'Luttinger liquid theory' of one-dimensional quantum fluids. I. Properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas

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'Luttinger liquid theory' of one-dimensional quantum fluids: I. Properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas

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Abstract. The explicitly soluble Luttinger model is used as a basis for the description of the general interacting Fermi gas in one dimension, which will be called 'Luttinger liquid theory', by analogy with Fermi liquid theory. The excitation spectrum of the Luttinger model is described by density-wave, charge and current excitations; its spectral properties determine a characteristic parameter that controls the correlation function exponents. These relations are shown to survive in non-soluble generalisations of the model with a non-linear fermion dispersion. It is proposed that this low-energy structure is universal to a wide class of 1D systems with conducting or fluid properties, including spin chains.

1. Introduction

This paper is the first in a series that will present a general description of the low-energy properties of a wide class of one-dimensional quantum many-body systems, which I will call 'Luttinger liquids'. The work to be described was originally motivated by the search for a replacement for Fermi liquid theory in one dimension, where it fails because of the infrared divergence of certain vertices it assumes to remain finite; these divergences make an approach based on conventional fermion many-body perturbation theory useless. However, there is a certain model of an interacting one-dimensional spinless fermion system, the Luttinger model (Luttinger 1963), which has been explicitly solved (Mattis and Lieb 1965). This solution, by a Bogoliubov transformation, in effect resums all the divergences encountered in perturbation theory. The excitation spectrum of the diagonalised model is described in terms of *non-interacting* boson collective modes.

The feature of the Luttinger model that allows its solution is its exactly linear fermion dispersion. What will be demonstrated in this paper is that correction terms representing non-linearity of the fermion dispersion can be added to the model, and give rise to non-linear boson couplings between the collective modes. A *boson* many-body perturbation expansion in these terms is shown to be completely regular, so the Bogoliubov transformation technique that solves the Luttinger model is shown to provide a general method for resumming *all* the infrared divergences present, at least in the spinless

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Fermi gas. The name 'Luttinger liquid' has been chosen to reflect the idea that such systems have a low-energy excitation spectrum similar to the Luttinger model spectrum, but with *interactions* between the elementary excitations. This resembles the relation between the Fermi liquid theory and the soluble model on which it is based, the free Fermi gas.

This paper is perhaps the most technical of the planned series. It sets up the essential machinery for working with the Luttinger model and its generalisations, and uses it to discuss the effects of a non-linear fermion dispersion. The previous treatments of the model in the literature are often ambiguous on certain points, and there has been a certain amount of confusion, particularly associated with the role of cut-offs. I have therefore aimed to present a completely self-contained and precise treatment of the original Luttinger model, in particular emphasing the key role played by charge and current excitations (as opposed to collective density wave modes) which have in general been neglected in previous treatments. It was attention to these details that allowed the identification of a key part of the underlying structure of the solution that proved to remain valid in the 'Luttinger liquid' generalisation, with applications to be described in future papers.

The characteristic properties of a 'Luttinger liquid' that have emerged are: (i) a conserved charge; (ii) a characteristic 'Kohn anomaly' wavevector '$2k_F$', varying linearly with charge density; (iii) persistent currents at low temperatures, quantised in units that carry momentum $2k_F$; (iv) a spectrum of collective density wave elementary excitations, with a dispersion linear in $|q|$ at long wavelengths that defines a sound velocity v_S; (v) two additional velocities, v_N and v_J, associated with charge and current excitations, obeying $v_S = (v_N v_J)^{1/2}$; (vi) power-law decay of correlation functions at $T = 0$, with coupling-strength-dependent exponents that depend only on $\exp(-2\varphi)$, where $v_N = v_S \exp(-2\varphi)$ and $v_J = v_S \exp(2\varphi)$. It should be emphasised that this means that $\exp(-2\varphi)$ is a measure of the essential renormalised coupling constant, and can thus be obtained from knowledge of v_S and the change of ground state energy with charge, which gives v_N.

1D systems with this Luttinger liquid structure so far identified include: (*a*) interacting spinless fermions; (*b*) interacting spin-$\frac{1}{2}$ fermions (and those with higher internal symmetries; (*c*) the Bose fluid (including systems with internal symmetries); (*d*) the finite-density gas of solitons of the Sine–Gordon theory; (*e*) uniaxially anisotropic spin systems (the 'charge' here is azimuthal spin)—antiferromagnets (only in the case of finite azimuthal magnetisation in the easy-axis case) and ferromagnets (easy-plane only). For many of these classes there exist models exactly soluble by the Bethe *ansatz* (Bethe 1930), and the Luttinger liquid structure can then be explicitly tested and verified (Haldane 1981). Subsequent papers will present such 'case studies' (see also Haldane 1980).

The Luttinger liquid has a characteristic instability if a multiple of its fundamental momentum $2k_F$ is equal to a reciprocal lattice vector reflecting an underlying periodicity. For large enough values of the parameter $\exp(-2\varphi)$, a gap opens in the spectrum, and the system becomes insulating. This instability can be studied in detail using the precise operator machinery set up in this paper, and will be the subject of paper II in this series. A universal description of the behaviour of the strongly renormalised Luttinger liquid near this instability emerges. The precise agreement between the predictions of this description and the features found in many of the 'test case' models solvable by the Bethe *ansatz* will provide strong evidence for the universality of the 'Luttinger liquid' description.

The organisation of this paper is as follows. To avoid confusion, it deals only with the *spinless* form of the model. The necessary generalisation to spin-$\frac{1}{2}$ fermions, and from these to the Bose fluid and spin systems, will be dealt with in subsequent articles. In § 2 there is a brief introduction to specifically one-dimensional features of the Fermi gas. Section 3 contains the bulk of the technical development, and describes the structure of the *non-interacting* Luttinger model. The machinery set up in § 3 leads quickly to the solution of the *interacting*-fermion Luttinger model in § 4. Section 5 uses the machinery to discuss the effects of a non-linear fermion dispersion. Finally, § 6 summarises the results, and formulates the hypothesis that the 'Luttinger liquid' structure is universal to conducting spinless fermion systems in 1D.

2. The one-dimensional Fermi gas

Fermion systems in one dimension have features quite distinct from those in higher dimensions. This is because the one-dimensional Fermi surface consists of two discrete points, while in higher dimensions it is continuous. The special spectral structure resulting from this can be seen by examining the full spectrum of excited states above a ground state with Fermi wavevector k_F. Figures 1(a) and (b) show the single-particle dispersion (and ground state occupancy) and the particle–hole pair spectrum of the (spinless) 1D Fermi gas with periodic boundary conditions on a length L. The distinctive one-dimensional feature of the pair spectrum is the non-existence of low-energy pairs for $0 < |k| < 2k_F$; in higher dimensions this region of 'missing' states is filled in. The full spectrum of excited states with zero excited charge (with respect to a ground state with odd charge $N_0 = k_F L/\pi$) is obtained by using figure 1(b) to determine the allowed energies of multiple-pair states, and is shown in figure 1(c).

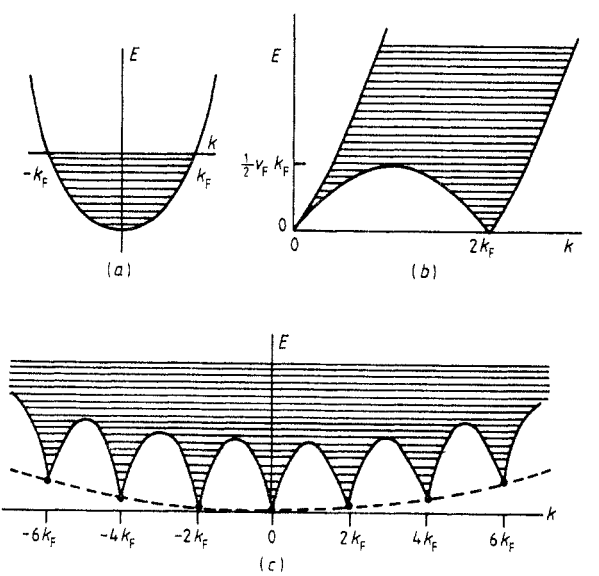

Figure 1. (a) Single-particle spectrum of the free Fermi gas in 1D; (b) Particle–hole pair spectrum; (c) full zero-charge (multiple particle–hole) excitation spectrum (energy differences $E(n) = 2\pi v_F n^2/L$ of extremal states at $k = 2nk_F$ greatly exaggerated).

At low energies $E \ll v_F k_F$, where v_F is the Fermi velocity $d\varepsilon(k_F)/dk$, the spectrum splits up into separate sectors that can be labelled by an even integer J, and can be described as excitations about a set of extremal states with momentum $k_F J$. These states have an energy $\frac{1}{2}(\pi/L)v_F J^2$, and this quadratic energy dependence (valid for energies $\ll v_F k_F$) is shown on a greatly exaggerated scale in figure 1(*c*). The spectrum of excitations with non-zero but even values of extra charge $(N - N_0)$ is similar, but with k_F replaced by $(k_F + \pi(N - N_0)/L)$, and the addition of a term $\frac{1}{2}(\pi/L)v_F(N - N_0)^2$ to the energy. The spectrum of excitations with odd values of $(N - N_0)$ differs only in that extremal states correspond to odd values of J. The form of the excitation spectrum suggests that at low energies it can be described by linear boson ('sound wave') excitations about the extremal states labelled by integers N and J. Such a classification breaks down at higher energies not only because of the non-linearity of the electron dispersion, but also because there is no longer any unambiguous operational way of assigning the quantum number J.

These observations can be summarised by the hypothesis that the *low-energy* spectrum can be represented by the form (where $b_q^\dagger$ are boson creation operators)

$$H = v_S \sum_q |q|\, b_q^\dagger b_q + \tfrac{1}{2}(\pi/L)[v_N(N - N_0)^2 + v_J J^2] \tag{2.1}$$

$$P = [k_F + (\pi/L)(N - N_0)]J + \sum_q q b_q^\dagger b_q; \qquad k_F = \pi N_0/L, \tag{2.2}$$

where $qL/2\pi = \pm 1, \pm 2, \ldots$; N and J are integers, subject to the selection rule (for periodic boundary conditions)

$$(-1)^J = -(-1)^N. \tag{2.3}$$

The parameters are identified as

$$v_S = v_N = v_J = v_F, \tag{2.4}$$

where v_F is the Fermi velocity. Such a spectral form is obviously compatible with the above discussion, but only has the status of a plausible hypothesis until it has been verified that it gives the correct multiplicity of states. This can in fact be verified, as is shown in the next section, by examination of the *non-interacting Luttinger model*, for which the spectral form (2.1)–(2.3) *holds exactly at all energies.*

Though the three parameters v_S, v_N and v_J are all equal to the Fermi velocity in the case of the non-interacting Fermi gas, they describe quite distinct properties of the spectrum. It is thus natural to wonder whether in fact *interacting* gapless fermion systems also have a low-energy spectrum described by (2.1)–(2.3), but with renormalised and unequal values of the three velocity parameters v_S, v_N and v_J. This can in fact be confirmed by the study of the *interacting Luttinger model* (described in § 4, which is explicitly soluble. Though it has the feature that v_S, v_N and v_J are no longer equal, they are not independent, and their ratios are determined by a parameter that characterises the essential interaction strength and low-energy physical properties such as the asymptotic forms of the various correlation functions. I will argue that these relations, together with the spectral form (2.1)–(2.3) are universally valid for the description of the low-energy properties of gapless interacting one-dimensional spinless fermion systems. The assignment of different values to the parameters v_S, v_N and v_J in this 'Luttinger liquid theory' will be analogous to the assignment of different effective masses to the quasiparticles for the characterisation of different physical properties in Fermi liquid theory.

3. The Luttinger model and its solution: I. The non-interacting limit

3.1. Historical development

The Luttinger (1963) model is an exactly soluble model of interacting fermions in one dimension with the following key features:

(i) its elementary excitations are non-interacting bosons;
(ii) the mean fermion current j is a good quantum number;
(iii) all its correlation functions can be explicitly evaluated.

The complete solubility of this model only emerged over a period of a decade; because the resolution of certain ambiguities in versions of the solution developed in the literature over this period turned out to be a key step in the work reported in this series of papers, I will merely cite some of the key papers in the literature, and then present a detailed version of the solution without further reference to its historical development.

The model was proposed by Luttinger (1963), but this first step in its correct solution was taken by Mattis and Lieb (1965), who discovered the free boson elementary excitations. Soon after, Overhauser (1965) pointed out that these bosons could be used to construct a complete set of eigenstates. Theumann (1967) and Dover (1968) gave early calculations of the single-particle correlation function, but the systematic calculation of correlation functions became trivial after the simultaneous discovery of the existence of a simple representation of the fermion operators in terms of the boson fields by Mattis (1974) and Luther and Peschel (1974). In fact, these fields are not on their own sufficient for the full construction of fermion operators in the diagonal basis, and both these early forms have problems associated with the characterisation of $q = 0$ modes. In particular, Luther and Peschel (1974) introduced a certain cut-off parameter α in their version, with the stipulation that it only became an exact operator identity in the limit $\alpha \to 0$. The necessity for any such limiting procedure has been entirely eliminated in the exact formulation reviewed below. The first completely precise formulation in the solid-state literature (though from a field-theory viewpoint) was given by Heidenreich *et al* (1975), though there has been an entirely parallel development in the field-theoretical literature on the related 'massless Thirring model' which I will not review here. The first construction of the important unitary charge-raising operators in terms of the bare fermions was apparently given by Haldane (1979). An important paper essentially parallel to, but not part of, the above developments is that of Dzyaloshinskii and Larkin (1973) who studied the spin-$\frac{1}{2}$ version of the (originally spinless-fermion) model, and provided an interpretation of the Mattis–Lieb solution from the point of view of conventional many-body diagrammatic perturbation theory. Similarly, Everts and Schulz (1974) have shown how the power-law character of the correlation functions can be simply recovered by the standard equation-of-motion techniques. Below, I give a description of the spinless fermion form of the model; the simple extension to the spin-$\frac{1}{2}$ case will be discussed elsewhere.

3.2. The fermion description

It is useful to begin a discussion of the Luttinger model by characterising its Hilbert space: this is *not* the usual electron Hilbert space, but has been expanded to include a branch of *'positron'* states as well. This second, unphysical set of fermions will require high energies for their excitation, so will not qualitatively affect low-energy properties, but are absolutely necessary for the construction of the new basis of eigenstates given

here. Note that definition of the Hilbert space does *not* require any precise specification of the electron and positron dispersions, only that these energies are bounded below, and increase without limit as the momentum $|k| \to \infty$. The model is defined on a finite ring of length L; only periodic fermion boundary conditions will be considered. It is then useful in developing the formalism to take the ground-state charge $N_0 = k_F L/\pi$ to be *odd*, so the ground state is non-degenerate (this restriction is eventually dropped). The Hilbert space worked in is spanned by the set of finite-energy eigenstates of the free Luttinger model, measured from a ground state with electron states from $-k_F$ to k_F filled, and all positron states empty.

A correct definition of the Hilbert space is required before any operator acting in it, such as the Hamiltonian, is defined. It gives meaning to 'operator identities' such as $\hat{A} = \hat{B}$, shorthand for $\langle\alpha|\hat{A}|\beta\rangle = \langle\alpha|\hat{B}|\beta\rangle$ for all $|\alpha\rangle$, $|\beta\rangle$ forming a set that spans the Hilbert space. Operators are only well defined if $\langle\alpha|\hat{A}|\beta\rangle$ is finite for all α, β; the problem of ill-defined operators does not arise in finite-dimensional Hilbert spaces such as in lattice systems, but problems can arise with infinite-dimensional spaces arising from continuum problems. This type of problem flawed Luttinger's original solution of the model. One standard way to ensure all operators worked with are finite is to consider only quantities that are normal-ordered in a set of creation operators that create excited states out of the ground state.

Instead of working directly with charge $+1$ electron states, and charge -1 positron states, it is useful to describe the Luttinger model in terms of charge $+1$ 'right-' and 'left-moving' fermions labelled by $p = \pm 1$ (note that this label should *not* be confused with a momentum label, for which k is used here). The kinetic part of the Luttinger Hamiltonian is then given (using units where $\hbar = 1$) by

$$H^0 = v_F \sum_{kp} (pk - k_F)(n_{kp} - \langle n_{kp}\rangle_0) \qquad \langle n_{kp}\rangle_0 = \theta(k_F - pk) \tag{3.1}$$

$$= v_F \int_0^L \mathrm{d}x \sum_p {:}\,\psi_p^\dagger(x)(\mathrm{i}p\nabla - k_F)\psi_p(x)\,{:} \tag{3.2}$$

where :(...): means fermion normal-ordering with respect to the ground state of (3.1). The term k_F is essentially a chemical potential to fix the ground state charge. The spectral diagrams corresponding to figure 1 for the non-interacting Luttinger model (3.1) are shown in figure 2. The fermion field $\psi_p^\dagger(x)$ is given by

$$\psi_p^\dagger(x) = \lim_{\varepsilon \to 0^+} \left[L^{-1/2} \sum_k \mathrm{e}^{\mathrm{i}kx} \exp(-\varepsilon|kL/2\pi|)\, c_{kp}^\dagger \right]. \tag{3.3}$$

The limiting procedure $\varepsilon \to 0^+$ is usually left implicit, but has been explicitly included here to emphasise the similarity with a analogous construction that will appear later. It is necessary for the definition of the periodic delta function generated by the anticommutation relations:

$$\{\psi_p^\dagger(x), \psi_{p'}(x')\} = \delta_{pp'} \left[\lim_{\varepsilon \to 0^+} \left(L^{-1} \sum_k \exp[-k(x - x')] \exp(-\varepsilon|kL/\pi|) \right) \right]$$

$$= L^{-1}\delta_{pp'} \lim_{\varepsilon \to 0^+} [(1 - z\mathrm{e}^{-2\varepsilon})^{-1} + (1 - z^*\mathrm{e}^{-2\varepsilon})^{-1} - 1]$$

$$= \delta_{pp'} \sum_{n=-\infty}^{\infty} \delta(x - x' - nL). \tag{3.4}$$

where $z = \exp(2\pi\mathrm{i}x/L)$, and allowed k-values in the sum satisfy $\exp(\mathrm{i}kL) = 1$.

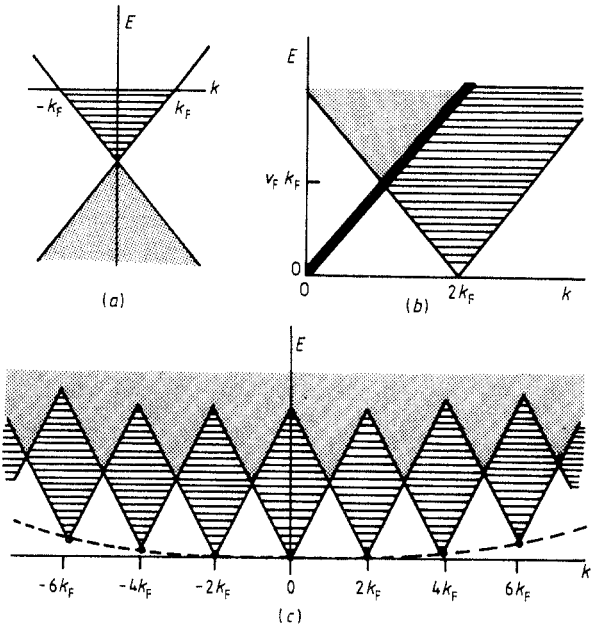

Figure 2. Diagrams corresponding to figure 1, this time for the spectrum of the non-interacting Luttinger model (3.1). Dotted areas indicate the presence of 'unphysical' states involving excited 'positrons'.

Note that the quantity ε appears as a dimensionless infinitesimal quantity necessary for controlling the sums over the infinite range of values of k, and in no way plays the role of a 'cut-off length'.

The fundamental electron and positron fields are related to the fields $\psi_p^\dagger(x)$ in a *non-local* way: in terms of the c_{kp}, they are given by

$$\psi^\dagger(x) = L^{-1/2} \sum_{kp} \theta(kp)\, e^{ikx} c_{kp}^\dagger$$

$$\overline{\psi}^\dagger(x) = L^{-1/2} \sum_{kp} \theta(-kp)\, e^{-ikx} c_{kp}. \tag{3.5}$$

When expressed in terms of $\psi_p^\dagger(x)$, the electron field $\psi^\dagger(x)$ is given by

$$\psi^\dagger(x) = \frac{1}{2\pi i} \sum_p p \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dy\, K(y)\, \psi_p^\dagger(x+y), \tag{3.6}$$

where $K(y) = (\pi/L)[\tan(\pi y/L)]^{-1}$.

3.3. *The boson description: construction from fermion operators*

A central role in the theory of the Luttinger model is played by the density operator for type-p fermions:

$$\rho_{qp} = \sum_k c_{k+qp}^\dagger c_{kp} \qquad (q \neq 0)$$

$$\equiv N_p = \sum_k n_{kp} - \langle n_{kp} \rangle_0 \qquad (q = 0). \tag{3.7}$$

Note that the subtraction of the (infinite) ground state density of type-p fermions means

that the $q = 0$ component of ρ_{qp} is a well defined operator in the sense discussed above: this procedure is equivalent to normal-ordering in the fermion variables. The commutation relations of the ρ_{qp} are

$$[\rho_{qp}, \rho_{-q'p'}] = \delta_{pp'}\delta_{qq'}(Lpq/2\pi). \tag{3.8}$$

This is easily established by direct evaluation of the commutator, then writing pair operators $c^\dagger_{kp}c_{k'p}$ as $(c^\dagger_{kp}c_{k'p} - \langle c^\dagger_{kp}c_{k'p}\rangle_0) + \langle c^\dagger_{kp}c_{k'p}\rangle_0$, where $\langle c^\dagger_{kp}c_{k'p}\rangle_0 \equiv \delta_{kk'}\langle n_{kp}\rangle_0$: this guarantees that operator quantities are effectively normal-ordered, and hence well defined, so they can be manipulated safely. The commutator (3.8) trivially vanishes when $p \neq p'$; for equal indices p, it is given by

$$\hat{O}_{pqq'} - \hat{O}_{pqq'} + \delta_{qq'}\sum_k (\langle n_{k+qp}\rangle_0 - \langle n_{kp}\rangle_0);$$

$$\hat{O}_{pqq'} \equiv \sum_k (c^\dagger_{k+q-q'p}c_{kp} - \langle c^\dagger_{k+q-q'p}c_{kp}\rangle_0).$$

Being well defined, the operators $\hat{O}_{pqq'}$ can safely be cancelled, and the remaining c-number term gives the RHS of (3.8). A second important commutation relation is that with the fermion fields:

$$[\rho_{qp}, \psi^\dagger_{p'}(x)] = \delta_{pp'}\, e^{-iqx}\psi^\dagger_p(x). \tag{3.9}$$

It will prove useful to define the following partial Fourier transforms:

$$\rho^{(\pm)}_p(x) = L^{-1}\sum_q \theta(\pm pq)\, e^{iqx}\, \rho_{qp}. \tag{3.10}$$

These operators have the property that $(\rho^{(-)}_p(x))^\dagger = \rho^{(+)}_p(x)$, and that $\rho^{(-)}_p(x)$ annihilates the vacuum state of (3.1). They satisfy periodic boundary conditions, $\rho^{(\pm)}_p(x + L) = \rho^{(\pm)}_p(x)$. The local density of p-type fermions, with respect to the ground state density, is

$$\lim_{a\to 0}[\psi^\dagger_p(x + a)\psi_p(x) - \langle\psi^\dagger_p(x + a)\psi_p(x)\rangle_0] \equiv \rho_p(x) = [\rho^{(+)}_p(x) + \rho^{(-)}_p(x)]. \tag{3.11}$$

The limiting procedure $a \to 0$ is to avoid direct reference to the infinite quantity $\langle\psi^\dagger_p(x)\psi_p(x)\rangle_0$.

The commutation algebra (3.8) immediately suggests the construction of boson operators. For $q \neq 0$,

$$a^\dagger_q = (2\pi/L|q|)^{1/2}\sum_p \theta(pq)\rho_{qp} \qquad (q \neq 0; q = 2\pi n/L, n = \pm 1, \pm 2, \ldots). \tag{3.12}$$

These obey exact boson commutation relations, and have the property that a_q annihilates the ground state of (3.1). There is no $q = 0$ boson mode (indeed, the form (3.12) is undefined at $q = 0$); the $q = 0$ mode is represented by the number operator N_p, which commutes with the bosons $a^\dagger_q$. The density operators are then expressed by

$$\rho_{qp} = N_p\delta_{p0} + (L|q|/2\pi)^{1/2}\{\theta(pq)a^\dagger_q + \theta(-pq)a_{-q}\}. \tag{3.13}$$

The algebra of the operators $(a^\dagger_q, a_q, N_p)$ that have been constructed so far is incomplete: it lacks a ladder operator U_p that raises the fermion charge N_p in unit steps, while commuting with the bosons a_q. The ladder of allowed values of N_p has no upper or lower limit, so (in contrast to the case of boson or finite spin ladder algebras) the number operator cannot be expressed in terms of the raising operator and its conjugate lowering operator. The raising operator can be chosen to be *unitary*, $U_p^{-1} = (U_p)^\dagger$. Finally, the

fermion nature of the ladder operators means that U_p will *anti*commute with U_{-p} and U_{-p}^{-1}.

It is useful to study the construction of the ladder operators U_p in detail. It is important that these operators be given in a well defined form. A heuristic understanding of their form can be gained from the following argument. A special subset of eigenstates of (3.1) are those with occupations $n_{kp} = \theta(k_F + (2\pi N_p/L) - pk)$; these states $|\{N_p\}\rangle$ include the vacuum, and share with it the property that they are annihilated by a_q. The ladder operator U_p must have the property that

$$U_p |N_p, N_{-p}\rangle = \eta(p, N_p, N_{-p}) |N_p + 1, N_{-p}\rangle, \quad \eta = \pm 1. \dagger$$

A construction with this property is

$$\sum_k c_{kp}^\dagger \delta(pk - [k_F + (2N_p + 1)\pi/L]).$$

Writing this in a more symmetrical form, using an integral representation of the Kronecker delta function, this becomes:

$$U_p = L^{-1/2} \int_0^L dx \exp(-ipk_F x) \exp[-i\phi_p^\dagger(x)] \psi_p^\dagger(x) \exp[-i\phi_p(x)];$$

$$\phi_p(x) = p(\pi x/L) N_p. \tag{3.14}$$

The ladder operator U_p must also have the property that it commutes with the boson operators a_q, or equivalently with the density operators ρ_{qp}, when $q \neq 0$. The commutation relation (3.9) means that the trial form above does *not* have this property. However, it would if the operators $\phi_p(x)$ were modified so that

$$[\rho_{qp}, \phi_p(x)] = -i\delta_{pp'}\theta(pq)\, e^{-iqx}(1 - \delta_{q0}). \tag{3.15}$$

Because this commutator is a c-number,

$$[\rho_{qp}, \exp(-i\phi_{p'}(x))] = -\theta(pq)(\delta_{pp'} e^{-iqx}) \exp\{-i\phi_p(x)\}(1 - \delta_{q0}),$$
$$[\rho_{pq}, \exp\{-i\phi_{p'}^\dagger(x)\}] = -\theta(-pq)(\delta_{pp'} e^{-iqx}) \exp\{-i\phi_p^\dagger(x)\}(1 - \delta_{q0}). \tag{3.16}$$

When $q \neq 0$, these terms exactly counterbalance the commutator (3.9), so when U_p is given by the form (3.14) with a $\phi_p(x)$ that satisfies (3.15),

$$[\rho_{qp}, U_{p'}] = \delta_{pp'}\delta_{q0} U_p. \tag{3.17}$$

The explicit construction of the quantities $\phi_p(x)$ is now easily given; using the property (3.8):

$$\phi_p(x) = (2\pi p/L)\left(\tfrac{1}{2} x N_p + i \sum_{q=0} \theta(-pq)(e^{-iqx}/q)\rho_{qp}\right). \tag{3.18}$$

The first term in this is just the $q = 0$ component of the sum, with the limit $q \to 0$ properly taken, so this can be re-expressed as

$$\phi_p(x) = \lim_{\varepsilon \to 0^+} \left((2\pi i p/L) \sum_q \theta(-pq)\, e^{-iqx}/q) \exp[-\varepsilon(|q|L/2\pi)]\, \rho_{qp}\right). \tag{3.19}$$

The limiting procedure $\varepsilon \to 0^+$ has been included in order to properly define the sum in much the same spirit as in equation (3.3), and it will be needed to properly define the periodic delta function. ε is a positive dimensionless infinitesimal, and in no way should

† η depends on the ordering convention used in constructing $|\{N_p\}\rangle$.

be interpreted as a cut-off length, despite the formal similarity to the length parameter α introduced by Luther and Peschel (1974). The field $\phi_p(x)$ is easily found to have the following properties:

$$\nabla\phi_p(x) = 2\pi p\rho_p^{(-)}(x) \tag{3.20}$$

$$[\phi_p(x), \phi_{p'}(x')] = [\phi_p^\dagger(x), \phi_{p'}^\dagger(x')] = 0;$$

$$[\phi_p(x), \phi_{p'}^\dagger(x')] = \lim_{\varepsilon\to 0^+} \{\delta_{pp'}(-\ln(1 - e^{-2\varepsilon z^{-2p}}))\} \tag{3.21}$$

where $z \equiv \exp(i\pi(x - x')/L)$. $\phi_p(x)$ has the property that it annihilates the vacuum state of (3.1); this is easily seen when $\phi_p(x)$ is written in terms of N_p and a_q:

$$\phi_p(x) = p(\pi x/L)N_p + i\sum_{q\neq 0} \theta(pq)(2\pi/L|q|)^{1/2} e^{-iqx}a_q. \tag{3.22}$$

This means that the operator U_p defined by (3.14) plus (3.19) is normal-ordered in the boson operators that annihilate the vacuum state of (3.1). As will shortly be seen, this guarantees that this construction of U_p defines a well defined operator.

It is useful to give a representation of the unitary operators U_p in terms of Hermitian phase variables $\bar{\theta}_p = \bar{\theta}_p^\dagger$, conjugate to N_p:

$$U_p = (-1)^{(\frac{1}{2}pN_{-p})}\bar{U}_p, \qquad \bar{U}_p = \exp(i\bar{\theta}_p); \tag{3.23}$$

$$[N_p, \bar{\theta}_{p'}] = i\delta_{pp'}, \qquad [\bar{\theta}_p, \bar{\theta}_{p'}] = [N_p, N_{p'}] = 0. \tag{3.24}$$

The prefactor $(-1)^{(\frac{1}{2}pN_{-p})}$ ensures that U_p and U_{-p} anticommute, so that the unitary operators $\tilde{U}_p$ and $\tilde{U}_{-p}$ can commute. This choice of anticommutation factor corresponds to a particular ordering convention in constructing states $|\{N_p\}\rangle$ in terms of fermion operators; other choices are possible.

For some purposes, it is useful to introduce a *local* phase field $\theta_p(x)$:

$$\theta_p(x) = \bar{\theta}_p + \phi_p(x) + \phi_p^\dagger(x). \tag{3.25}$$

This has the property

$$\nabla\theta_p(x) = 2\pi p\rho_p(x). \tag{3.26}$$

It can easily be seen that $\rho_p(x)$ and $\theta_p(x)$ are canonical conjugate fields:

$$[\rho_p(x), \theta_{p'}(x')] = i\delta_{pp'}\sum_n \delta(x - x' + nL). \tag{3.27}$$

3.4. Boson form of the Hamiltonian

So far, no use whatsoever has been made of the fact that the Hamiltonian (3.1) has a linear fermion dispersion, and the above discussion has only depended on identification of its vacuum state and the structure of the associated Hilbert space. The linear spectral property will now be used to construct a new basis of eigenstates that will be shown to be complete and hence to span the Hilbert space defined by all finite-energy eigenstates of (3.1). The operators N_p commute with (3.1), but $a_q^\dagger$ and U_p do not: first note the commutator of the density operator with the Hamiltonian:

$$[H^0, \rho_{qp}] = v_F pq\rho_{qp}. \tag{3.28}$$

With this, the definition (3.12) of the boson operators a_q leads to

$$[H^0, a_q^\dagger] = v_F |q| a_q^\dagger. \tag{3.29}$$

Instead of attempting to directly evaluate the commutator $[H^0, U_p]$, I use the following argument: the special set of states $|\{N_p\}\rangle$ can be constructed from the vacuum by acting on it with U_p:

$$|\{N_p\}\rangle = \pm \prod_p (U_p)^{N_p} |0\rangle; \tag{3.30}$$

this is verified by explicitly showing that U_p as constructed indeed has the property $U_p|N_p, N_{-p}\rangle = \eta|N_p + 1, N_p\rangle$. Because a_q annihilates the states $|\{N_p\}\rangle$, (3.22) implies that

$$U_p|\{N_p\}\rangle = L^{-1} \sum_k \int_0^L dx \exp[i(k - pk_F)x] \exp[-i\phi_p^\dagger(x)] c_{kp}^\dagger \exp[-ip(\pi x/L)N_p] |\{N_p\}\rangle$$

$$= \prod_{q \neq 0} \left(1 + \theta(pq) \sum_{n_q=1}^{\infty} (2\pi/L|q|)^{\frac{1}{2}n_q} (n_q!)^{-1} (a_q^\dagger)^{n_q}\right)$$

$$\times \sum_k c_{kp}^\dagger \delta\left(pk - [k_F + (2N_p + 1)\pi/L] + \sum_q |q| n_q\right) |\{N_p\}\rangle.$$

The Kronecker delta ensures that no states containing boson excitations survive in this sum, and (3.30) is verified.

The energies of the eigenstates $|\{N_p\}\rangle$ are easily obtained by examining their construction: $E(\{N_p\}) = v_F(\pi/L) \Sigma_p (N_p)^2$. A larger set of eigenstates is obtained by acting these with the boson operators:

$$|\{N_p\}, \{n_q\}\rangle = \prod_{q \neq 0} \left(\frac{(a_q^\dagger)^{n_q}}{(n_q!)^{1/2}}\right) \prod_p (U_p)^{N_p} |0\rangle. \tag{3.31}$$

In its action on *these* eigenstates, the Hamiltonian is given by

$$H^0 = v_F \left[\sum_q |q| a_q^\dagger a_q + (\pi/L) \sum_p (N_p)^2\right]. \tag{3.32}$$

This can be written in terms of the phase fields $\theta_p(x)$ as

$$H^0 = v_F \frac{1}{\pi} \int_0^L dx \sum_p :(\nabla\theta_p(x))^2:, \tag{3.33}$$

where *boson* normal-ordering is implied. The momentum operator is similarly given by

$$P = \sum_p p[k_F + (\pi/L)N_p]N_p + \sum_q q a_q^\dagger a_q \tag{3.34}$$

$$= \frac{1}{\pi} \int_0^L dx \sum_p p[k_F \nabla\theta_p(x) + :(\nabla\theta_p(x))^2:]. \tag{3.35}$$

The question arises: *are the eigenstates* (3.31) *a complete set?* If so, U_p is proved to be a well defined operator, and (3.32) and (3.34) have the status of identities in the full Hilbert space based on the vacuum of (3.1). The possibility of two such different sets of eigenstates of the free Luttinger Hamiltonian (3.1) arises because of the high degree of degeneracy of the spectrum due to the linear dispersion: all eigenstates with even (odd) fermion charge $N - N_0$ have energies that are even (odd) multiples of $\pi v_F/L$ with respect to the ground state. One way to check the completeness of the set (3.31) is to directly

investigate the degeneracy of states at a given energy. An equivalent, more elegant, way is to compute the grand partition sum of the Hamiltonian at arbitrary inverse temperature β, first using the 'obvious' set of fermion excitation states, then the set (3.31). This is a sum over positive definite quantities, so if any states were missing from (3.31), the result of the second calculation would be less than the first.

Defining $w = \exp(-\beta\pi v_F/L)$, the direct evaluation of the partition function using the free fermion basis gives

$$Z(w) = \left(\prod_{n=1}^{\infty}(1 + w^{2n-1})^2\right)^2. \tag{3.36a}$$

Using the set (3.31), one obtains

$$Z(w) = \left(\prod_{n=1}^{\infty}(1 - w^{2n})^{-1}\right)^2\left(\sum_{m=-\infty}^{\infty} w^{(n^2)}\right)^2. \tag{3.36b}$$

These apparently different expressions are in fact both equal, since the elliptic theta function $\vartheta_3(0; w)$ (Gradsteyn and Ryzhik 1965, p 921) has both a series and a product representation:

$$\vartheta_3(0; w) = \sum_{n=-\infty}^{\infty} w^{(n^2)} = \prod_{n=1}^{\infty}(1 + w^{2n-1})^2(1 - w^{2n}).$$

The set (3.31) is thus complete, and spans the full Hilbert space.

3.5. *Boson form of fermion operators*

With the completeness of the set of eigenstates (3.31) established, the remaining task is to construct the representation of the fermion operators $\psi_p^\dagger(x)$ in this basis. The ground work has been laid: $\psi_p^\dagger(x)$ is trivially obtained by inverting the expression (3.14) for U_p:

$$\begin{aligned}\psi_p^\dagger(x) &= L^{-1/2}\exp(ipk_Fx)\{\exp[i\phi_p^\dagger(x)]U_p\exp[i\phi_p(x)]\}\\ &= (-1)^{(\frac{1}{2}pN-p)}L^{-1/2}\exp(ipk_Fx)\{\exp[i\phi_p^\dagger(x)]\exp(i\bar{\theta}_p)\exp[i\phi_p(x)]\},\end{aligned} \tag{3.37}$$

where $\phi_p(x)$ is now defined directly by (3.22). This is an explicitly well defined operator, since it is normal-ordered in terms of the bosons a_q. The anticommutation relations can be explicitly verified; here the limiting procedure defined in (3.19) and (3.21) is required: the procedure is to construct the anticommutators, and then re-normal-order the resulting products in terms of the bosons, so they become explicitly well defined operators that can be manipulated and cancelled. The anticommutation of fields with different labels p is trivially assured by the anticommuting properties of U_p; for equal p, the anticommutator $\{\psi_p^\dagger(x), \psi_p^\dagger(x')\}$ is given by

$$\begin{aligned}&L^{-1}\exp[ipk_F(x + x')]\exp[i\hat{O}_1^\dagger(x, x')]U_p^2\exp[i\hat{O}_1(x, x')]F_1(x, x')\\ &\hat{O}_1(x, x') = \phi_p(x) + \phi_p(x')\\ &F_1(x, x') = G_1(x - x') + G_1(x' - x)\\ &G_1(x - x') = \exp[i\pi p(x - x')/L]\exp\{-[\phi_p(x), \phi_p^\dagger(x')]\}.\end{aligned} \tag{3.38}$$

The c-number function $F_1(x, x')$ is multiplying a well defined (i.e. normal-ordered)

operator expression. $F_1(x,x')$ can be evaluated using equation (3.21): setting $z = \exp[i\pi(x-x')/L]$,

$$F_1 = \lim_{\varepsilon\to 0^+} \{z^p(1-e^{-2\varepsilon}z^{-2p}) + z^{-p}(1-e^{-2\varepsilon}z^{2p})\} = 0. \tag{3.39}$$

The anticommutator $\{\psi_p^\dagger(x), \psi_p^\dagger(x')\}$ thus vanishes correctly. The anticommutator $\{\psi_p(x), \psi_p^\dagger(x')\}$ is given by

$$\begin{aligned} &L^{-1}\exp[ipk_F(x'-x)]\exp[i\hat{O}_2^\dagger(x,x')]\exp[i\hat{O}_2(x,x')]F_2(x,x') \\ &\hat{O}_2^\dagger(x,x') = \phi_p(x) - \phi_p(x') \\ &F_2(x,x') = G_2(x-x') + G_2(x'-x) \\ &G_2(x-x') = \exp[i\pi p(x-x')/L]\exp\{+[\phi_p(x), \phi_p^\dagger(x')]\}. \end{aligned} \tag{3.40}$$

Again this is a normal-ordered operator expression, times a c-number function $F_2(x,x')$. Again using (3.21),

$$\begin{aligned} F_2 &= \lim_{\varepsilon\to 0^+} \{z^p(1-e^{-2\varepsilon}z^{-2p})^{-1} + z^{-p}(1-e^{-2\varepsilon}z^{2p})^{-1}\} \\ &= L\sum_{n=-\infty}^{\infty}(-1)^n\delta(x-x'+nL). \end{aligned} \tag{3.41}$$

when $x - x' = nL$, the operator-valued expression that multiplies F_2 in (3.41) takes the simple c-number values $L^{-1}\exp(-inpk_FL) = L^{-1}(-1)^n$. The anticommutator is thus correctly given by the periodic delta function as in (3.3).

This completes the derivation of the operator algebra needed to describe the model using the alternative basis set of eigenstates (3.31). This algebra is a precise tool, and I now use it to recover the expressions (3.32) and (3.34) for H^0 and P directly from the fermion representation (3.37). Consider the quantity

$$\int_0^L dx \exp(-ipk_Fa)\,\psi_p^\dagger(x+\tfrac{1}{2}a)\psi_p(x-\tfrac{1}{2}a). \tag{3.42}$$

Using the expression (3.3) for $\psi_p(x)$, this is easily found to be

$$L(2\pi ipa')^{-1} + \sum_k \exp[i(k-pk_F)a](n_{kp} - \langle n_{kp}\rangle_0) \tag{3.43}$$

where $a' = (L/\pi)\sin(\pi a/L)$. Using the alternative expression (3.37), and then normal-ordering, it is found to be

$$\begin{aligned} &(2\pi ipa')^{-1}\left(L + \int_0^L dx\{\exp[2\pi ip(a/L)N_p]\exp[i\Phi_p^\dagger(x)]\exp[i\Phi_p(x)] - 1\}\right) \\ &\Phi_p(x) = i\sum_q (2\pi/L\,|q|)^{1/2}e^{-iqx}\theta(qp)\,2\sin(\tfrac{1}{2}qa)a_q. \end{aligned} \tag{3.44}$$

Cancelling the divergent term $L(2\pi ipa')^{-1}$ and comparing the term $O(a)$ in the expansions of the two expressions, one directly obtains

$$\sum_k (pk - k_F)(n_{kp} - \langle n_{kp}\rangle_0) = (\pi/L)N_p^2 + \sum_q pq\,\theta(qp)a_q^\dagger a_q. \tag{3.45}$$

The expressions for H^0 and P are now trivial to obtain.

3.6. The charge and current formalism

So far, the formalism has been developed in terms of operators labelled by $p = \pm 1$, corresponding to the right- and left-going fermions. It is convenient for some purposes to introduce the symmetric and antisymmetric combinations, labelled by N and J respectively, which will be related to charge and current variables. It is also useful to include the ground state electronic charge (number of electrons minus number of positrons) $N_0 = k_F L/\pi$ in the charge variables. The following combinations are defined:

$$N = N_0 + \sum_p N_p \qquad J = \sum_p p N_p \tag{3.46}$$

$$\bar{\theta}_N = \sum_p \bar{\theta}_p \qquad \bar{\theta}_J = \sum_p p\bar{\theta}_p \tag{3.47}$$

$$\rho_N(x) = (N_0/L) + \sum_p \rho_p(x) \qquad \rho_J(x) = \sum_p p\rho_p(x) \tag{3.48}$$

$$\phi_N(x) = \sum_p \phi_p(x) \qquad \phi_J(x) = \pi(N_0/L)x + \sum_p p\phi_p(x). \tag{3.49}$$

Phase fields $\theta_N(x)$ and $\theta_J(x)$ are then defined by, e.g.,

$$\theta_N(x) = \bar{\theta}_N + \phi_N(x) + \phi_N^\dagger(x). \tag{3.50}$$

The following relations are found:

$$\nabla\theta_N(x) = 2\pi\rho_J(x) \qquad \nabla\theta_J(x) = 2\pi\rho_N(x) \tag{3.51}$$

$$[\rho_N(x), \theta_N(x')] = [\rho_J(x), \theta_J(x')] = \mathrm{i}\sum_n \delta(x - x' + nL); \tag{3.52}$$

$$[\rho_N(x), \theta_J(x')] = [\rho_J(x), \theta_N(x')] = 0. \tag{3.53}$$

The fields $(\rho_N(x), \theta_N(x))$ and $(\rho_J(x), \theta_J(x))$ are canonically conjugate pairs. Note however that $[\rho_N(x), \rho_J(x')]$ and $[\theta_J(x), \theta_N(x')]$ do *not* vanish, except at equal positions, $x = x'$. On the other hand, $[\theta_N(x), \theta_N(x')]$, $[\rho_N(x), \rho_N(x')]$, etc, *do* vanish.

When the quantities $\rho_N(x)$, $\phi_N(x)$, etc are expressed in terms of boson variables, they are explicitly given by

$$\begin{aligned}
\rho_N(x) &= (N/L) + \sum_q (|q|/2\pi L)^{1/2}\, \mathrm{e}^{\mathrm{i}qx}\,(a_q^\dagger + a_{-q}) \\
\phi_N(x) &= \pi(J/L)x + \mathrm{i}\sum_q (2\pi/L\,|q|)^{1/2}\, \mathrm{e}^{-\mathrm{i}qx} a_q \\
\rho_J(x) &= (J/L) + \sum_q (|q|/2\pi L)^{1/2}\, \mathrm{sgn}(q)\, \mathrm{e}^{\mathrm{i}qx}(a_q^\dagger - a_{-q}) \\
\phi_J(x) &= \pi(N/L)x + \mathrm{i}\sum_q (2\pi/L\,|q|)^{1/2}\, \mathrm{sgn}(q)\, \mathrm{e}^{-\mathrm{i}qx} a_q.
\end{aligned} \tag{3.54}$$

Note how $\mathrm{sgn}(q)$ characteristically appears in the boson part of J-labelled quantities.

The commuting unitary operators $U_N = \exp(\mathrm{i}\bar{\theta}_N)$ and $U_J = \exp(\mathrm{i}\bar{\theta}_J)$ respectively raise N and J by one. In this basis, the fermion field operator $\psi_p^\dagger(x)$becomes

$$\begin{aligned}
\psi_p^\dagger(x) = L^{-\frac{1}{2}}(-1)^{\frac{1}{2}(pJ-N)}\{&\exp[\tfrac{1}{2}\mathrm{i}(p\phi_J^\dagger + \phi_N^\dagger)]\, \exp[\tfrac{1}{2}\mathrm{i}(p\bar{\theta}_J + \bar{\theta}_N)] \\
&\times \exp[\tfrac{1}{2}\mathrm{i}(p\phi_J + \phi_N)]\}.
\end{aligned} \tag{3.55}$$

The dependence on k_F has been absorbed into the definition of $\phi_J(x)$.

The Hamiltonian takes the form

$$H^0 = v_F \left(\sum_q |q|\, a_q^\dagger a_q + \tfrac{1}{2}(\pi/L)\left((N - N_0)^2 + J^2\right) \right) \qquad (-1)^J = -(-1)^N; \tag{3.56}$$

$$P = [k_F + \pi(N - N_0)/L]\, J + \sum_q q a_q^\dagger a_q \qquad k_F = \pi N_0/L. \tag{3.57}$$

This is just the form postulated in § 2. The selection rule linking allowed values of J and N arises because N_p and N_{-p} are both integral.

In the phase field variables, the Hamiltonian can be written

$$H = v_F \frac{1}{\pi} \int_0^L \mathrm{d}x :(\nabla\theta_N(x))^2 + (\nabla\theta_J(x))^2:$$

$$P = \frac{1}{\pi} \int_0^L \mathrm{d}x :\nabla\theta_N(x)\nabla\theta_J(x): + \text{HC}. \tag{3.58}$$

Since $\rho_N(x) = (2\pi)^{-1}\nabla\theta_J(x)$ is the canonical conjugate to $\theta_N(x)$, and $\rho_J(x)$ to $\theta_J(x)$, the Hamiltonian (3.58) can be written as a Klein–Gordon field Hamiltonian in either the N or the J variables. The periodic fermion boundary conditions that must be satisfied by (3.55) imply that $\exp[\mathrm{i}\theta_N(x + L)] = \exp[\mathrm{i}\theta_N(x)]$, etc, so $\theta_N(x + L) = 2\pi J + \theta_N(x + L)$, and $\theta_J(x + L) = 2\pi N + \theta_J(x)$. The quantum numbers N and J thus can be related to *topological* excitations of the phase fields $\theta_N(x)$ and $\theta_J(x)$, while the bosons relate to their small fluctuations.

The physical interpretation of the quantum number N is simple: it is just the total electronic charge (electrons minus positrons). Similarly, it will now be shown that J is proportional to the mean *current*. It would be tempting to identify $\rho_N(x)$ with the *local* charge density operator $\rho(x)$; unfortunately, this is not correct, due to the non-local relation between the electron field and $\psi_p^\dagger(x)$. The fundamental definition of the local electronic density in terms of the electrons and positrons leads to

$$\rho(x) = \rho_N(x) + \tau(x),$$

$$\tau(x) = \sum_{kk'pp'} \exp[\mathrm{i}(k - k')x]\,(-pp')\,\theta(-kk')\,c_{kp}^\dagger c_{k'p'}. \tag{3.59}$$

At low energies, the extra term $\tau(x)$ involves only fluctuations with $q \sim 2k_F$. The fundamental definition of the current $j(x)$ is through the continuity equation for local charge:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(x) \equiv \mathrm{i}[H, \rho(x)] = \nabla j(x). \tag{3.60}$$

The mean current j is then given by

$$j \equiv L^{-1} \int_0^L \mathrm{d}x\, j(x) = \lim_{q\to 0} \{(qL)^{-1}[H, \rho_q]\}, \tag{3.61}$$

ρ_q being the Fourier transform of $\rho(x)$. In a low-energy subspace, and provided k_F is finite, the contribution from $\tau(x)$ can be neglected, and ρ_{Nq} substituted for ρ_q in (3.61). Then it is easily found that

$$j = v_F(J/L). \tag{3.62}$$

Actually, this is exact in the case of the free Luttinger model, but in a more general model extra terms will be present, and a linear relation like (3.62) will only be valid in

a low-energy subspace, where the presence of long-wavelength boson excitations does not affect the current.

Finally, I note a useful low-energy, finite k_F *approximation* for $\tau(x)$:

$$\tau(x) \sim \sum_p \psi_p^\dagger(x)\psi_{-p}(x). \tag{3.63}$$

4. The Luttinger model and its solution: II. The interacting model

The full Luttinger model is obtained by taking the kinetic term (3.1), (3.32) and adding the fermion two-particle interaction:

$$H^1 = (\pi/L)\sum_{pp'q}(V_{1q}\delta_{pp'} + V_{2q}\delta_{p,-p'})\rho_{qp}\rho_{-qp'}. \tag{4.1}$$

The density operators ρ_{qp} are defined by (3.7) and (3.13). The coupling constants $V_{1q} \equiv V_1(|q|R)$ and $V_{2q} \equiv V_2(|q|R)$ have dimensions of velocity. They will be required to satisfy the following conditions:

(i) $V_1(0)$, $V_2(0)$ are finite;
(ii) $V_{2q}/(v_F + V_{1q}) \to 0$ as $|q| \to \infty$, faster than $|q|^{-\frac{1}{2}}$;
(iii) $|V_{2q}| < (v_F + V_{1q})$ for all q.

Conditions (i) could be relaxed somewhat, but this would alter the physics of the model. Conditions (ii) and (iii) are necessary to ensure that the Hilbert space of the model $H^0 + H^1$ remains the same as that of H^0. The conditions (i) and (ii) imply the existence of some length scale R that controls the crossover from the small-q to large-q regimes. The inclusion of this length scale in (4.1) means that V_1 and V_2 can be written as functions with a dimensionless argument. R is an effective range of the interaction in real space.

Using the phase-field formalism of § 3.6, the *low-energy* ($E \ll v_F/R$) form of the Hamiltonian can be written

$$H \simeq \frac{1}{\pi}\int_0^L dx :v_N(\nabla\theta_N(x))^2 + v_J(\nabla\theta_J(x))^2: \tag{4.2}$$

where

$$v_N = v_F + V_1(0) + V_2(0), \qquad v_J = v_F + V_1(0) - V_2(0). \tag{4.3}$$

It is also useful to define the quantities

$$\omega_q = |(v_F + V_{1q})^2 - (V_{2q})^2|^{1/2}|q| \tag{4.4}$$

$$\tanh(2\varphi_q) = -V_{2q}/(v_F + V_{1q}). \tag{4.5}$$

Then the quantities v_S and φ are defined by

$$v_S = \lim_{q\to 0}(\omega_q/|q|); \qquad \varphi = \lim_{q\to 0}(\varphi_q). \tag{4.6}$$

The definitions (4.2)–(4.5) imply the relations

$$v_N = v_S\exp(-2\varphi); \qquad v_J = v_S\exp(2\varphi). \tag{4.7}$$

It will be convenient to represent φ_g as $\varphi g(|q|R)$, where the function $g(y)$ has the properties

$$g(0) = 1; \qquad y^{1/2}g(y) \to 0 \quad \text{as} \quad y \to \infty. \tag{4.8}$$

The conditions (i)–(iii) assure this, and also that φ_q is finite, ω_q is positive definite (except at $q = 0$) and v_N, v_J are positive definite. The model is fully parametrised by L, k_F, ω_q, and φ_q (or φ, R and $g(y)$). R has not been defined up to a multiplicative factor: it should be chosen so the crossover in $g(y)$ is around $y \sim 1$ (a unique definition might be provided by demanding that $(g(y))^2$ is normalised, for example).

When the full Hamiltonian $H^0 + H^1$ is written out in terms of a_q and the number operators N, J, it takes the simple bilinear form

$$H = -\tfrac{1}{2}\Big(\sum_q v_F|q|\Big) + \tfrac{1}{2}(\pi/L)(v_N N^2 + v_J J^2)$$
$$+ \tfrac{1}{2}\sum_q |q|[(v_F + V_{1q})(a_q^\dagger a_q + a_q a_q^\dagger) + V_{2q}(a_q^\dagger a_{-q}^\dagger + a_q a_{-q})]. \tag{4.9}$$

This is trivially diagonalised by a Bogoliubov transformation. The new ground state is given by

$$|GS\rangle = \exp[-(A^2 L/R)] \exp\Big(\sum_{q>0} \tanh(\varphi_q) a_q^\dagger a_{-q}^\dagger\Big)\,|0,0\rangle$$

$$A^2(\varphi g(y)) = \frac{1}{2\pi}\int_0^\infty dy\, \ln[\cosh(\varphi g(y))]. \tag{4.10}$$

For the ground state to belong to the Hilbert space of H^0, the normalisation constant must be finite; this means that the limit $R \to 0$ cannot be taken. The condition (4.8) assures that the constant A^2 is finite.

The Hamiltonian is diagonal in terms of the new boson operators

$$b_q^\dagger = \cosh(\varphi_q) a_q^\dagger - \sinh(\varphi_q) a_{-q} = \sum_p \alpha(pq, -\varphi_q)\,\rho_{qp} \qquad (q \neq 0); \tag{4.11}$$

$$\alpha(q, \varphi_q) = (2\pi/L\,|q|)^{1/2}[\theta(q)\cosh(\varphi_q) + \theta(-q)\sinh(\varphi_q)]. \tag{4.12}$$

The diagonalised Hamiltonian is given by substituting these into (4.9):

$$H = E_0 + \sum_q \omega_q b_q^\dagger b_q + \tfrac{1}{2}(\pi/L)(v_N N^2 + v_J J^2),$$

$$E_0 = \tfrac{1}{2}\sum_q (\omega_q - v_F|q|). \tag{4.13}$$

The ground state energy shift E_0 may well be divergent if ω_q does not tend to $v_F|q|$ fast enough as $|q| \to \infty$; however, in contrast to the case of a divergence of the ground state normalisation parameter A^2, this divergence is subtractable, and causes no problems. The form of the momentum operator remains essentially unchanged:

$$P = [k_F + \pi(N/L)]J + \sum_q q b_q^\dagger b_q. \tag{4.14}$$

The relation between the 'true' Fermi momentum $[k_F + \pi(N/L)]$ and the total charge N is unaffected by the interactions.

In addition to the total charge N with respect to the ground state remaining a good quantum number, the current quantum number J is also conserved. This reflects an invariance of the Hamiltonian, under which it is unchanged by independent global gauge transformations of the 'right-' and 'left-moving' ('clockwise' and 'anticlockwise') fermion fields $\psi_p(x)$, $p = \pm 1$. The density operators ρ_{qp} are given by

$$\rho_{qp} = \tfrac{1}{2}(N + pJ)\,\delta_{q0} + (L\,|q|/2\pi)[\alpha(pq, \varphi_q) b_q^\dagger + \alpha(-qp, \varphi_q) b_{-q}],$$

$$[H, \rho_{qp}] = p\,\mathrm{sgn}(q)\omega_q[\cosh(2\varphi_q)\rho_{qp} - \sinh(2\varphi_q)\rho_{q-p})]. \tag{4.15}$$

Following the arguments of (3.60)–(3.62), the mean current j is given by

$$j \simeq \lim_{q\to 0}\left\{(qL)^{-1}\sum_p [H, \rho_{qp}]\right\} = \lim_{q\to o}\left(L^{-1}(\omega_q/|q|)\,\mathrm{e}^{2\varphi_q}\sum_p p\rho_{qp}\right). \tag{4.16}$$

From (4.7) the mean current is found to be $j = v_J(J/L)$; v_J thus plays the role of the renormalised Fermi velocity for fermion currents, as well as controlling the energy of $2k_F$ excitations. Note that it is somewhat unphysical for V_1 and V_2 to differ: if they are set equal, as would be the case if the model was derived as an effective Hamiltonian for a model where only the total charge density was coupled, v_J remains equal to the bare value v_F due to the kinetic term, and is not renormalised.

It is now necessary to transcribe the fermion field $\psi_p(x)$ (3.37) into a form normal-ordered in the new basis. First the definition of the quantities $\phi_p(x)$ (3.22) must be generalised:

$$\phi_p(x, \varphi_q) = p(\pi x/L)N_p + \mathrm{i}\sum_{q\neq 0}\alpha(pq, -\varphi_q)\,\mathrm{e}^{-\mathrm{i}qx}b_q; \tag{4.17}$$

note that the phase field $\theta_p(x)$ is still given by

$$\theta_p(x) = \bar{\theta}_p + \phi_p(x, \varphi_q) + \phi_p^\dagger(x, \varphi_q) \tag{4.18}$$

independent of φ_q. Then the fermion field is given by

$$\psi_p(x) = \mathrm{e}^{-\frac{1}{2}\bar{A}}L^{-\nu}R^{\nu-1/2}\exp(\mathrm{i}pk_F x)\exp[\mathrm{i}\phi_p^\dagger(x, \varphi_p)]U_p\exp[\mathrm{i}\phi_p(x, \varphi_q)] \tag{4.19}$$

where $\nu = \frac{1}{2}\cosh(2\varphi)$, and the cut-off-dependent constant $\bar{A}(\varphi, g(y))$ and a similar quantity $\bar{B}$ are given by

$$\begin{aligned}\bar{A} &= \lim_{\varepsilon\to 0^+}\left[2\sinh^2(\varphi)\left(C + \ln(\varepsilon/2\pi) + \int_\varepsilon^\infty \mathrm{d}y\, y^{-1}2\sinh^2(\varphi g(y))\right)\right],\\ \bar{B} &= \lim_{\varepsilon\to 0^+}\left[-\sinh(2\varphi)\left(C + \ln(\varepsilon/2\pi) - \int_\varepsilon^\infty \mathrm{d}y\, y^{-1}\sinh(2\varphi g(y))\right)\right];\end{aligned} \tag{4.20}$$

(C here is Euler's constant).

It is also useful to define two cut-off-dependent and φ-dependent functions $A_1(u) \equiv A_1(u; \varphi g(y'))$ and $B_1(u)$:

$$\begin{aligned}A_1(u) &= \int_0^\infty \mathrm{d}y\, y^{-1}\sinh^2(\varphi g(y))[2\sin(\tfrac{1}{2}uy)]^2,\\ B_1(u) &= -\tfrac{1}{2}\int_0^\infty \mathrm{d}y\, y^{-1}\sinh(2\varphi g(y))[2\sin(\tfrac{1}{2}uy)]^2.\end{aligned} \tag{4.21}$$

The even functions $A_1(u)$ and $B_1(u)$ vanish as $u\to 0$; for large $|u|$, they behave as

$$\begin{aligned}A_1(u) &\sim \bar{A} + 2\sinh^2(\varphi)\ln(2\pi|u|) + \mathrm{O}(|u|^{-1}),\\ B_1(u) &\sim \bar{B} - \sinh(2\varphi)\ln(2\pi|u|) + \mathrm{O}(|u|^{-1}).\end{aligned} \tag{4.22}$$

Together with $\bar{A}$ and $\bar{B}$, they vanish in the non-interacting limit $\varphi\to 0$.

These quantities characterise the commutation algebra of the quantities $\phi_p(x, \varphi_q)$, which I henceforth write as $\phi_p(x)$, suppressing the explicit dependence on φ_q: in the limit $L \gg R$,

$$[\phi_p(x), \phi_{p'}(x')] = [\phi_p^\dagger(x), \phi_{p'}^\dagger(x')] = 0;$$

$$[\phi_p(x), \phi_p^\dagger(x')] = \lim_{\varepsilon \to 0^+} \{-\ln[1 - e^{-2\varepsilon} \exp(-2\pi i p(x - x')/L)]\}$$
$$-A_1(d(x - x')/R) + \overline{A} - 2\sinh^2(\varphi) \ln(R/L),$$

$$[\phi_p(x), \phi_{-p}^\dagger(x')] = -B_1(d(x - x')/R) + \overline{B} + \sinh(2\varphi) \ln(R/L). \tag{4.23}$$

Here $d(x) \equiv (L/\pi)|\sin(\pi x/L)|$ is the *chord* distance between points with separation x along the circumference of the ring of length L.

The necessary mechanism for calculation of correlation functions has now been established: the desired quantity must be constructed in terms of the fermion operators (4.19), and then manipulated into normal-ordered form in boson variables. The limit $L \to \infty$ can then be taken. As an example, the *electron* single-particle correlation function is easily constructed (using (3.5) to construct the electron field in terms of $\psi_p(x)$); the finite-temperature terms are easily evaluated using the familiar property that $\langle \exp(\alpha b^\dagger) \exp(\alpha' b)\rangle = \exp(\alpha\alpha' \langle b^\dagger b\rangle)$ if $H = \omega b^\dagger b$:

$$\langle \psi^\dagger(x)\psi(0)\rangle_{T=0} = (k_F/\pi)[\sin(k_F x)/(k_F x)] \exp[-A_1(|x|/R)]; \tag{4.24}$$

$$\langle \psi^\dagger(x), \psi(0)\rangle = \langle \psi^\dagger(x)\psi(0)\rangle_{T=0} \exp[-F(|x|)],$$

$$F(x) = \int_0^\infty dq\, q^{-1} [\exp(\beta\omega_q) - 1]^{-1} \cosh(2\varphi_q)[2\sin(\tfrac{1}{2}qx)]^2. \tag{4.25}$$

At $T = 0$ the familiar free-electron result is reduced at large separations by a factor $\exp(-\overline{A})|2\pi x/R|^{1-2\nu}$. At low but finite temperatures $T \ll v_S/R$, it is further reduced at separations $|x| \gg v_S/T \gg R$ by a factor $\exp(-2\nu|x|/\xi)$, where $\xi = (v_S \pi T)$. Note that when models with the same sound velocity v_S are compared, the single-particle correlation function of the interacting model is always reduced below that of the free model.

The recipe for such calculations of correlation functions was first given by Luther and Peschel (1974). The calculation is easily extended to give the dynamic correlation functions, as shown by these authors. In table 1 I summarise the low-energy properties of the spinless fermion Luttinger model, and list the static single-particle, density, and pair correlation functions. In the Luttinger model itself, the linear relation (3.6) between the electron field $\psi(x)$ and the fields $\psi_p(x)$ means that the single-particle correlation only has a k_F oscillatory component, while the density and pair correlations only have 0 and $2k_F$ components, just as in the case of the free Fermi gas. However, in a more general model where J is not strictly conserved, interaction effects will give rise to additional periodic components with extra multiples of $2k_F$ in the period. For example, in addition to the two components $\psi_p^\dagger(x)$ and $\psi_{-p}^\dagger(x)$ making up the operator representing the electron field $\psi^\dagger(x)$, there will be admixture of terms like $\psi_p^\dagger(x)\, \nabla\psi_p^\dagger(x)\, \psi_{-p}(x)$ which adds a $3k_F$ oscillatory term to the single-particle correlation function. Charge conservation allows terms with periodicity $(2m + 1)k_F$ in the single-particle correlation function, and $2mk_F$ in the density and pair correlation functions, and the relevant terms are listed in table 1.

To conclude the discussion of the Luttinger model solution, I note that the low-energy properties of the diagonalised model depend on five distinct parameters: v_S, v_N and v_J parametrise the Hamiltonian, k_F the momentum operator, and φ the fermion field operator. A fundamental result is the relations $v_N = v_S \exp(-2\varphi)$, $v_J = v_S \exp(2\varphi)$, which were deduced from the structure of the solution. The question arises: are these relations fundamental, in that they can be deduced solely from the low-energy structure

Table 1. Summary of 'Luttinger liquid' properties of the spinless 1D Fermi gas. $[\psi_p(x)]^m$ means

$$\lim_{a\to 0}[a^{-\frac{1}{2}m(m-1)}\psi_p(x)\psi_p(x+a)\ldots\psi_p(x+(m-1)a)].$$

Higher harmonics of $2k_F$ allowed by charge conservation, and likely to be present in a more general model, are also included in the list of correlation functions. The phase (cos or sin) of the asymptotic oscillations is also indicated.

1. Interaction parameter (>1 for repulsive forces): $\exp(-2\varphi)$
2. Relation of Fermi vector k_F to charge density $\rho = N/L$: $k_F = \pi\rho$
3. Density fluctuation sound velocity: v_S
4. Change of chemical potential with Fermi vector: $v_N \equiv d\mu/dk_F = v_S e^{-2\varphi}$
5. Fermi velocity (for currents): $v_J = v_S e^{2\varphi}$
6. Asymptotic form of low-temperature correlation functions:

$$\langle A^\dagger(x)A(x')\rangle \sim \sum_i C_i \begin{Bmatrix}\cos(n_1 k_F|x-x'|)\\ \sin(n_1 k_F|x-x'|)\end{Bmatrix}[|x-x'|^{-1}\exp(-\pi T|x-x'|/v_S)]^{\eta_i}.$$

Correlation	$A^\dagger(x)$	Luttinger model form	n	η
Single-particle (sin)	$\psi^\dagger(x)$	$\psi_p^\dagger(x)$	1	$\frac{1}{2}e^{-2\varphi}+\frac{1}{2}e^{2\varphi}$
		$[\psi_p^\dagger(x)]^{m+1}[\psi_{-p}(x)]^m$	$(2m+1)$	$\frac{1}{2}e^{-2\varphi}+2(m+\frac{1}{2})^2e^{2\varphi}$
Density (cos)	$[\psi^\dagger(x)\psi(x)-\rho]$	$\rho_p(x)$	0	2
		$[\psi_p^\dagger(x)]^m[\psi_{-p}(x)]^m$	$2m(\geq 2)$	$2m^2e^{2\varphi}$
Pair (cos)	$\psi^\dagger(x)\nabla\psi^\dagger(x)$	$\psi_p^\dagger(x)\psi_{-p}^\dagger(x)$	0	$2e^{-2\varphi}$
		$[\psi_p^\dagger(x)]^{m+1}[\psi_{-p}(x)]^{m-1}$	$2m$	$2e^{-2\varphi}+2m^2e^{2\varphi}$

of the diagonalised form of the Hamiltonian, without reference to the 'bare' form of the model? The answer is yes: the relations (4.7) can be obtained by considering the static response functions of the density components $\Sigma_p\rho_{pq}$ and $\Sigma_p p\rho_{pq}$; when $q\neq 0$, the calculation only involves the boson variables, and v_S and $\exp(-2\varphi)$. In the limit $q\to 0$, the results must go over into the results $1/2\pi v_N$ and $1/2\pi v_J$ calculated when $q=0$, and the relations (4.7) are recovered.

In addition to the above five characteristic parameters, various multiplicative factors appear in the asymptotic form of the various correlation functions. These depend only on the length scale R, and the two constants A and B; however, in contrast to (4.7), the relation between these various multiplicative factors is likely to be a model-dependent feature of the Luttinger model, as R, A and B depend on the high-energy structure of the model (i.e., the cut-off function $g(y)$).

5. Generalisation to non-soluble models: the 'Luttinger liquid' concept

The complete solubility of the Luttinger model makes it a fascinating example of an interacting one-dimensional system. Nevertheless, its solubility rests on quite specific properties that are lost if the model is modified. However, I will argue that its low-energy structure still provides a model of the most important features of more general, non-soluble models. As an example I consider a generalisation of the Luttinger model that incorporates a non-linear fermion dispersion relation:

$$\varepsilon(kp) = v_F(kp-k_F) + (1/2m)(kp-k_F)^2 + \lambda(1/12m^2v_F)(kp-k_F)^3. \tag{5.1}$$

For stability reasons, it is necessary to include the cubic term: the ground state of the

non-interacting model is altered unless $\lambda > \frac{3}{4}$, when $\mathrm{sgn}(\varepsilon(kp)) = \mathrm{sgn}(kp)$; $\varepsilon(kp)$ increases monotonically if $\lambda > 1$. In general, the interacting model will remain stable for λ greater than some positive limit λ_c. This modification of the model retains the feature that J is a good quantum number; though the non-linear dispersion means that the mean current operator j is no longer simply proportional to J, it remains so in a low-energy subspace.

The procedure for translating this generalised Luttinger model into normal-ordered boson form is extremely simple. An expansion technique as in equations (3.42)–(3.45) can be used to transcribe the non-linear fermion dispersion terms. The general fermion representation (4.19) should be used, with arbitrary parameter φ_q. The final result is a boson normal-ordered Hamiltonian with quadratic boson terms that depend on N and J, plus new cubic and quartic boson interaction terms. The parameter $\varphi_q(N/L, J/L)$ is then chosen to diagonalise the quadratic boson terms, giving a Luttinger model with N- and J-dependent parameters, *plus* irreducible boson interaction terms. The dependence of the Luttinger model parameters on N and J merely reflects the change in Fermi velocity for non-zero N and J, so in order to show up more clearly the other new feature (the boson–boson interaction), I give the new Hamiltonian only in the subspace $N = J = 0$; when $J \neq 0$, the structure of the boson spectrum is slightly altered in that $\varphi_q(N/L, J/L)$ and $\omega_q(N/L, J/L)$ are no longer even functions of q because the right- and left-travelling fermions then have different Fermi velocities. The boson part of the Hamiltonian has the form

$$H(N, J=0) = \sum_q \omega_q b_q^\dagger b_q + \sum_p \frac{1}{2\pi} \int_0^L \mathrm{d}x [:(1/6m)(\Phi_p(x))^3 + (\lambda/48m^2 v_F)(\Phi_p(x))^4:]$$

$$\Phi_p(x) = \sum_p pq\alpha(pq, -\varphi_q)(\mathrm{e}^{\mathrm{i}qx} b_q^\dagger + \mathrm{e}^{-\mathrm{i}qx} b_q). \tag{5.2}$$

The colons :(...): mean boson normal-ordering. The parameters ω_q and φ_q are now given by modified versions of the expressions (4.4) and (4.5), where v_F has been replaced by $\tilde{v}_{Fq} = v_F + (\lambda/4m^2 v_F)(c_1 + \frac{1}{6}q^2)$; the equation for φ_q must be solved self-consistently, since the constant term c_1 itself depends on φ_q:

$$c_1 = \frac{2\pi}{L} \sum_q |q| \sinh^2(\varphi_q) \equiv A_1''(0)/R^2.$$

The constant c_1 exists provided the large-q behaviour of the fermion interaction matrix elements is sufficiently good for $yg(y)$ to vanish as $y \to \infty$. In fact, as will be seen, the requirement that the renormalisation of the ground state of the quadratic part of (5.2) by the boson interactions be finite imposes the stronger requirement $y^3 g(y) \to 0$ as $y \to \infty$. Assuming $V_1(q)$ does not diverge as $q \to \infty$, this implies the condition $qV_2(q) \to 0$ as $q \to \infty$, a slightly stronger condition than in the absence of a non-linear dispersion ($q^{1/2}V_2(q) \to 0$).

With the explicit construction (5.2) of the boson–boson interaction terms induced by a non-linear fermion dispersion, it is possible to construct an expansion in m^{-1} for the changes in the model properties due to the modification. This is particularly interesting in the case of the correlation functions: it allows the rigorous proof, at least for this type of generalised model, that the relations (4.7) between the spectral parameters v_S, v_N, and v_J and the parameter φ, and the relation between φ and the correlation exponents,

remain unchanged from those found in the unmodified Luttinger model. This provides evidence in favour of the universal nature of these relations which will be proposed in this paper.

The relation between the spectral parameters is easiest to demonstrate; I give the form of the Hamiltonian in the subspace where no boson modes are excited:

$$H(n_q = 0) = \tfrac{1}{2}(\pi/L)(v_N N^2 + v_J J^2) + (1/6m)(\pi/L)^2(N^3 + 3NJ^2)$$
$$+ (\lambda/48m^2 v_F)(\pi/L)^3(N^4 + 6N^2J^2 + J^4);$$
$$v_N = \bar{v}_{F0} + V_1(0) + V_2(0); \qquad v_J = \bar{v}_{F0} + V_1(0) - V_2(0). \tag{5.3}$$

The relations (4.7) between v_N, v_J, v_S and φ are clearly unchanged. The stability condition giving the lower bound λ_c to allowed values of λ is clearly obtained by demanding that the ground state of (5.3) has $N = J = 0$. A necessary condition is that v_N and v_J are positive definite, i.e., that $|V_2(0)| < v_F + V_1(0) + (\lambda/4m^2 v_F)c_1(\lambda)$; since $c_1(\lambda)$ is positive, this is a less restrictive condition than that in the original Luttinger model with $m^{-1} = 0$. The condition $\lambda/v_F > \max(3/4v_N, 1/v_J)$ ensures (5.2) has no stationary points other than $N = J = 0$, and is sufficient to guarantee stability.

The effect of the non-linear dispersion on the correlation functions will now be discussed. I study the single-electron correlation function $\langle\psi^\dagger(x)\psi(0)\rangle_{T=0}$ (4.24) discussed earlier, as an example. Following that discussion, this is given (after a little manipulation) by

$$\langle\psi^\dagger(x)\psi(0)\rangle_{T=0} = (k_F/\pi)(k_F x)^{-1}\exp[-A_1(x/R)]\frac{1}{2i}\sum_p p\exp(ipk_F x)$$
$$\times\langle\exp[i\chi_p^\dagger(x)]\exp[i\chi_p(x)]\rangle$$
$$\chi_p(x) = \sum_q \alpha(pq, -\varphi_q)\,2\sin(\tfrac{1}{2}qx)b_q. \tag{5.4}$$

$\langle\psi^\dagger\psi\rangle = (k_F/\pi)$, so the relation between k_F and electron density is unaffected by the non-linear dispersion. The expectation value is of course taken in the ground state of the interacting boson system (5.2), and hence differs from unity when m^{-1} is non-zero. A perturbation expansion in m^{-1} can be developed; the ground state expansion is

$$|\mathrm{GS}\rangle = \mathcal{N}\left(1 + \frac{1}{6mv_S}\sum_{q_1+q_2+q_3=0} f(q_1, q_2, q_3)b_{q_1}^\dagger b_{q_2}^\dagger b_{q_3}^\dagger + O(m^{-2})\right)|0\rangle$$
$$f(q_1, q_2, q_3) = \frac{L}{2\pi}\frac{q_1q_2q_3}{|q_1| + |q_2| + |q_3|}\sum_p p\prod_{i=1}^{3}\alpha(pq_i, -\varphi_{q_i}). \tag{5.5}$$

The normalisation constant $\mathcal{N}$ is given by

$$\mathcal{N} = 1 - \tfrac{1}{2}(1/mv_S R)^2(L/2\pi R)c_2[\varphi g(y)] + O(m^{-4});$$
$$c_2 = \frac{R^3}{6}\frac{2\pi}{L}\sum_{q_1+q_2+q_3=0} f(q_1, q_2, q_3)^2 \tag{5.6}$$

$c_2[\varphi g(y)]$ is a positive dimensionless constant that is finite provided $y^3 g(y) \to 0$ as $y \to \infty$, as mentioned earlier:

$$c_2 = \int_0^\infty dx \int_0^x dy\, x^{-1}(x^2 - y^2)\,h(x, y)^2$$

$$h(x,y) = \tfrac{1}{2}[cg(x+y)cg(x-y)sg(2x) + sg(x+y)sg(x-y)cg(2x)] \tag{5.7}$$

where $cg(x)$ and $sg(x)$ are $\cosh(\varphi g(y))$ and $\sinh(\varphi g(y))$. Note that c_2 vanishes in the absence of fermion interactions ($\varphi = 0$), when there is no renormalisation of the ground state by the boson interactions.

I now calculate the single-electron correlation function to $O(m^{-1})$. From (5.4) and (5.5), this is given by

$$(k_F/\pi)\exp[-A_1(|x|/R)](k_F x)^{-1}\left[\sin(k_F x) - \cos(k_F x)\left(\tfrac{1}{6}\left\langle\left(\sum_p p(\chi_p(x))\right)^3\right\rangle + \text{HC}\right) + O(m^{-2})\right];$$

$$\left\langle\left(\sum_p p(\chi_p(x))\right)^3\right\rangle = -(1/mv_S x)F(|x|/R) + O(m^{-2});$$

$$F(u) = 4\int_0^\infty dx \int_0^x dy\, x^{-1}\sin x\,(\cos x - \cos y)h(x/u, y/u)^2.$$

The function $F(u)$ vanishes at $u = 0$, and remains bounded as $u \to \infty$; the corrections to the correlation function thus do not affect the asymptotic behaviour of the correlation functions. Physically, this is because the factors $|q_i|^{1/2}$ in the interaction matrix elements of (5.2) kill the effects of the boson interactions at long wavelengths. The relation between the various correlation exponents and the parameter φ is thus identical to that in the original Luttinger model; the *value* of the parameter φ, on the other hand, *is* affected by the interaction terms, and varies with the ground state charge density.

6. Discussion: the Luttinger liquid concept

To summarise the results of this paper: it has been shown that the low-energy excitation of the soluble Luttinger model of interacting fermions in one dimension consists of three parts: the well known collective density fluctuation boson modes, plus charge and current excitations, which have not previously been emphasised. Associated with these three types of excitations are three velocities, v_S, v_N and v_J, which obey the relation $v_S = (v_N v_J)^{1/2}$. The current of the Luttinger model is a good quantum number, and is quantised in units $2v_J/L$, each unit carrying momentum $2k_F$. $v_N = d\mu/dk_F$ describes the rate of change of chemical potential with the Fermi vector, which is unrenormalised by interactions, and given by the charge density, $k_F = \pi(N/L)$; v_S is the density excitation sound velocity. The relation between the three velocities defines a parameter φ: $v_N = v_S\exp(-2\varphi)$, $v_J = v_S\exp(2\varphi)$. This parameter φ is the intrinsic renormalised coupling constant of the model, and determines the non-integer power laws characterising the asymptotic behaviour of the correlation functions. The elementary excitations of the Luttinger model are non-interacting, which explains why it can be explicitly solved. An important tool for working with the model and its generalisation is the representation of the fermion fields in terms of the elementary excitations: this is given here in a fully precise form.

A generalisation of the Luttinger model with a non-linear fermion dispersion, but where the current quantum number J is still conserved, was considered here. It was shown that the characteristic low-energy structure of the Luttinger model was preserved,

including the relations between its velocities and correlation exponents, but that its renormalised parameters now depend on the position of the Fermi level, and non-linear couplings appear between the elementary excitations.

On the basis of this demonstration that this structure remains valid in a much wider class of models than the Luttinger model itself, I will propose that it is generally valid for conducting spinless fermion systems in one dimension. For full generality, it is necessary to consider models where the current quantum number J is no longer a good quantum number: this will be done in the next paper in this series. What emerges is that unless a multiple of the fundamental wavevector $2k_F$ is some multiple of a reciprocal lattice vector reflecting an underlying periodicity of the system, momentum conservation eventually inactivates a non-J-conserving term at low energies (though such terms will give rise to renormalisations of the low-energy spectral parameters), and the low-energy structure is again of the form described here. If $2k_F = (n/m)G$, this remains valid provided $\exp(-2\varphi)$ is less than a critical value $\frac{1}{2}m^2$, above which an instability against an insulating pinned charge-density-wave state occurs. If such Umklapp processes are present, but $\exp(-2\varphi) < \frac{1}{2}m^2$, there is a characteristic non-analytic scaling dependence of the renormalised $\exp(-2\varphi)$ on $|2k_F - (n/m)G|$, reflecting the power laws of the correlation functions.

A very important test of the universality of the Luttinger model structure is provided by the class of models exactly soluble by the Bethe *ansatz*, mentioned in the Introduction. For these models, v_S, v_N, and v_J can be explicitly calculated, though their correlation functions have not as yet been obtained. As described in Haldane (1981), the relation $v_S = (v_J v_N)^{1/2}$ can be explicitly verified, and the parameter $\exp(-2\varphi)$ obtained from these velocities shows the characteristic behaviour due to Umklapp processes when $2k_F \sim (n/m)G$ mentioned above, providing additional confirmation that the relation between $\exp(-2\varphi)$ and the correlation exponents is valid (Haldane 1980).

It is obviously possible to generalise the discussion to the case of spin-$\frac{1}{2}$ fermions; the spin-$\frac{1}{2}$ Fermi gas has a characteristic instability against a gap opening in the spin excitation spectrum in zero magnetic field, if $2k_F$ exchange (backscattering) processes are attractive (Luther and Emery 1974); the resulting state is the one-dimensional analogue of superconductivity, though no long-range order is involved, and can be related to the 1D Bose fluid. Similarly, when Umklapp processes open up a gap in the charge density excitation spectrum, leaving gapless low-energy spin-wave modes (Emery *et al* 1976), the resulting system models the antiferromagnetic chain. This in turn can be related to a ferromagnetic chain by a sublattice rotation. In this way, the apparently diverse collection of systems mentioned in the Introduction can be brought into the framework of what I propose to call 'Luttinger liquid theory', which can be tested on those models soluble by the Bethe *ansatz*. Of course, this description of these models is only valid in those regimes where they have a gapless linear density wave excitation, and are conductors of a locally conserved charge, with associated quantised persistent currents at $T = 0$. This underlying unity explains the rather bizarre fact that spin systems and Bose fluids in one dimension have the fermion-like property of a characteristic momentum $2k_F$, as seen in the equivalence of the $S = \frac{1}{2}XY$ spin chain and hard core bose lattice gas to a spinless fermion system (Lieb *et al* 1961, Matsubara and Matsuda 1956). These generalisations will be discussed in detail in subsequent papers.

The emphasis here has been on spectral properties and correlation functions. As a final comment, I note that the approach introduced here could be used as the basis of a theory of transport processes in 'Luttinger liquids'; for example, in the Luttinger model itself, transport of energy by the boson modes would be purely ballistic, since they are

non-interacting. The boson interactions due to a non-linear fermion dispersion would introduce lifetime effects and dissipative behaviour.

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