

Regularization methods for delta-function potential in two-dimensional quantum mechanics

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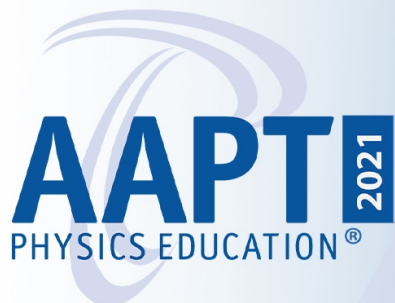
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Regularization methods for delta-function potential in two-dimensional quantum mechanics

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The quantum mechanics of a bound particle in the delta-function potential in two dimensions is studied with a discussion of its regularization and renormalization. A simple regularization approach is considered with the introduction of several regularizing functions for defining the quantum system. More systematic regularization is introduced from the mathematical viewpoint of the theory of distributions. The renormalization scheme independence of the physical observable is demonstrated. © 2000 American Association of Physics Teachers.

I. INTRODUCTION

Simple solvable models in quantum mechanics¹ play an important role in illustrating physical concepts²⁻⁶ and provide useful applications for models that make use of perturbation theory, in which a solvable part is required. For example, the simple harmonic oscillator has provided the analytic and algebraic methods for many calculations in quantum theory.

The delta-function potentials in quantum mechanics³⁻⁶ are exactly solvable models, which enjoy many useful applications. In two and higher dimensions, they provide a pedagogical introduction to the techniques of regularization in quantum field theory. Here we shall concentrate on the two-dimensional case, which exhibits a logarithmic divergent property and requires regularization before renormalization can be carried out. Thus, this model provides a simple illustration of dimension deficiency, such that an additional scale has to be introduced. In one dimension, the quantum system needs no regularization. For other dimensions, we refer to Refs. 3-6, where other technical and physical aspects of the models are studied. Here we shall consider only the bound state, while the scattering state can be easily worked out as outlined in Refs. 5 and 6.

We organize this paper as follows. In Sec. II, we consider the Schrödinger equation for a bound particle in the delta-function potential in two dimensions with an outline of the usual solution. In Sec. III, a simple regularization approach is considered for the delta function with the introduction of several regularizing functions for the divergent integral in the problem. The renormalization scheme independence of the physical observable of the model is mentioned. In Sec. IV, the theory of distributions is briefly introduced for defining the delta-function potential. Section V provides a discussion.

II. THE DELTA-FUNCTION POTENTIAL IN TWO-DIMENSIONAL QUANTUM MECHANICS

The Schrödinger equation for a particle in the delta-function potential in two dimensions is given by

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - \lambda\delta^2(\mathbf{r}) \right] \psi(\mathbf{r}) = E\psi(\mathbf{r}), \quad (1)$$

where, for the bound state, $\lambda > 0$. At first sight, we would naively expect that the physical binding energy $E = -B < 0$ can only depend on the parameters in the equation, λ and

\hbar^2/m , which have the same dimensions. But, dimensional analysis for the energy scale of the system indicates that the binding energy cannot just be a combination of the two parameters. There is a dimension deficiency, and an additional parameter with dimensions of squared momentum is required.

For convenience, we rewrite Eq. (1) as

$$[\nabla^2 + \lambda_0\delta^2(\mathbf{r})]\psi(\mathbf{r}) = B_0\psi(\mathbf{r}), \quad (2)$$

where $\lambda_0 = 2m\lambda/\hbar^2$ is a dimensionless parameter and $B_0 = 2mB/\hbar^2$ is an arbitrary dimensional constant. Equation (2) can be easily solved by Fourier transform with^{4,5}

$$\psi(\mathbf{r}) = \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{i\mathbf{k}'\cdot\mathbf{r}} \phi(\mathbf{k}'), \quad (3)$$

$$\psi(0) = \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \phi(\mathbf{k}'). \quad (4)$$

The Fourier transform of the left-hand side of Eq. (2) gives

$$\begin{aligned} \int d^2\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} [\nabla^2 + \lambda_0\delta^2(\mathbf{r})]\psi(\mathbf{r}) \\ = \int \frac{d^2\mathbf{k}'}{(2\pi)^2} [-\mathbf{k}'^2\phi(\mathbf{k}')(2\pi)^2\delta^2(\mathbf{k}'-\mathbf{k}) + \lambda_0\phi(\mathbf{k}')] \\ = -\mathbf{k}^2\phi(\mathbf{k}) + \lambda_0\psi(0), \end{aligned} \quad (5)$$

while the transform of the right-hand side leads to

$$\int d^2\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} B_0\psi(\mathbf{r}) = B_0\phi(\mathbf{k}). \quad (6)$$

Equating Eq. (5) to Eq. (6), we then get

$$\phi(\mathbf{k}) = \frac{\lambda_0\psi(0)}{\mathbf{k}^2 + B_0}. \quad (7)$$

Integrating both sides over \mathbf{k} and using Eq. (4), we obtain the integral

$$\begin{aligned} \frac{1}{\lambda_0} &= \frac{1}{4\pi^2} \int \frac{d^2\mathbf{k}}{\mathbf{k}^2 + B_0} \\ &= \lim_{\Lambda \rightarrow \infty} \frac{1}{4\pi} \int_0^{\Lambda^2} \frac{d\mathbf{k}^2}{\mathbf{k}^2 + B_0} \\ &= \lim_{\Lambda \rightarrow \infty} \frac{1}{4\pi} \ln\left(\frac{\Lambda^2 + B_0}{B_0}\right). \end{aligned} \quad (8)$$

Thus the integral diverges logarithmically as $\Lambda \rightarrow \infty$. For large but finite cutoff momentum Λ , which acts as a regulator, we then have

$$\frac{1}{\lambda_0} \approx \frac{1}{4\pi} \ln\left(\frac{\Lambda^2}{B_0}\right), \quad (9)$$

or the binding energy

$$E = -\frac{\hbar^2 \Lambda^2}{2m} e^{-4\pi/\lambda_0}, \quad (10)$$

which is a physical observable and should be independent of Λ . We now define the renormalized dimensionless coupling constant λ_R by

$$\frac{1}{\lambda_R} = \frac{1}{\lambda_0} - \frac{1}{4\pi} \ln\left(\frac{\Lambda^2}{\mu^2}\right), \quad (11)$$

where the momentum scale μ is an arbitrary renormalization or subtraction scale. The constant λ_R defined as such is finite as $\Lambda \rightarrow \infty$ and reads

$$\frac{1}{\lambda_R} = -\frac{1}{4\pi} \ln\left(\frac{B_0}{\mu^2}\right), \quad (12)$$

from which the β function is given by ($dB_0/d\mu=0$)

$$\beta(\mu) \equiv \frac{d\lambda_R}{d \ln \mu} = -\frac{\lambda_R^2}{2\pi} < 0, \quad (13)$$

which exhibits an asymptotic free behavior: $\lambda_R \rightarrow 0$ as $\mu \rightarrow \infty$.

Now from Eq. (12) the binding energy is given by

$$E = -\frac{\hbar^2 \mu^2}{2m} e^{-4\pi/\lambda_R}, \quad (14)$$

which is independent of μ as E is a physical observable.

III. SIMPLE REGULARIZATION

Instead of Eq. (8), which is undefined as $\Lambda \rightarrow \infty$, we consider regularizing it by

$$\frac{1}{\lambda_0} = \frac{1}{4\pi^2} \int \frac{d^2 \mathbf{k} \rho_\epsilon(\mathbf{k})}{\mathbf{k}^2 + B_0}, \quad (15)$$

where $\rho_\epsilon(\mathbf{k})$ is a regularizing or smearing function with $1/\epsilon$ acting as an ultraviolet cutoff, such that $\lim_{\epsilon \rightarrow 0} \rho_\epsilon(\mathbf{k}) = 1$. The limit $\epsilon \rightarrow 0$ is to be taken after the integration over \mathbf{k} and, for $\epsilon \neq 0$, the integral is finite. For illustrative examples, we consider the following regularizing functions.

A. $\rho_\epsilon(\mathbf{k}) = \theta^2(1/\epsilon + \mathbf{k}) \theta^2(1/\epsilon - \mathbf{k})$

Here $\theta^2(1/\epsilon \pm \mathbf{k})$ denotes $\theta(1/\epsilon \pm k_x) \theta(1/\epsilon \pm k_y)$, which is a product of step functions and corresponds to implementing cutoffs on the integration limits. Let $1/\epsilon = \Lambda \gg 1$, Eq. (15) becomes

$$\begin{aligned} \frac{1}{\lambda_0} &= \frac{1}{4\pi^2} \int_{-\Lambda}^{\Lambda} \int_{-\Lambda}^{\Lambda} \frac{dk_x dk_y}{k_x^2 + k_y^2 + B_0} \\ &\approx \frac{1}{4\pi^2} \int_0^{2\pi} d\theta \int_0^{\Lambda} \frac{k dk}{k^2 + B_0} \\ &= \frac{1}{4\pi} \ln\left(\frac{\Lambda^2 + B_0}{B_0}\right), \end{aligned} \quad (16)$$

which is identical to Eq. (8) as it should be.

B. $\rho_\epsilon(\mathbf{k}) = \exp(-\epsilon \mathbf{k}^2)$

This choice leads to the Laplace transform⁷

$$\frac{1}{\lambda_0} = \frac{1}{4\pi} \int_0^\infty \frac{dx e^{-\epsilon x}}{x + B_0} = \frac{1}{4\pi} e^{B_0 \epsilon} \text{Ei}(B_0 \epsilon), \quad (17)$$

where $\text{Ei}(x)$ is the exponential integral

$$\text{Ei}(x) = \int_x^\infty \frac{e^{-u}}{u} du = -\gamma_E - \ln x - \sum_{n=1}^\infty \frac{(-x)^n}{nn!} \quad (x \ll 1), \quad (18)$$

with $\gamma_E = 0.5772\dots$ being Euler's constant. For $\epsilon = 0$, Eq. (17) becomes the unregularized form (8), while for $\epsilon \neq 0$, the integral (17) is finite. From Eq. (17), we see that for a given binding energy E , the coupling constant depends on the cutoff scale $\Lambda \equiv 1/\sqrt{\epsilon}$. For $\Lambda \gg 1$,

$$\frac{1}{\lambda_0} \approx -\frac{1}{4\pi} \ln B_0 \epsilon + C = \frac{1}{4\pi} \ln\left(\frac{\Lambda^2}{B_0}\right) + C, \quad (19)$$

where $C = -\gamma_E/4\pi$. Thus, in the limit $\Lambda \rightarrow \infty$, we recover Eq. (9), but there is an additional constant C . We now define the renormalized constant by

$$\frac{1}{\lambda_R} = \frac{1}{\lambda_0} - \frac{1}{4\pi} \ln\left(\frac{\Lambda^2}{\mu^2}\right) - C' = -\frac{1}{4\pi} \ln\left(\frac{B_0}{\mu^2}\right) + (C - C'), \quad (20)$$

where the choice of C' corresponds to a renormalization scheme; for example, we may choose $C' = 0$ or $C' = C$ for convenience. However, the physical observable E is independent of the renormalization scale and hence the choice of a scheme, since we can always make a rescaling and write

$$\frac{1}{\lambda_R} = -\frac{1}{4\pi} \ln\left(\frac{B_0}{\mu'^2}\right), \quad \mu'^2 = \mu^2 e^{4\pi(C - C')}. \quad (21)$$

C. $\rho_\epsilon(\mathbf{k}) = \exp(-\epsilon |\mathbf{k}|)$

With this choice, we get from Eq. (15)⁷

$$\begin{aligned} \frac{1}{\lambda_0} &= \frac{1}{2\pi} \int_0^\infty k dk e^{-\epsilon k} \frac{1}{k^2 + B_0} \\ &= -\frac{1}{2\pi} \frac{d}{ds} \int_0^\infty \frac{dx e^{-sx}}{x^2 + 1} \Bigg|_{s=\sqrt{B_0}\epsilon} \\ &= -\frac{1}{2\pi} \frac{d}{ds} \left(\left[\frac{\pi}{2} - \text{Si}(s) \right] \cos s + \text{Ci}(s) \sin s \right) \Bigg|_{s=\sqrt{B_0}\epsilon}, \end{aligned} \quad (22)$$

where $\text{Si}(s)$ and $\text{Ci}(s)$ are the Sine and Cosine integrals defined by

$$\text{Si}(s) = \int_0^s \frac{\sin t}{t} dt, \quad \text{Ci}(s) = \gamma_E + \ln s + \int_0^s \frac{\cos t - 1}{t} dt. \quad (23)$$

For $\epsilon = 1/\Lambda \ll 1$, we have

$$\frac{1}{\lambda_0} \approx \frac{1}{4\pi} \ln\left(\frac{\Lambda^2}{B_0}\right) + C''. \quad (24)$$

Here we have a constant, $C'' = -\gamma_E/2\pi$, but it can be absorbed by redefining the scale μ in λ_R .

IV. REGULARIZED DELTA FUNCTIONS

In the previous sections, the quantum system with the delta-function potential in two dimensions was shown to be an exactly solvable model but requires a regularization. We first considered the problem by using a cutoff as an ultraviolet regulator and then introduced a more general and simple regularization approach. Here we discuss a more systematic approach from the mathematical point of view.

We first recall that the Dirac delta function in one dimension is defined by $\delta(x) = 0$ for $x \neq 0$ and $\int_{-a}^a \delta(x) dx = 1$ for arbitrary $a > 0$. Technically, the function is ill-defined since it is neither differentiable nor continuous at $x = 0$. Mathematically, it is defined as a generalized function or distribution in the theory of distributions. Here we give a brief introduction to the theory.⁸

We refer to $f(x)$ as a test function if it is infinitely differentiable. For our purpose, we take the Schwartz space \mathcal{S} , which is a space of test functions of rapid descent. For $f(x) \in \mathcal{S}$, then $f(x)$ and all its derivatives decrease to zero faster than every negative power of x . We define $\tilde{\phi}(x)$ as a generalized function or distribution and represent it by a sequence $\{\phi_\epsilon; \epsilon = 1/j, j = 1, 2, \dots\}$. Then $\phi_\epsilon(x) \in \mathcal{S}$, and

$$\int_{\Omega} \tilde{\phi}(x) f(x) dx \equiv \lim_{\epsilon \rightarrow 0} \int_{\Omega} \phi_\epsilon(x) f(x) dx \quad (25)$$

exists for $f(x) \in \mathcal{S}, x \in \Omega$.

Now the Dirac delta function can be defined by

$$\delta(x) \equiv \lim_{\epsilon \rightarrow 0} \delta_\epsilon(x), \quad (26)$$

which may be expressed as the Fourier transform

$$\delta_\epsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \rho_\epsilon(k) e^{ikx} dk, \quad \int_{-\infty}^{+\infty} \delta_\epsilon(x) dx = 1, \quad (27)$$

where ϵ is a dimensional parameter playing the role of a cutoff. In Eq. (26), the limit $\epsilon \rightarrow 0$ should only be taken after the integration over x . For example, it is easy to verify that the delta-function satisfies the operational property:

$$\delta: f \equiv \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{+\infty} \delta_\epsilon(x) f(x) dx = f(0). \quad (28)$$

The regularizing function $\rho_\epsilon(k)$ satisfies the condition $\lim_{\epsilon \rightarrow 0} \rho_\epsilon(k) = 1$. Then different regularizing functions correspond to different mathematical deformations of the model. In the limit $\epsilon \rightarrow 0$, we should expect that the deformed models lead to the same physical result, provided that the regularization does not destroy the symmetries of the original model in the limit $\epsilon \rightarrow 0$. Depending on the physical problem of interest, a regularizing function is chosen to facilitate

Table I. The regularizing functions $\rho_\epsilon(k)$ and their corresponding regularized delta functions $\delta_\epsilon(x)$ in one dimension.

	$\rho_\epsilon(k) (\epsilon > 0)$		
	$\theta(1/\epsilon + k)\theta(1/\epsilon - k)$	$e^{-\epsilon k^2}$	$e^{-\epsilon k }$
$\delta_\epsilon(x)$	$\frac{1}{\pi x} \sin\left(\frac{x}{\epsilon}\right)$	$\frac{1}{\sqrt{4\pi\epsilon}} \exp\left(-\frac{x^2}{4\epsilon}\right)$	$\frac{1}{\pi x^2 + \epsilon^2}$

a given calculation. Some commonly used regularizing functions with their corresponding regularized delta functions are given in Table I.

We now consider the two-dimensional case. With the regularized delta-function potential, the reduced Schrödinger equation (2) becomes

$$[\nabla^2 + \lambda_0 \delta_\epsilon^2(\mathbf{r})] \psi(\mathbf{r}) = B_0 \psi(\mathbf{r}). \quad (29)$$

For $\epsilon \neq 0$, B_0 depends on λ_0 and the regularizing parameter ϵ , and different regularizing functions correspond to different but related models. We expect that in the limit $\epsilon \rightarrow 0$, the binding energy reduces to Eq. (14).

The Fourier transform of the regularized delta-function potential in the left-hand-side of Eq. (29) gives

$$\begin{aligned} \lambda_0 \int d^2\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \delta_\epsilon^2(\mathbf{r}) \psi(\mathbf{r}) \\ = \lambda_0 \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \phi(\mathbf{k}') \int d^2\mathbf{r} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \delta_\epsilon^2(\mathbf{r}) \\ = \frac{\lambda_0}{4\pi^2} \int d^2\mathbf{k}' \rho_\epsilon(\mathbf{k}-\mathbf{k}') \phi(\mathbf{k}'), \end{aligned} \quad (30)$$

where

$$\rho_\epsilon(\mathbf{k}) = \int d^2\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \delta_\epsilon^2(\mathbf{r}). \quad (31)$$

Hence, the Fourier transform of Eq. (29) leads to the integral equation

$$\phi(\mathbf{k}) = \frac{\lambda_0}{4\pi^2} \frac{1}{\mathbf{k}^2 + B_0} \int d^2\mathbf{k}' \rho_\epsilon(\mathbf{k}-\mathbf{k}') \phi(\mathbf{k}'). \quad (32)$$

Integrating both sides over \mathbf{k} , we obtain

$$\psi(0) = \frac{\lambda_0}{16\pi^4} \int d^2\mathbf{k}' \phi(\mathbf{k}') \int \frac{d^2\mathbf{k} \rho_\epsilon(\mathbf{k}-\mathbf{k}')}{\mathbf{k}^2 + B_0}, \quad (33)$$

where interchange of integrations has been assumed. If we take \mathbf{k}' fixed and choose $1/\epsilon \gg |\mathbf{k}'|$, then we can show that

$$I = \int \frac{d^2\mathbf{k} \rho_\epsilon(\mathbf{k}-\mathbf{k}')}{\mathbf{k}^2 + B_0} \approx \int \frac{d^2\mathbf{k} \rho_\epsilon(\mathbf{k})}{\mathbf{k}^2 + B_0}, \quad (34)$$

so that Eq. (33) reduces to Eq. (15). To see this, we write

$$I = \int_0^\infty d\alpha e^{-\alpha B_0} \int d^2\mathbf{k} \rho_\epsilon(\mathbf{k}) e^{-\alpha(\mathbf{k}'-\mathbf{k})^2}, \quad (35)$$

where it is clear that for $1/\epsilon \gg |\mathbf{k}'|$, we have $\rho_\epsilon(\mathbf{k}) \approx 1$, and

$$\int d^2\mathbf{k} \rho_\epsilon(\mathbf{k}) e^{-\alpha(\mathbf{k}'-\mathbf{k})^2} \approx \frac{\pi}{\alpha}, \quad (36)$$

which becomes independent of \mathbf{k}' as $\epsilon \rightarrow 0$. Now consider the regularizing functions given in Table I for the two-dimensional case; the corresponding integrals Eq. (35) given in the following are then trivial.

A. $\rho_\epsilon(\mathbf{k}) = \theta^2(1/\epsilon + \mathbf{k})\theta^2(1/\epsilon - \mathbf{k})$

For $1/\epsilon = \Lambda \gg |\mathbf{k}'|$, we get

$$\begin{aligned} I &= \int_{-\Lambda}^{\Lambda} \int_{-\Lambda}^{\Lambda} \frac{dk_x dk_y}{(\mathbf{k}' - \mathbf{k})^2 + B_0} \\ &= \int_{k'_y - \Lambda}^{k'_y + \Lambda} \int_{k'_x - \Lambda}^{k'_x + \Lambda} \frac{dk_x dk_y}{\mathbf{k}^2 + B_0} \\ &\approx \int_{-\Lambda}^{\Lambda} \int_{-\Lambda}^{\Lambda} \frac{dk_x dk_y}{\mathbf{k}^2 + B_0}, \end{aligned} \quad (37)$$

which is further evaluated in Eq. (16).

B. $\rho_\epsilon(\mathbf{k}) = \exp(-\epsilon \mathbf{k}^2)$

Using this function in Eq. (35), we have, for vanishingly small ϵ ,

$$\begin{aligned} I &= \int_0^\infty d\alpha e^{-\alpha B_0} \int d^2\mathbf{k} e^{-\alpha(\mathbf{k}' - \mathbf{k})^2 - \epsilon \mathbf{k}^2} \\ &= \int_0^\infty d\alpha \exp\left(-\frac{\alpha\epsilon}{\alpha + \epsilon} \mathbf{k}'^2 - \alpha B_0\right) \int d^2\mathbf{k} \\ &\quad \times \exp\left[-(\alpha + \epsilon) \left(\mathbf{k} - \frac{\alpha}{\alpha + \epsilon} \mathbf{k}'\right)^2\right] \\ &= \pi \int_0^\infty \frac{d\alpha}{\alpha + \epsilon} \exp\left(-\frac{\alpha\epsilon}{\alpha + \epsilon} \mathbf{k}'^2 - \alpha B_0\right) \\ &\approx \pi \int_0^\infty \frac{d\alpha}{\alpha + \epsilon} e^{-\alpha B_0} \\ &= \pi e^{B_0\epsilon} \int_\epsilon^\infty \frac{du}{u} e^{-u B_0} \\ &= \pi e^{B_0\epsilon} \text{Ei}(B_0\epsilon), \end{aligned} \quad (38)$$

which leads to Eq. (17).

C. $\rho_\epsilon(\mathbf{k}) = \exp(-\epsilon|\mathbf{k}|)$

In this instance, we get from Eq. (35), for $1/\epsilon \gg |\mathbf{k}'|$,

$$\begin{aligned} I &= \int_0^\infty d\alpha e^{-\alpha B_0} \int d^2\mathbf{k} e^{-\alpha(\mathbf{k}' - \mathbf{k})^2} e^{-\epsilon|\mathbf{k}|} \\ &= \int_0^\infty d\alpha e^{-\alpha B_0} \int d^2\mathbf{k} e^{-\alpha \mathbf{k}^2} e^{-\epsilon|\mathbf{k} - \mathbf{k}'|} \\ &\approx \int_0^\infty d\alpha e^{-\alpha B_0} \int d^2\mathbf{k} e^{-\alpha \mathbf{k}^2} e^{-\epsilon|\mathbf{k}|} \\ &= 2\pi \int_0^\infty \frac{k dk e^{-\epsilon k}}{k^2 + B_0}, \end{aligned} \quad (39)$$

which can then be evaluated according to Eq. (22).

From the above mentioned examples, we observe that not all $\rho_\epsilon(\mathbf{k})$'s lead to the same calculational difficulties.

V. DISCUSSION

In this paper, different regularizing functions parametrized by an infinitesimal parameter ϵ have been considered for defining the delta-function potential in two-dimensional Schrödinger equation. We note that different functions correspond to different regularization methods, and hence allow different renormalization schemes to be used. Thus, different regularization methods correspond to different models in the limit $\epsilon \rightarrow 0$ with renormalization scheme dependent coupling constants. In essence, a regularization is a mathematical deformation of a system. In this regard, we note that regularization is a general procedure for defining a physical system. In the study of a gauge theory, implementation of a gauge condition is the use of a regularization. A good choice of a gauge condition can facilitate our analysis of a gauge problem. Physics is not affected by our choice.

A regularization may not respect all the symmetries of a system if the deformed system does not enjoy the symmetries of the original system. In this instance, the regularization is not suitable. The usefulness of a particular technique of regularization is usually limited to a certain theory or model, so care should be exercised to choose a consistent regularization method.

In this paper, we have concentrated only on the bound state and demonstrated the renormalization scheme independence of the binding energy E . This fixed, but arbitrary, binding energy only serves to define the energy scale of the quantum system. At fixed B_0 , Eq. (12) or (14) provides a relation between the renormalized coupling constant $\lambda_R(\mu)$ and the renormalization scale μ . Further study of the system includes the consideration of the scattering solution, or the physical quantities, such as the differential or total cross section. Detailed calculation of the scattering solution to the Schrödinger equation with a positive scattering energy may be found in Refs. 5 and 6. The scattering cross section calculated from the scattering solution appears to depend on the scale μ and $\lambda_R(\mu)$, but they are related according to Eq. (12) and can be simultaneously eliminated for the fixed B_0 . The cross section is then explicitly renormalization scheme independent and depends on the fixed binding energy as reference scale.

Finally, it should be noted that renormalization is not limited to quantum systems; it appears also in classical systems. We refer to Ref. 9 for a discussion with a simple classical example, where dimensional regularization was used.

ACKNOWLEDGMENT

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QUEEN ELIZABETH

When you wish to attack a colleague’s claim, criticise a world-view, modalise a statement you cannot *just* say that Nature is with you; ‘just’ will never be enough. You are bound to use other allies besides Nature. If you succeed, then Nature will be enough and all the other allies and resources will be made redundant. A political analogy may be of some help at this point. Nature, in scientists’ hands, is a constitutional monarch, much like Queen Elizabeth the Second. From the throne she reads with the same tone, majesty and conviction a speech written by Conservative or Labour prime ministers depending on the election outcome. Indeed she *adds* something to the dispute, but only after the dispute had ended; as long as the election is going on she does nothing but wait.

Bruno Latour, *Science in Action—How to Follow Scientists and Engineers Through Society* (Harvard University Press, Cambridge, Massachusetts, 1987), p. 98.