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Effective Harmonic-Fluid Approach to Low-Energy Properties of One-Dimensional Quantum Fluids

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A universal description of the low-energy properties of one-dimensional quantum fluids, based on a harmonic theory of long-wavelength density fluctuations with use of renormalized parameters, is outlined. The structure of long-distance correlations of a spinless fluid is obtained, showing the essential similarity of one-dimensional Bose and Fermi fluids. The results are illustrated by application to the one-dimensional Bose fluid with δ-function interaction.

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A recent study[1,2] of the one-dimensional (1D) Fermi fluid led to a simple low-energy description of it as a "Luttinger liquid": The low-energy effective Hamiltonian could be based on the spectrum of the Luttinger[3] model (which has a non-interacting elementary excitation spectrum of harmonic density fluctuations), plus residual anharmonic couplings that vanished at low energies; the structure of the theory was reminiscent of Fermi-liquid theory. In this Letter I report that the concept of an effective harmonic-fluid description applies quite generally to 1D quantum fluids independent of statistics, and the structure of their correlations becomes clear once a representation of the density operator has been constructed to reflect correctly the discrete particle nature of the fluid. Planar spin chains with axial symmetry can also be understood as a Bose fluid of "magnon" excitations about a fully aligned state. The theory of 1D quantum fluids can be applied to extract information from the solutions of exactly solvable (but opaque) models such as the Bose gas with finite-strength δ-function interaction.[4]

The Fermi fluid results obtained in Ref. 2 through Luttinger model techniques can be extended to the Bose fluid by considering the spin-$\frac{1}{2}$ Luttinger model with attractive $2k_F$ scattering

which induces pairing and a gap in the spin excitation spectrum.[5] The remaining low-energy charge degrees of freedom are decoupled from spin degrees of freedom, and can be identified with a Bose fluid of Cooper pairs. However, it emerged that the low-energy structure of the Bose results thus obtained (and the original fermion results) can be independently reproduced from the study of symmetrized and antisymmetrized states of a harmonic chain.[6] The high-energy structures of the Luttinger and harmonic-chain models are quite different; the equivalence of their low-energy properties suggests that a simple model-independent derivation is possible. This is presented here.

I consider a spinless Bose or Fermi fluid with density $\rho_0 = N/L$, and periodic boundary conditions $\Psi(x+L) = \Psi(x)$:

$$H = (\hbar^2/2m)\int dx\,|\nabla\Psi|^2 + \tfrac{1}{2}\int\int dx\,dy\,V(x-y)\rho(x)\rho(y). \tag{1}$$

I first discuss the Bose fluid. It will be useful to use the density-phase representation $\Psi_B^\dagger(x) = |[\rho(x)]^{1/2}|\exp[i\varphi(x)]$, where

$$[\rho(x), \exp\{i\varphi(x')\}] = \delta(x-x')\exp[i\varphi(x)].$$

Low-energy properties of the fluid are dominated by long-wavelength zero-point fluctuations of density with wave numbers $\ll\rho_0$, and a local fluctuation field $\Pi(x)$ can be introduced by considering the "smeared" local density $\rho(x) \sim \rho_0 + \Pi(x)$, averaged over lengths $\gg\rho_0^{-1}$. In the subspace of low-energy states where fluctuations of $\Pi(x)$ are small, $\varphi(x)$ and $\Pi(x)$ can be treated as conjugate canonical fields, $[\varphi(x), \Pi(x')] = i\delta(x-x')$. I note that periodic boundary conditions on the particle field $\Psi(x)$ allow *topologically excited* states of the phase field with $\varphi(x+L) = \varphi(x) + \Pi J$, J an even integer.

It is essential to construct a representation of the unsmeared density operator that reflects its discrete character. This is achieved by introducing a new field $\theta(x)$ satisfying $\nabla\theta(x) = \pi[\rho_0 + \Pi(x)]$. Its boundary conditions are $\theta(x+L) = \theta(x) + \pi N$. $\theta(x)$ increases monotonically by π each time x passes the location of a particle. Particles are thus taken to be located at the points where $\theta(x)$ is a multiple of π, allowing the density operator to be expressed as $\rho(x) = \nabla\theta(x)\{\sum_n \delta[\theta(x) - n\pi]\}$, or, equivalently,

$$\rho(x) = [\rho_0 + \Pi(x)]\sum_{m=-\infty}^{\infty}\exp[i2m\theta(x)]. \tag{2}$$

The $m = 0$ term of this sum is just the "smeared" or long-wavelength approximation. Since (up to a multiplicative renormalization factor) the square root of a δ function is also a δ function, we have

$$\Psi_B^\dagger(x) \sim [\rho_0 + \Pi(x)]^{1/2}\{\sum_{m\ \text{even}}\exp[im\theta(x)]\}\exp[i\varphi(x)]. \tag{3}$$

(The multiplicative factor is not determined independent of the high-energy cutoff structure.)

The commutation relation between $\varphi(x)$ and $\theta(x)$ is $[\varphi(x), \theta(x')] = (\tfrac{1}{2}\pi i)\mathrm{sgn}(x-x')$, and $\exp[i\theta(x)]$ alternates between values ±1 at the locations of consecutive particles. The Fermi field is thus easily constructed as $\Psi_F^\dagger(x) = \Psi_B^\dagger(x)\exp[i\theta(x)]$ (essentially a Jordan-Wigner transformation[7]).

$$\Psi_F^\dagger(x) \sim [\rho_0 + \Pi(x)]^{1/2}\{\sum_{m\ \text{odd}}\exp[im\theta(x)]\}\exp[i\varphi(x)]. \tag{4}$$

The selection rule on the topological quantum number J due to periodic boundary conditions is slightly modified: $N+J$ must be even, i.e., $(-1)^J = (-1)^N$.

It is now appropriate to go to Fourier-transformed field variables:

$$\theta(x) = \theta_0 + Nx/L - i\sum_{q\neq0}\left|\frac{2\pi}{qL}\right|^{1/2} e^{\varphi(q)}\mathrm{sgn}(q)e^{iqx}(b_q^\dagger + b_{-q}),$$

$$\varphi(x) = \varphi_0 + Jx/L - i\sum_{q\neq0}\left|\frac{2\pi}{qL}\right|^{1/2} e^{-\varphi(q)}e^{iqx}(b_q^\dagger - b_{-q}), \tag{5}$$

where $b_q^\dagger$, $q\neq0$, are boson creation operators for the long-wavelength density-fluctuation modes, $\varphi(q)$ is a free Bogoliubov transformation parameter, and (since density fluctuations with wavelengths $\ll\rho_0^{-1}$ are excluded) sums over q are restricted to $|q| \lesssim \rho_0$. The pairs (N, φ_0) and (J, θ_0) are conjugate *action-angle* variables satisfying $[N, \exp\{i\varphi_0\}] = \exp[i\varphi_0]$, etc. Note that a global shift of θ_0 moves all the

particles in one direction, and so the mean current operator j is $\pi^{-1}d\theta_0/dt$.

If the Hamiltonian (1) is now linearized in $\Pi(x)$, a harmonic form is obtained, and easily diagonalized by choice of $\varphi(q)$:

$$H \approx \frac{\hbar}{2\pi}\int dx\,[v_J(\nabla\varphi)^2 + v_N(\nabla\theta - \pi\rho_0)^2] = \hbar\{\sum_{q\neq 0}\omega_q b_q^{\dagger} b_q + \tfrac{1}{2}(\pi/L)[v_N(N-N_0)^2 + v_J J^2]\};$$

$$P = \hbar(\pi NJ/L + \sum_{q\neq 0} q b_q^{\dagger} b_q). \tag{6}$$

Here $v_J = \pi\hbar\rho_0/m$, and $v_N = (\pi\hbar\rho_0^2)^{-1}\kappa$, where κ is the compressibility per unit length; $\omega_q \sim v_s|q|$ as $q \to 0$, where the sound velocity v_s of density fluctuations is given by $v_s = (v_N v_J)^{1/2} = (\kappa/m\rho_0)^{1/2}$, the standard result for a 1D fluid. The coefficient v_J is independent of the interaction term, as a consequence of Galilean invariance,[8] but v_N is controlled by the *true* compressibility κ, which includes the effects of renormalizations due to short-wavelength nonlinear terms not included in the linearized form of the Hamiltonian. As $q \to 0$, the Bogoliubov transformation parameter is given by $\exp[2\varphi(q)] \to (v_J/v_N)^{1/2}$.

In the linear approximation (6), the current $j = \pi^{-1}d\theta_0/dt$ is given by $v_J(J/L)$; the topological excitations of the phase field are thus elementary current quanta. Note that the elementary excitation $J \to J+2$ carries an intrinsic momentum $2\pi\hbar\rho_0$.

Asymptotic correlation functions at separations $x \gg \rho_0^{-1}$ are easily evaluated with use of (2)–(5); at $T = 0$,

$$\langle\rho(x)\rho(0)\rangle \sim \rho_0^2\{1 + \eta(2\pi\rho_0 x)^{-2} + \sum_{m=1}^{\infty} A_m(\rho_0 x)^{-m^2\eta}\cos(2\pi m\rho_0 x)\}, \tag{7}$$

$$\langle\Psi_B^{\dagger}(x)\Psi_B(0)\rangle \sim \rho_0(\rho_0 x)^{-1/\eta}\{\sum_{m=0}^{\infty} B_m(\rho_0 x)^{-m^2\eta}\cos(2\pi m\rho_0 x)\}, \tag{8}$$

$$\langle\Psi_F^{\dagger}(x)\Psi_F(0)\rangle \sim \rho_0(\rho_0 x)^{-1/\eta}\{\sum_{m=0}^{\infty} C_m(\rho_0 x)^{-(m+1/2)^2\eta}\sin[2\pi(m+\tfrac{1}{2})\rho_0 x]\}. \tag{9}$$

The *correlation exponent* $\eta = 2(v_J/v_N)^{1/2}$. For free fermions, $\eta = 2$; for free bosons ($\kappa \to 0$), $\eta \to \infty$. The coefficients A_m, B_m, and C_m are model-dependent quantities dependent on the short-wavelength structure of the fluid. These results show the fundamental similarity of boson and fermion systems in 1D where (in contrast to more dimensions) the symmetry of a wave function cannot be tested by a continuous change of coordinates that exchanges particles without close approach (collision). Thus interaction and statistics effects cannot be separated. The low-energy properties of the fluid are fixed once its particle density, kinetic mass density, and compressibility are given (though the relation of the latter to the microscopic interaction *does* depend on statistics); the only remaining difference between Fermi and Bose systems is attributable to the different selection rules on the current quantum number J. The leading terms of the sums in (4), (7), and (9) are essentially the results first obtained by Luther and Peschel[9] in studies of the Luttinger model.

The expression (2) for $\rho(x)$ allows the effect of a periodic substrate potential $V \sim \int dx\cos(\lambda x)\rho(x)$ to be studied. Note that in (6), $\pi^{-1}\nabla\varphi(x)$ is the conjugate field to $\theta(x)$, and if $\lambda = 2\pi n\rho_0$, terms of sine-Gordon[10] type are generated. The fluid state is only stable if the sine-Gordon coupling parameter[10] satisfies $\beta^2 < 8\pi$, i.e., $\eta < \frac{1}{4}n^2$. Even if the potential is incommensurate with the mean particle spacing, it gives rise to renormalizations of v_N and v_J that drive the latter away from the Galilean-invariance value.

The simple low-energy structure of 1D quantum fluids is exhibited in detail not only by the Luttinger and harmonic-chain models, but by the less trivial integrable models solved by the *Bethe Ansatz*.[11] The models have anharmonic interactions, but the asymptotic low-energy spectrum of models with a fluid ground state has the harmonic form (6): The parameters v_N, v_J, and v_s can be explicitly calculated from the Bethe-*Ansatz* equations,[12] and quite generally satisfy the harmonic relation $v_s = (v_N v_J)^{1/2}$.

In general, the correlation functions of the integrable models have not so far been obtained (this is an outstanding, up-to-now intractable problem); an exception is Sutherland's model of the nonrelativistic spinless gas with $1/r^2$ interaction.[13] In this case, at particular coupling strengths such that $\eta = 1$ or 4, density correlations are known[13]; the $\eta = 4$ Bose single-particle correla-

tions were also found.[13] Asymptotic properties of these correlations are in complete agreement with (7) and (8), as the spectrum is with (6). Free spinless fermion systems ($\eta=2$) are also in agreement, as is the hard-core δ-function interaction Bose gas, which is related to free fermions by a Jordan-Wigner-type transformation. In the latter case, recent asymptotic expansions[14] up to x^{-12} for the Bose single-particle correlations are in full agreement with (8).

A system of long-standing interest is the earliest integrable fluid, the nonlinear Schrödinger equation, or nonrelativistic Bose fluid with finite-strength δ-function interaction, $V(x)=c\,\delta(x)$, for which the spectrum was exactly obtained by Lieb and Liniger.[4] The dimensionless coupling $\gamma=\hbar^2c/m\rho_0$ diverges in the dilute limit, where the system becomes effectively hard-core, equivalent to a free spinless Fermi gas. The harmonic relation $v_s=(v_N v_J)^{1/2}$ was implicitly found by Lieb,[4] who reports a relation equivalent to $v_s\propto(v_N)^{1/2}$ (v_J is fixed by Galilean invariance). Lieb examined the Bogoliubov approximation results for v_s: (i) $v_s\simeq v_J(\gamma^{1/2}/\pi)$, (ii) $(v_N v_J)^{1/2}\simeq v_J(\gamma^{1/2}/\pi)[1-\frac{1}{2}(\gamma^{1/2}/\pi)]^{1/2}$; the second expression, derived from the Bogoliubov approximation for the ground-state energy, was indistinguishable from the exact numerically obtained result up to an apparently large coupling $\gamma\lesssim 10$. Actually, since $v_s=v_J\exp[-2\varphi(0)]$, and $\exp[-2\varphi(0)]=2\eta^{-1}$ rises to the "hard-core" limit 1 in the strong-coupling limit, it is seen that $\gamma^{1/2}/\pi\lesssim 1$, or $\gamma\lesssim\pi^2$, is the natural limit on the weak-coupling regime. {An expansion in the strong-coupling limit, treating the system as equivalent to a spinless Fermi gas with weak attractive coupling, gives $v_s\simeq v_J[1-8/\gamma+O(\gamma^{-2})]^{1/2}$, leading to similar conclusions.} Bethe-*Ansatz* equations for the correlation exponent η are easily obtained from the general discussion in Ref. 12: in terms of an integration range parameter Λ,

$$2\pi/\gamma=\int_{-\Lambda}^{\Lambda}dx\,f(x;\Lambda);\quad \eta=2[f(\Lambda;\Lambda)]^{-2}, \tag{10}$$

where $f(x;\Lambda)$ is the solution of a Fredholm equation:

$$f(x;\Lambda)=1+\pi^{-1}\int_{-\Lambda}^{\Lambda}dy[1+(x-y)^2]^{-1}f(y;\Lambda). \tag{11}$$

The picture described here can be extended to fluid systems without Galilean invariance, where v_J is no longer coupling independent, but still defines an *effective* kinetic mass density for bulk flow of the fluid. The treatment of fluid Bethe-*Ansatz*-solvable systems in Ref. 12 applies equally well to lattice fluids such as the spinless fermion chain studied in Ref. 1 (for which the calculated phase diagram of v_J as a function of fluid density shows the effects of departures from Galilean invariance), and Lorentz-invariant systems such as the sine-Gordon soliton fluid, for which consideration of a Lorentz transformation gives $v_J=\pi\hbar c^2\rho_0/\mu$, where μ is the chemical potential.[15] A treatment of axially symmetric spin chains with arbitrary S, very much along the lines of that described above, will be reported elsewhere.

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