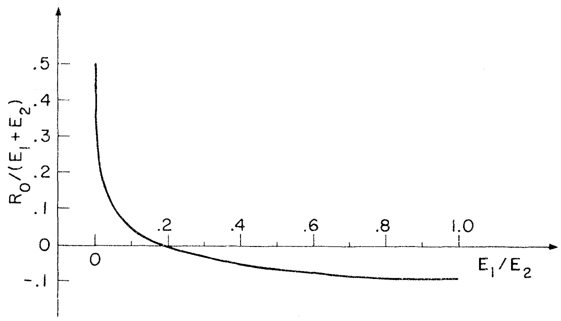

FIG. 1. The dimensionless quantity $R_0/(E_1+E_2)$ as a function of $x=E_1/E_2$. When $x=1$, the value is $-\sqrt{2}/16$. As $x\to 0$ the function behaves as $1/2+3/2\ln x$. The curve crosses zero at ~ 0.1993.

$=\sqrt{2}-1$, $\beta_c=\frac{1}{2}\ln(1+\sqrt{2})$, and numerically evaluate (13)–(15) to obtain[14]

$$C_{0-}=0.025\,536\,971\,9\ldots,$$

$$C_{0+}=0.962\,581\,732\,2\ldots,$$

$$C_{1-}=-0.001\,989\,410\,7\ldots,$$

$$C_{1+}=0.074\,988\,153\,8\ldots.$$

These results give very good agreement with Sykes *et al.*[15] above T_c, $C_{0+}=0.962\,59\pm 3\times 10^{-5}$, $C_{1+}=0.0742$, and with Guttmann[16] below T_c, $C_{0-}=0.0256\pm 1\times 10^{-4}$.

Arguments can be made that the constants C_{2+} and C_{2-} are equal. The value of this constant depends on correlation functions at short distances, and hence cannot be computed by the present method. Details will be published elsewhere.

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Wilson Theory for Spin Systems on a Triangular Lattice

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A special renormalization transformation is constructed for one-component spin systems on a two-dimensional triangular lattice. Fixed point, eigenvalues, and eigenvectors are determined in various approximations, which converge well to known Ising data.

Most of the specific results of the renormalization approach to critical phenomena have been obtained by the ϵ expansion for continuous spin systems interacting through a Landau-Ginzburg Hamiltonian[1] (with $\epsilon=4-d$, and d the dimensionality of the spin lattice). This Letter concerns an application of Wilson's[2] ideas to a general class of discrete spin Hamiltonians which comes closer to Kadanoff's[1] original derivation of the scaling laws and which avoids the ϵ expansion (which is presumably asymptotic rather than convergent).

The method is best illustrated[3] for a two-dimensional (2D) triangular lattice. In Fig. 1 the lattice is divided into cells (triangles) having an *odd*

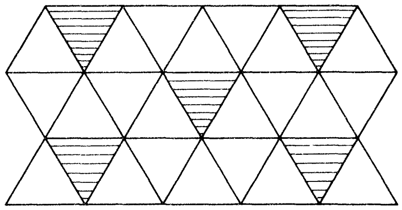


FIG. 1. Triangular lattice with cells shaded.

number (three) of sites such that the lattice of cells is again *triangular*. Other cell divisions are possible but this choice has the additional advantage that the cells are as small as possible (three sites) and that the cells fully occupy the lattice. Each of the N sites i has a spin $s_i = \pm 1$ which interacts through a general Hamiltonian $H(s)$ ($-\beta = -1/k_BT$ included). $H(s)$ is decomposed into its various types of interactions, viz., nearest-neighbor (nn) pair interactions $K_{nn} s_i s_j$, longer-ranged pair interactions, triple-spin interactions $K_{tr} s_i s_j s_k$, etc. Formally we write, where the sum over b runs over all subsets of sites,

$$H(s) = \sum_b K_b s_b, \qquad s_b = \prod_{i \in b} s_i. \tag{1a}$$

The strength parameters K_b can be obtained from $H(s)$ as

$$K_b = 2^{-N} \sum_{\{s\}} s_b H(s), \tag{1b}$$

where the sum over $\{s\}$ runs over all possible spin configurations. $H(s)$ is taken to have short-range interactions and to be invariant under the symmetries of the lattice. So the sum in (1) includes no sets with sites far apart, and the K_b of sets b of the same type β (e.g., nearest-neighbor pairs) have the same value K_β.

We associate with a cell i' a spin $s_{i'}'$ defined as the signature of the sum over all spins in cell i':

$$s_{i'}' = \mathrm{sgn}\Big(\sum_{i \in i'} s_i\Big). \tag{2}$$

Since a cell has an odd number of sites, $s_{i'}'$ is unambiguously ± 1. For a given value of $s_{i'}'$ there are a number (viz., four) of internal configurations $\sigma_{i'}$ for the spins of cell i'. Thus $H(s)$ can also be written as $H(s', \sigma)$. Then define a renormalization transformation from a site-spin Hamiltonian $H(s)$ to a cell-spin Hamiltonian $H'(s')$ as

$$\sum_{\{\sigma\}} \exp H(s', \sigma) \equiv \exp H'(s'). \tag{3}$$

Here it should be noted that even if one starts out with only simple interactions (e.g., only nearest-neighbor interactions) on the site lattice, (3) generates in principle all types of interactions on the cell lattice. The main point of this Letter is to show, by studying various approximations, that (3) exhibits a fixed point with properties to be expected for 2D spin systems. We view (3) as a map of the original interaction constants K_α to renormalized K_α' [belonging to $H'(s')$]:

$$K_\alpha' = K_\alpha'(K). \tag{4}$$

A fixed point is a set of values K_α^* such that $K_\alpha'(K^*) = K_\alpha^*$. The critical properties (exponents) can be expressed in terms of the eigenvalues and eigenvectors of the matrix

$$T_{\alpha\beta} = (\partial K_\alpha' / \partial K_\beta)_{K = K^*}. \tag{5}$$

From the known results for the 2D triangular Ising system, one expects two eigenvalues λ_T and λ_H to be

$$\lambda_T = l = \sqrt{3} = 1.732\,05, \qquad \lambda_H = l^{15/8} = 3^{15/16} = 2.800\,92 \tag{6}$$

(l being the cell spacing measured in units of the site spacing), and all others <1 in absolute value.

A fixed point is located in the surface of critical systems. The tangent plane in this fixed point is (for vanishing odd interactions) determined by the (left) eigenvector φ_α^T belonging to λ_T. Since the fixed point has no special physical significance, the critical surface will not be anomalous there. We found, in fact, very little curvature around the fixed point. So the tangent plane gives a good measure for the variation of the critical temperature $T_c(J)$ with the (even) interaction constants $J_\alpha = (k_BT)K_\alpha$. One may write the equation for the tangent plane in the form

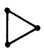
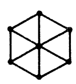
$$T_c(J) = T_c\Big[1 + \sum_{\alpha = \mathrm{nn}} (\varphi_\alpha^T / \varphi_{nn}^T)(J_\alpha / J_{nn})\Big], \tag{7}$$

where T_c is the Ising critical temperature (with only nearest-neighbor interactions present). We used the intercept of the Ising axis ($K_\alpha = 0$ except $\alpha = $ nn) with this tangent plane as an estimate for the Ising critical parameter K_c (Table I, third column).

In order to study (3) we must approximate the sum over the internal configurations σ. Most naively one separates $H(s)$ into a piece H^0 containing the intracell interactions and a perturbation V containing the intercell interactions. The results of first-order perturbation theory for λ_T, λ_H, and K_c are listed in the first line of Table I.

A more promising approximation uses the fact that the transformation (3) can be studied in any

TABLE I. Values of the "thermal" and "magnetic" eigenvalues λ_T and λ_H and the value K_c for an Ising system as deduced from the fixed-point tangent plane.

Approximation		λ_T	λ_H	K_c
1st order perturbation		1.634	3.036	0.336
cluster		1.544	3.036	0.365
		1.501	2.501	0.255
		1.567	2.497	0.253
		1.782	3.186	0.281
		1.7590	2.8024	0.27416
exact (Ising)		1.73205	2.80092	0.27465

detail on a small system. If c is a (specific) cluster of cells with Hamiltonian $H_c(s)$, we write

$$\sum_{\{\sigma\}} \exp H_c(s', \sigma) = \exp H_c'(s') = \exp \sum_{b\subseteq c} K_b'^c s_b', \quad (8)$$

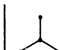
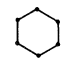
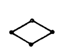
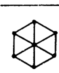
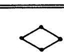
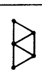
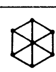
where the summation over b runs through the subsets of cluster c. Then we have to solve the combinatorial problem to account for the number of times a certain interaction can be a part of a cluster c of type γ. The result may be put in the form

$$K_\alpha'(K) = \sum_{\alpha\subseteq\beta\subseteq\gamma} C_\gamma(\beta) K_\alpha'^\beta(K), \quad (9)$$

where the combinatorial coefficients $C_\gamma(\beta)$ give the weight factors by which the coefficients $K_\alpha'^\beta$ of the subfigures β of γ have to be combined.

We have computed the transformation $K_\alpha'(K)$ for several figures. For the very small clusters (two or three cells) one easily evaluates the $K_\alpha'^\beta(K)$ analytically. For the larger clusters the transformation was obtained on a computer by generating all configurations σ compatible with a cell spin distribution s' and then selecting out the $K_\alpha'^\beta(K)$ by weighted sums over $\{s'\}$ as in (1). The basic limitation of such a cluster approximation is the fact that all interaction types are omitted which do not fit in the figure chosen. The advantage of the cluster approximation is that all interactions inside the cluster can be treated to any order and on equal footing, which turns out

TABLE II. Fixed-point values for the interaction parameters $K_\alpha{}^*$ and components $r_\alpha = \varphi_\alpha{}^T/\varphi_{nn}{}^T$ of the left eigenvector belonging to λ_T. Here, as in Eq. (4), α corresponds to the particular geometrical arrangement of interacting spins or cells as shown in Fig. 1.

Approximation \ α													
1st order pert.		K_α^*	0.3356	—	—	—	—	—	—	—	—	—	—
cluster		K_α^*	0.365	—	—	—	—	—	—	—	—	—	—
··		K_α^*	0.255	—	—	—	—	—	—	—	—	—	—
··		K_α^*	0.257	-0.0022	—	-0.00085	—	—	—	—	—	—	—
··		K_α^*	0.331	-0.0275	-0.0267	0.0086	0.0080	—	-0.0037	—	—	—	—
··		K_α^*	0.3069	-0.0183	-0.0214	0.0034	0.0066	0.0036	-0.0022	-0.0016	-0.0009	0.0003	0.00004
··		r_α	1	1.205	—	2.990	—	—	—	—	—	—	—
··		r_α	1	1.708	1.917	1.237	5.978	—	2.742	—	—	—	—
··		r_α	1	1.607	1.811	1.248	5.782	1.083	2.808	1.372	3.081	0.452	2.777

to be important because, e.g., pair interactions are partially compensated at the fixed point by quadruple interactions of the same range.

In Table II we have listed the locus $K_\alpha{}^*$ of the fixed point in the various approximations together with the left eigenvector $r_\alpha = \varphi_\alpha{}^T/\varphi_{\rm nn}{}^T$ belonging to λ_T. From this table one observes which interaction parameters are included and also that among the $K_\alpha{}^*$ the nearest-neighbor interaction stands out by a factor 15 over the other (negative) pair interactions, which again are a factor 4 larger than the four-spin interactions and many times larger than the six-spin interactions. Although the fixed point shifts notably, a lower approximation could very well be used as a guess for the fixed point of a higher approximation, indicating that the transformation does not develop singularities for larger and larger clusters at the fixed point (which is a basic assumption in the renormalization approach).

In Table I the values for λ_T, λ_H, and K_c are given and compared with the values for a triangular Ising system.[4] The last approximation (seven cells symmetrically arranged) gives particularly accurate values, in our opinion, not only because it is the largest basic figure but also because its symmetry is the same as that of the lattice. On the basis of these λ_T and λ_H we find, e.g., the critical exponents

$$\nu = \ln l/\ln\lambda_T = 0.973,$$
$$\delta = \ln\lambda_H/(2\ln l - \ln\lambda_H) = 15.017, \tag{10}$$

which should be compared with the exact values $\nu = 1$ and $\delta = 15$. The value of r_α may be compared with the coefficients giving the variation of $T_c(J)$ with J around the Ising system ($J_\alpha = 0$, $\alpha \neq \rm nn$) obtained either analytically[5] or numerically.[6] Dalton and Wood[6] find $r_{\rm nnn} = 1.35$. Our rather high value of 1.607 could be lowered only a few hundredths by accounting for the curvature of the surface of criticality. One must conclude that longer-range forces than fit in our largest cluster play a role in determining r_α.

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Tricritical Lines in Metamagnets

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Measurements of the magnetization of $FeBr_2$ and $FeCl_2$ as a function of magnetic field, temperature, and hydrostatic pressure establish lines of tricritical points T_{3c} with slopes $(T_{3c})^{-1}dT_{3c}/dP = -0.025$, $+0.021$, and $+0.040$ kbar^{-1} for the low- and high-pressure phases of $FeCl_2$ and for $FeBr_2$, respectively. The variation of the tricritical transition with pressure should provide sensitive tests of theories relating interaction constants in the Hamiltonian to tricritical behavior in magnetic systems.

Considerable interest has been aroused recently by the existence of tricritical points, which involve the meeting of a line of second-order transitions with a line of first-order transitions.[1] Metamagnets such as $FeCl_2$ and $FeBr_2$[2] provide typical examples of such tricritical points; other examples are the two-fluid critical mixing point in He^3-He^4, the order-disorder transitions in NH_4Cl and NH_4Br, thin superconducting films, and the metamagnet dysprosium aluminum garnet. There are two levels to the problem of tricritical points. The first is understanding their