear combination of Gaussian variables. Now the correlation function

$$\langle \phi(x,t)\phi(x',t)\rangle \equiv D(x-x';t,t')$$

can be easily computed. We find

$$D(x-x';t,t') = 2\int_0^\infty d\tau \int d^D y G(x-y,t-\tau)G(x'-y,t'-\tau). \tag{3.6}$$

In the momentum space we have, for $t' < t$,

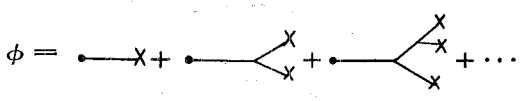
$$D(k;t,t') = \frac{\exp[-(k^2+m^2)(t-t')]}{k^2+m^2}\{1 - \exp[-2(k^2+m^2)t']\}. \tag{3.7}$$

When $t' \to \infty$, the second term can be neglected. At equal times ($t = t' \to \infty$) we obtain the equilibrium result $1/(k^2+m^2)$.

For $g \neq 0$, we can write

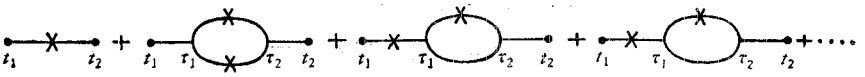
$$\phi(x,t) = \int_0^t d\tau \int d^D y G(x-y,t-\tau)[\eta(y,\tau) + g\phi^2(y,\tau)]. \tag{3.8}$$

If we denote G by a line, η by a cross and ϕ by a point, assign a factor g to each three-line vertex and integrate over the times and coordinates of all crosses and vertices, then we obtain by iterating Eq. (3.8)

$\phi =$  (3.9)

The mean over η is zero, if two crosses do not coincide. So we get up to the order

g^2 (if we neglect tadpole-like diagrams),

$\langle\phi(t_1)\phi(t_2)\rangle =$ 

(a) (b) (c) (d) (3.10)

The diagram (a) gives the free propagator Eq. (3.6) or (3.7), and the contributions of the diagrams (b), (c) and (d) are respectively:

$$b = g^2 \int \frac{d^D k_1}{(2\pi)^D} \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 G(k;t_1-\tau_1)G(k;t_2-\tau_2) \times D(k_1;\tau_1,\tau_2)D(k-k_1;\tau_1,\tau_2), \tag{3.11}$$

$$\begin{aligned} c+d = g^2 \int \frac{d^D k_1}{(2\pi)^D} \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 \{ & D(k-k_1;\tau_1,\tau_2) \\ & \times [D(k;t_1,\tau_1)G(k_1;\tau_2-\tau_1)G(k;t_2-\tau_2) \\ & + D(k;t_2,\tau_2)G(k_1;\tau_1-\tau_2)G(k;t_1-\tau_1)] \\ & + \text{terms obtained by } k_1 \rightleftarrows k-k_1\}. \end{aligned} \tag{3.12}$$

A simple computation shows that we recover the correct equilibrium result at equal large times ($t_1 = t_2 \to \infty$):

$$b = g^2 \frac{d^D k_1}{(2\pi)^D} \frac{1}{(k^2+m^2)(k_1^2+m^2)(k_2^2+m^2)(k^2+k_1^2+k_2^2+3m^2)}, \tag{3.13}$$

$$c+d = g^2 \int \frac{d^D k_1}{(2\pi)^D} \left(\frac{1}{k_1^2+m^2} + \frac{1}{k_2^2+m^2}\right) \frac{1}{(k^2+m^2)^2(k^2+k_1^2+k_2^2+3m^2)}, \tag{3.14}$$

$$b+c+d = g^2 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{(k_1^2+m^2)(k_2^2+m^2)(k^2+m^2)^2}, \tag{3.15}$$

where $k_2 = k - k_1$. This is an example showing explicitly the validity of Eq. (2.5) to the order g^2. That it is generally true for all orders follow from the generaf consideration of the previous section or, alternatively, from a diagrammatical prool given by De Dominicis for the case (3.1)[7].

IV. A Simple Example

In this section we will present a simple example of a more general phenomenon. Let us consider the potential

$$V(\boldsymbol{q}) = -\frac{1}{2}\mu^2\boldsymbol{q}^2 + \frac{1}{4}g(\boldsymbol{q}^2)^2, \tag{4.1}$$

where $\boldsymbol{q}$ is an n-dimensional vector, and $\boldsymbol{q}^2 = \sum_{i=1}^{n} (q^i)^2$.

If we want to compute

$$\langle q^2 \rangle \propto \int d[q] q^2 \exp\{-V(q)\} \tag{4.2}$$

in perturbation theory in g, we have first to find the minimum of $V(q)$. The minimum happens at $|q|^2 = \mu^2/g$, but owing to the $O(n)$ symmetry of the potential $V(q)$, it is not an isolated minimum. If we choose the minimum

$$q = q_0 \equiv \left(\sqrt{\frac{\mu^2}{g}}, 0, \cdots, 0\right),$$

we can write $q = q_0 + \tilde{q}$ and develop the exponent in powers of $\tilde{q}$, but we shall get divergent intergals. Indeed, let us write

$$q = \left(\sqrt{\frac{\mu^2}{g}} + \tilde{q}_L, q_T\right), \tag{4.3}$$

where q_T is an $(n-1)$-dimensional vector. We find

$$V(q) = \mu^2 \tilde{q}_L^2 + \sqrt{\mu^2 g}\tilde{q}_L(\tilde{q}_L^2 + q_T^2) + \frac{1}{4} g (\tilde{q}_L^2 + q_T^2)^2 - \frac{1}{4g}\mu^4, \tag{4.4}$$

and the integration over q_T is not damped for $g = 0$. If we add a regularizing term hq_T^2 and send h to zero, we would get the wrong result at the first order in g. The correct result can always be obtained by doing the nonlinear transformation to the variable $r = (q^2)^{1/2}$ and the set of angular coordinates of the $(n-1)$-dimensional sphere,

$$\langle q^2 \rangle = \frac{\int dr r^{N-1} r^2 \exp\left\{-\left(\frac{g}{4} r^4 - \frac{\mu^2}{2} r^2\right)\right\}}{\int dr r^{N-1} \exp\left\{-\left(\frac{g}{4} r^4 - \frac{\mu^2}{2} r^2\right)\right\}}. \tag{4.5}$$

We want to show that from the Langevin equation one can get the correct result without having to do the nonlinear transformation. Let us consider the first order in g. We know that we must have

$$\langle q^2 \rangle = \frac{\mu^2}{g} + A(n-1) + B. \tag{4.6}$$

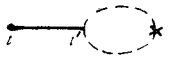
If $n = 1$, we obtain automatically the correct result, so we can only compute the term proportional to $(n-1)$. Now the Langevin equations are

$$\left.\begin{aligned} \dot{q}_T &= \eta_T + O(g^{1/2}), \\ \dot{\tilde{q}}_L &= -2\mu^2 \tilde{q}_L - \sqrt{\mu^2 g}(3\tilde{q}_L^2 + q_T^2) + O(g) + \eta_L. \end{aligned}\right\} \tag{4.7}$$

An easy computation shows that

$$\langle q_T^i(t) q_T^j(t') \rangle = 2\delta_{ij} \min(t, t'). \tag{4.8}$$

In order to get $\langle q^2 \rangle$, we must compute $\tilde{q}_L$ at the order $g^{1/2}$. The only diagram (for the term proportional to $(n-1)$) is

$\tilde{q}_L(t) \approx$ 

$$= \mu g^{1/2} \int_0^t dt' \langle \boldsymbol{q}_T^2(t') \rangle \exp\left[-2\mu^2(t-t')\right]$$
$$= -(n-1)\sqrt{g/\mu^2}(t - 1/2\mu^2), \tag{4.9}$$

where the dashed line stands for the transverse propagator, and we have neglected terms which vanish when $t \to \infty$. If we compute the terms proportional to $(n-1)$ in

$$\langle \boldsymbol{q}^2 \rangle = \frac{\mu^2}{g} + 2\sqrt{\frac{\mu^2}{g}} \langle \tilde{q}_L \rangle + \langle \tilde{q}_L^2 \rangle + \langle \boldsymbol{q}_T^2 \rangle, \tag{4.10}$$

we find they are

$$-2(n-1)\left(t - \frac{1}{2\mu^2}\right) + 2(n-1)t = \frac{n-1}{\mu^2}. \tag{4.11}$$

From this equation we see that the terms proportional to t coming from $\langle \tilde{q}_L \rangle$ and $\langle \boldsymbol{q}_T^2 \rangle$ cancel each other, and the finite contribution gives the correct result as can be easily checked from Eq. (4.5) by the saddle point method.

Of course, we can also apply the nonlinear transformations to the Langevin equation. Let us consider the case $n = 2$. It can be shown that if we go to the variables $r(t)$ and $\theta(t)$, the Langevin equation

$$\dot{\boldsymbol{q}} = -\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}} + \boldsymbol{\eta} \tag{4.12}$$

becomes[1]

$$\left.\begin{aligned} \dot{r} &= -\frac{d}{dr}\left[V(r) + \ln r\right] + \eta_r, \\ \dot{\theta} &= \eta_\theta, \end{aligned}\right\} \tag{4.13}$$

where

$$\langle \eta_r(t)\eta_r(t') \rangle = 2\delta(t-t'), \quad \langle \eta_\theta(t)\eta_\theta(t') \rangle = \frac{2}{r^2}\delta(t-t').$$

The evolution for r is decoupled from that for θ, and the perturbative expansion in g is uniform at all times. However, we do not need at any rate to do explicitly the nonlinear transformation. The solution of the Langevin equation is finite at all t (t plays the role of a regulator), and we get the correct results automatically. Of course, if we compute $\langle \boldsymbol{q}_T^2 \rangle = \langle r^2 \sin^2\theta \rangle$ we would get divergent results in perturbation theory:

$$\langle \boldsymbol{q}_T^2 \rangle \simeq t + t^2 g + \cdots \tag{4.14}$$

indicating that the equilibrium is reached only for times $t \gg 1/g$. Therefore, the

1) If one is not careful in doing the nonlinear transformation, one would miss the $\ln r$ term.

only distinction is between quantities which go fast to equilibrium ($\lambda_1 \sim O(1)$) and those which go slowly ($\lambda_1 \sim O(1/g)$); only the first ones can be correctly computed in perturbation theory. For example, we obtain

$$\langle \boldsymbol{q} \rangle \simeq O(\exp(-tg)). \tag{4.15}$$

The existence of an equation for r being decoupled from the equation for θ is a general consequence of the symmetry of the problem, and it implies that the approach to equilibrium must be fast for symmetric invariant quantities.

Indeed, let us consider the Fokker-Planck equation for $\phi = P(\boldsymbol{q},t)\exp\left(\frac{1}{2}V(\boldsymbol{q})\right)$. Using the radial variables r and θ we can see that the "S-wave" component of ϕ,

$$\phi_S(r,t) = \int d\theta \phi(r,\theta,t) \tag{4.16}$$

satisfies the radial equation

$$\frac{\partial}{\partial t}\phi_S(r,t) = \hat{H}_S \phi_S(r,t). \tag{4.17}$$

The eigenvalue λ_1 of $\hat{H}_S$ controls the approach to equilibrium of radially symmetric quantities. By general theorems λ_1 is a continuously differentiable function of the coupling constant, so that we expect that the approach to equilibrium must be uniform in g and time.

As the reader can see, this approach has the virtue of giving automatically the correct results in perturbation theory without worrying about the nonlinear transformations. This is in contrast with the approach consisting in adding to V a symmetry breaking term $h\boldsymbol{q}_T^2$ and firstly expanding in g and later sending h to zero.

V. Gauge Theories

Now we proceed to consider a gauge theory, the Euclidean Hamiltonian being

$$V = \int d^D x \left\{ (D_\mu \phi^+)(D_\mu \phi) + \frac{1}{2}\mathrm{Tr} F_{\mu\nu}^2 \right\}, \tag{5.1}$$

where

$$D_\mu \phi = (\partial_\mu - ieA_\mu)\phi, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]$$
$$A_\mu = A_\mu^a \tau_a, \quad F_{\mu\nu} = F_{\mu\nu}^a \tau_a, \quad \mathrm{Tr}(\tau_a \tau_b) = \frac{1}{2}\delta_{ab}.$$

The associate Langevin equations are

$$\begin{cases} \dot{\phi} = D^2\phi + \eta_\phi, \quad \dot{\phi}^+ = D^2\phi^+ + \eta_\phi^+, \\ \dot{A}_\mu = D_\nu F_{\nu\mu} + J_\mu + \eta_\mu, \end{cases} \tag{5.2}$$

where

$$\left.\begin{aligned} J_\mu &= J_\mu^a \tau_a, \quad J_\mu^a = ie\phi^+ \tau_a \partial_\mu \phi + e^2 \phi^+ \{\tau_a, A_\mu\}\phi, \\ \langle \eta_\phi(x,t)\eta_\phi^+(x',t')\rangle &= 2\delta(x-x')\delta(t-t'), \\ \langle \eta_\mu(x,t)\eta_\nu(x',t')\rangle &= 2\delta_{\mu\nu}\delta(x-x')\delta(t-t')C_2, \end{aligned}\right\} \tag{5.3}$$

and $C_2 = \delta^{ab}\tau_a\tau_b$ is the second Casimir operator, which is a multiple of the unit matrix.

Let us first consider the free Abelian case in which,

$$\dot{A}_\mu = \partial^2 A_\mu - \partial_\mu \partial_\nu A_\nu + \eta_\mu. \tag{5.4}$$

After being imposed the boundary condition

$$A_\mu(x,t)\,|_{t=0} = 0, \tag{5.5}$$

the solution of the Eq. (5.4) for $t > 0$ (in the momentum space) is,

$$A_\mu(k,t) = \int_0^t dt' G_{\mu\nu}(k,t-t')\eta_\nu(k,t'), \tag{5.6}$$

where the retarded Green function defined only for $t > t'$, is

$$G_{\mu\nu}(k,t-t') = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\exp\left[-k^2(t-t')\right] + \frac{k_\mu k_\nu}{k^2}. \tag{5.7}$$

The expectation value of $A_\mu(x,t)A_\nu(y,t')$ in the momentum space is

$$\begin{aligned} D_{\mu\nu}(k;t,t') = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}\{&\exp\left[-k^2|t-t'|\right] \\ &- \exp\left[-k^2(t+t')\right]\} + \frac{k_\mu k_\nu}{k^2}\,2\min(t,t'). \end{aligned} \tag{5.8}$$

For large equal times ($t = t' \to \infty$), we find

$$D_{\mu\nu}(k;t,t) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2} + 2t\frac{k_\mu k_\nu}{k^2}. \tag{5.9}$$

Diagrammatically, Eq. (5.7) is the (retarded) propagator without any cross and Eq. (5.8) is the propagator with a cross.

We note that Eq. (5.9) is just the usual Feynman propagator in the Landau gauge plus a longitùdinal term which is divergent as the time. Indeed, we can write

$$A_\mu(x,t) = A^T_\mu(x,t) + \partial_\mu\alpha(x,t), \tag{5.10}$$

where $A^T_\mu(x,t)$ satisfies

$$\partial_\mu A^T_\mu(x,t) = 0. \tag{5.11}$$

$A^T_\mu(x,t)$ is gauge invariant and $\alpha(x,t)$ looks like a gauge transformation. Then we find in the momentum space,

$$\left.\begin{aligned} \langle A^T_\mu(k,t)A^T_\nu(-k,t)\rangle &= \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}, \\ \langle\alpha(k,t)\alpha(-k,t)\rangle &= \frac{2}{k^2}\,t. \end{aligned}\right\} \tag{5.12}$$

This means that the evolution for gauge invariant quantities is fast, while the system undergoes a random walk in the gauge parameter space.

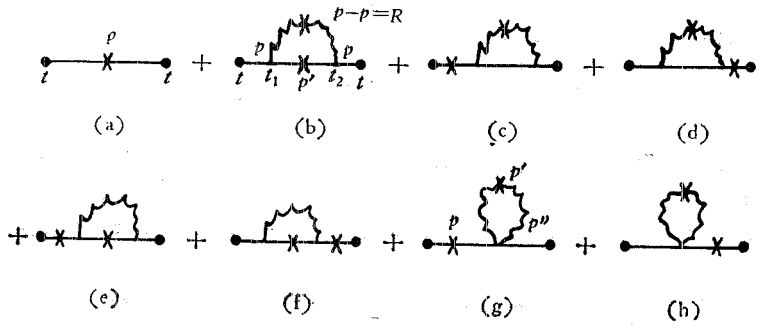


Fig. 1

Now let us consider the $\phi^+\phi$ propagator in the Abelian theory at equal times. The diagrammatical rules for scalar QED are very similar to those in Sec. III. In addition to the propagators given by Eqs. (3.7), (5.7) and (5.8), what we should add here are only the rules for the $A_\mu\phi^+\phi$ and $A_\mu A_\nu\phi^+\phi$ vertices, but they are the same as in usual Feynman diagrams. We shall do the computation up to the order e^2, keeping only the terms which survive at large t. The diagrams are shown in Fig. 1, where the straight line stands for the ϕ field, and the wavy line the photon. Their contributions at equal times t are given as follows for large t (after being integrated over the times, t_1 and t_2, of the vertices):

$$a = 2\int_0^t dt' \exp(-2p^2t') = \frac{1}{p^2},$$

$$b = e^2\int \frac{d^Dp'}{(2\pi)^D}\frac{1}{p^2p'^2k^2(p^2+p'^2+k^2)}\left[(p+p')^2 - \frac{(p^2-p'^2)^2}{k^2}\right]$$
$$+ e^2\int \frac{d^Dp'}{(2\pi)^D}\frac{1}{p^2p'^2(p^2+p'^2)}\left[2t - \frac{2}{p^2+p'^2} - \frac{1}{p^2}\right]\frac{(p^2-p'^2)^2}{k^2}, \qquad (5.13)$$

$$c + e = d + f = \frac{e^2}{2}\int \frac{d^Dp'}{(2\pi)^D}\frac{1}{p^2}\left(\frac{1}{p'^2}+\frac{1}{k^2}\right)\frac{1}{p^2+p'^2+k^2}\left[(p+p')^2 - \frac{(p^2-p'^2)^2}{k^2}\right]$$
$$+ \frac{e^2}{2}\int \frac{d^Dp'}{(2\pi)^D}\frac{1}{p^4(p^2+p'^2)}\left[\frac{1}{p'^2} + 2t - \frac{2}{p^2+p'^2} - \frac{1}{p^2}\right]\frac{(p^2-p'^2)^2}{k^2}, \qquad (5.14)$$

$$g + h = -3e^2\int \frac{d^Dp'}{(2\pi)^D}\frac{1}{p^4p'^2} - e^2\int \frac{d^Dp'}{(2\pi)^D}\frac{1}{p^4}\left[2t - \frac{1}{p^2}\right]. \qquad (5.15)$$

Here $k = p - p'$; the first and the second integrals represent respectively the contributions of the transverse and longitudinal part of the A_μ field.

If we sum all the contributions of the transverse part, we find the usual result in the Landau gauge. For the contribution of the longitudinal part, at large times the term proportional to t is equal to the variation of the equilibrium $\phi^+\phi$ propagator induced by adding the gauge term $t(k_\mu k_\nu/k^2)$ to the equilibrium $A_\mu A_\nu$ propagator. Indeed, the main contribution to the diagrams comes from the region of integration where $t - t_1, t - t_2$ are of order 1, so that at the leading order in t we can substitute t for t_1, t_2. In this way we lose terms of $O(1)$ when $t \to \infty$, but we compute correctly the terms of order t. This argument can be generalized to the leading order

in t at fixed e^2. Using the standard theorem on the variation of Green functions under a gauge transformation[8], we get

$$\langle \phi^+(x,t)\phi(y,t)\rangle \underset{t\to\infty}{\sim} \langle \phi^+(x)\phi(y)\rangle_f \exp\left[-e^2 t\omega(x-y)\right], \tag{5.16}$$

where $\omega(x) \propto 1/|x|^{D-2}$ is the Fourier transform of $1/k^2$, and $\langle \phi^+(x)\phi(y)\rangle_f$ is the free propagator. Eq. (5.16) implies that for large times the charged field propagator is very near to zero, but this happens only for times t greater than $1/e^2\omega(x-y)$. In the asymptotic limit $t\to\infty$, the sum of the leading terms in t for each order in e^2 goes to zero. However, a completely different result would be obtained considering only a finite number of terms in the perturbation expansion.

Let us consider a gauge invariant quantity. The simplest one is $\phi^+(x,t)\phi(x,t)$. The contribution proportional to t is obviously zero, because it corresponds to a gauge transformation. This can be explicitly checked from Eqs. (5.13)—(5.15) by using dimensional regularization. As for the remaining finite terms, after some algebraic operation and having eliminated terms odd in $k = p - p'$, we find they are equal to the well-known contribution in the Landau gauge plus the following term,

$$2\int \frac{d^Dp}{(2\pi)^D}\frac{d^Dp'}{(2\pi)^D}\frac{p^2-p'^2}{p^2p'^2(p^2+p'^2)(p-p')^2}. \tag{5.17}$$

Being odd under the exchange $p\longleftrightarrow p'$, this term is equal to zero, as required by consistency.

We have seen that also in gauge theories, at least in this simple example, the correct results for gauge invariant quantities can be obtained in our approach with the Langevin equation by expanding in powers of the coupling constant without having to fix the gauge as is done in the conventional approach. The only breaking of gauge invariance is in the boundary conditions at $t=0: A_\mu(x,t)|_{t=0}=0$, but the large t behaviour is independent of the boundary conditions.

What happens at higher orders? Diagrams must be regularized by dimensional regularization or lattice regularization (i.e., by a gauge invariant procedure). In principle, it is known that for a renormalizable theory dynamic correlation functions are finite only after a renormalization of the matrix $M(x,y)$ in Eq. (2.15). This phenomenon which is wellknown in the theory of the second-order phase transitions[4], does not modify the static (equal-time) correlation functions at large times. We have therefore two possibilities:

1) Add the counterterms in $M(x,y)$ in order to have finite results at all times.

2) Take the limit $t\to\infty$ before sending the cutoff to infinity or the space dimensions D to the physical one. The first alternative would be the best, if we want to find explicitly the convergence rate of the Langevin equation (e.g., to compare with Monte Carlo procedures), while the second alternative is the simplest if we are interested only in equilibrium properties.

In the next section we will argue that also in non-Abelian theories we obtain the correct results which correspond to the effect of the Faddeev-Popov ghost.

VI. General Considerations

The general theorems on the approaching to equilibrium of the solution of the Langevin (or the Fokker-Planck) equation can be applied to our case, therefore there is no doubt that the large-time behaviour of the correlation functions is the correct one. The main problem is to show that for gauge invariant quantities equilibrium is approached uniformly in g and in t so that the Taylor expansion in g and the limit $t \to \infty$ can be freely exchanged. As is seen in the previous sections, this is not true for quantities which are not gauge invariant. Therefore, it is better to present the argument in detail (although it is rather similar to that in Sec. IV).

It is convenient to introduce the quantities $A_\mu^T(x,t)$ $\alpha(x,t)$ and $\phi^T(x,t)$ defined by

$$\left.\begin{aligned} &\partial_\mu A_\mu^T(x,t) = 0, \\ &-ieA_\mu^T(x,t) = \exp[-ie\alpha(x,t)](\partial_\mu = ieA_\mu)\exp[ie\alpha(x,t)], \\ &\phi^T(x,t) = \exp[-ie\alpha(x,t)]\phi(x,t). \end{aligned}\right\} \tag{6.1}$$

In perturbation theory in e these equations fix uniquely $\alpha(x,t), A_\mu^T(x,t)$ and $\phi^T(x,t)$ in terms of $A_\mu(x,t)$ and $\phi(x,t)$[1]. $A_\mu^T(x,t)$ and $\phi^T(x,t)$ are gauge invariant quantities, i.e., they do not change under a gauge transformation for $A_\mu(x,t)$. All gauge invariant quantities can be written in terms of $A_\mu^T(x,t)$ and $\phi^T(x,t)$.

The gauge invariance of the Langevin equation or of the associated Fokker-Planck equation implies that the evolution of $A_\mu^T(x,t)$ and $\phi^T(x,t)$ is independent of the evolution of $\alpha(x,t)$. Therefore we can write

$$\left.\begin{aligned} \dot{A}_\mu^T(x,t) &= F_1(A_\mu^T, \phi^T, \eta^T), \\ \dot{\phi}^T(x,t) &= F_2(A_\mu^T, \phi^T, \eta^T). \end{aligned}\right\} \tag{6.2}$$

We are not interested in the detailed form of these equations or of the associated Fokker-Planck equations. From Eq. (4.2) we have already seen that the approach to equilibrium is controlled by the smallest positive eigenvalues λ_1 of an operator H which acts on gauge invariant quantities, so that the slow approach to equilibrium for gauge non-invariant quantities like the distribution of $\alpha(x,t)$ has no effect on the evolution of gauge invariant quantities. Thus, apart from possible ultraviolet or infrared divergences[2], equilibrium is approached in the interacting theory at roughly the same rate as in the free theory. Divergences should be eliminated by introducing the needed cutoffs and counterterms in order to obtain finite results.

These arguments show that no term linear in t appears in the expectation value of gauge invariant quantities, and that one finds automatically the correct results which corresponds to the contribution of the Faddeev-Popov ghost. Of course, this

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1) More precisely, the uniqueness of $\alpha(x,t)$ holds only for fixed $A_\mu(x,t)$ and sufficiently small e. The existence of $A_\mu(x,t)$ such that there are many solutions for $\alpha(x,t)$ is just the Gribov ambiguity.

2) In order to make the argument complete, one should first consider the case of finite lattice and impose the appropriate boundary conditions in such a way that the spectrum of H becomes discrete and that the corrections to the equilibrium distributions are exponentially small.

contribution would show up as a finite remainder of incomplete cancellation of terms which are dependent on t, if we do perturbative expansions directly for $A_\mu(x,t)$.

VII. Conclusions

The method we have proposed here is not very useful for practical computation. The number of diagrams is much higher than that in the conventional approach and the algebraic operation is longer. Some additional difficulties arise in the computation of the S matrix for charged fields. Indeed, the Green functions of charged fields are zero at equilibrium, so that the LSZ formalism cannot be used. We have not investigated the possibility of using the gauge invariant path-dependent operators of Mandelstam in the LSZ formalism. In principle, all physical measurable informations, like cross sections, can be extracted from the Green functions of gauge invariant quantities, although this operation is very complicated in practice. It is important, however, to know that, at least in principle, we can avoid the use of the Faddeev-Popov trick, whose correctness has been questioned beyond perturbation theory.

We note that the equality between the Green functions of a field theory and the equal-time stochastic correlation functions of the Langevin equation can be the starting point of a reformulation of field theory using a different language. In this paper, we show that this formulation may be useful to construct a perturbative expansion for systems for which a nonlinear transformation is needed in the conventional approach to obtain the correct results. It is possible that the same technique can be of a wider application. Indeed, the Langevin equation (or its discretized version, the Monte Carlo procedure) is a really constructive approach to field theory in the sense that it can be used as a practical starting point for computer simulations.

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