国际单位制下的电磁场

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1 国际单位制下的电磁场 [1]

定义:

$$x^\mu = (ct, \vec{x}) \tag{1}$$

$$\partial_\mu = (\frac{1}{c}\partial_t, \triangledown), \partial^\mu = (-\frac{1}{c}\partial_t, \triangledown) \tag{2}$$

进而我们定义相对论情况下的 4 矢量

$$p^\mu = (\frac{mc}{\sqrt{1-u^2/c^2}}, \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}) = (\frac{E}{c}, \vec{p}) \tag{3}$$

$$k^\mu = (\frac{\omega}{c}, \vec{k}), k_\mu = (-\frac{\omega}{c}, \vec{k}) \tag{4}$$

定义点乘符号形式满足

$$k \cdot x = k_\mu x^\mu \tag{5}$$

我们定义波的传播因子为

$$e^{-ik_\mu x^\mu} = e^{i\omega t - i\vec{k}\cdot\vec{x}} \tag{6}$$

定义电磁场相关的物理量为

$$A_\mu=(-\frac{\varphi}{c},\vec{A}),j_\mu=(-c\rho,\vec{j}) \tag{7}$$

可定义电流守恒方程

$$\partial_\mu j^\mu=0 \tag{8}$$

于是我们重新定义电场和磁场

$$\vec{\epsilon}=-\frac{1}{c}\nabla\varphi-\frac{1}{c}\partial_t\vec{A}=\frac{\vec{E}}{c}\rightarrow\frac{E_i}{c}=F^{0i} \tag{9}$$

$$B^{ij}=F^{ij}=\partial^i A^j-\partial^j A^i \tag{10}$$

电子和电磁场相互作用表达式

$$\triangle L=j_\mu A^\mu \tag{11}$$

以及得到麦克斯韦方程的张量形式

$$\sum_\nu\partial_\nu F^{\mu\nu}=\mu_0 j^\mu \tag{12}$$

$$\sum_{\nu mn}\epsilon_{\mu\nu mn}\partial_\nu F^{mn}=0 \tag{13}$$

以及电磁场的拉格朗日量密度

$$L=-\int d^3x\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} \tag{14}$$

2 国际单位制下的狄拉克方程和 KG 方程

国际单位制下的 KG 方程形式为

$$[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2+\frac{m^2c^2}{\hbar^2}]\psi=0 \tag{15}$$

$$[\partial_\mu\partial^\mu-\frac{m^2c^2}{\hbar^2}]\psi=0 \tag{16}$$

以及 Dirac 方程，形式为

$$(i\gamma^\mu\partial_\mu-\frac{mc}{\hbar})\psi=0 \tag{17}$$

根据电子静质量能量为 mc^2，动能为 0 时波函数不随时间和空间改变，大胆猜测国际单位制下的电子拉格朗日密度为

$$L=\int d^3x\hbar c\bar{\psi}(i\gamma^\nu\partial_\nu-\frac{mc}{\hbar})\psi \tag{18}$$

下面我们叙述从电磁场的拉格朗日量推导 Maxwell 方程。整体电磁场的拉格朗日量为

$$L=-\int d^3x\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu}+\int d^3x j_\mu A^\mu \tag{19}$$

3 推导 SU(2) 规范场和同位旋守恒 [6]

3.1 推导 SU(2) 规范场 [6]

$\psi \to U\psi$，只不过这里 $U \neq exp(-ie\int A\cdot dr)$，即不是简单的 U(1) 规范场。SU(2) 规范场与 U(1) 规范场类似，$\hat{p}-e\hat{A}$ 在规范变换下不变，即

$$U^{\dagger}(\partial_\mu - ieA'_\mu/\hbar)U = \partial_\mu - ieA_\mu/\hbar \tag{20}$$

$$(\partial_\mu - ieA'_\mu/\hbar)U = U(\partial_\mu - ieA_\mu/\hbar)[U^{\dagger}U] \tag{21}$$

$$A_\mu(x) \to A'_\mu(x) = UA_\mu U^{\dagger} + \frac{i\hbar}{e}U\partial_\mu U^{\dagger} \tag{22}$$

定义

$$D_\mu = \partial_\mu - ieA_\mu/\hbar, D'_\mu = \partial_\mu - ieA'_\mu/\hbar, UD_\mu U^{\dagger} = UD'_\mu U^{\dagger} \tag{23}$$

$$F'_{\mu\nu} = D'_\mu A'_\nu - D'_\nu A'_\mu = \partial_\mu A'_\nu - \partial_\nu A'_\mu - i\frac{e}{\hbar}[A'_\mu, A'_\nu] \tag{24}$$

相应的 SU(2) 规范不变量为

$$\int (D_\mu\varphi)^2 - V(\varphi^{\dagger}\varphi) \tag{25}$$

可以计算出

$$\begin{aligned}
F'_{\mu\nu} &= D'_\mu A'_\nu - D'_\nu A'_\mu \\
&= \partial_\mu(UA_\nu U^{\dagger} + \frac{i\hbar}{e}U\partial_\nu U^{\dagger}) - \partial_\nu(UA_\mu U^{\dagger} + \frac{i\hbar}{e}U\partial_\mu U^{\dagger}) \\
&- \frac{ie}{\hbar}[UA_\mu U^{\dagger} + \frac{i\hbar}{e}U\partial_\mu U^{\dagger}][UA_\nu U^{\dagger} + \frac{i\hbar}{e}U\partial_\nu U^{\dagger}] \\
&+ \frac{ie}{\hbar}[UA_\nu U^{\dagger} + \frac{i\hbar}{e}U\partial_\nu U^{\dagger}][UA_\mu U^{\dagger} + \frac{i\hbar}{e}U\partial_\mu U^{\dagger}] \\
&= U(\partial_\mu A_\nu - \partial_\nu A_\mu - i\frac{e}{\hbar}[A_\mu, A_\nu])U^{\dagger} \\
&= U(D_\mu A_\nu - D_\nu A_\mu)U^{\dagger}
\end{aligned} \tag{26}$$

由于 $\partial_\mu U = -U(\partial_\mu U^{\dagger})U, \partial_\mu U^{\dagger} = -U^{\dagger}(\partial_\mu U)U^{\dagger}$，可将上式化简为

$$F'_{\mu\nu} = UF_{\mu\nu}U^{\dagger} \tag{27}$$

定义 $U = exp(-i\theta^a T^a)$，要求 $[T^a, T^b] = \epsilon_{abc}T^c$。在 SU(2) 规范场中

$$T^a = \sigma^a/2, A_\mu = \sum_c A^c_\mu T^c, F_{\mu\nu} = \sum_c F^c_{\mu\nu}T^c \tag{28}$$

可得

$$A^{a\prime}_\mu = A^a_\mu + \epsilon_{abc}\theta^b A^c_\mu - \partial_\mu\theta^a \tag{29}$$

$$F'^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \frac{e}{\hbar}\epsilon_{abc}A^b_\mu A^c_\nu \tag{30}$$

我们常常应用时间规范下的形式，即规范势的时空分量为 0(即不考虑含时演化) 形式，有

$$A_0(x) = 0, A_{i\neq 0}(x) = \frac{i\hbar}{e}U\partial_i U^{\dagger}, \partial_\mu = \partial^\mu \tag{31}$$

下面我用自己的方式证明一个表达式 $F = d\bigwedge A - iA\bigwedge A$[2, 4]，我定义在原子单位制下

$$d\bigwedge A - iA\bigwedge A = \epsilon_{\nu\mu}\partial_\nu A^\mu - i\epsilon_{\nu\mu}\frac{e}{\hbar}A_\nu A^\mu \tag{32}$$

$$A'_\mu(x) = UA_\mu U^\dagger + \frac{i\hbar}{e}U\partial_\mu U^\dagger \tag{33}$$

下面我们证明幺正变换的连续性

$$U = U_a U_b \tag{34}$$

$$A'_\mu(x) = UA_\mu U^\dagger + \frac{i\hbar}{e}U\partial_\mu U^\dagger = U_a[U_b A_\mu U_b^\dagger + \frac{i\hbar}{e}U_b\partial_\mu U_b^\dagger]U_a^\dagger + \frac{i\hbar}{e}U_a\partial_\mu U_a^\dagger \tag{35}$$

经过一系列计算步骤得

$$d\bigwedge A' - iA'\bigwedge A' = U(d\bigwedge A - iA\bigwedge A)U^\dagger \tag{36}$$

于是我们可以得出 $F_{\mu\nu}$ 的另一种等价定义形式

$$F_{\mu\nu} = (d\bigwedge A - iA\bigwedge A)_{\mu\nu} = -(\epsilon_{\nu\mu}\partial_\nu A^\mu - i\epsilon_{\nu\mu}\frac{e}{\hbar}A_\nu A^\mu) = \epsilon_{\mu\nu}\partial_\mu A^\nu - i\epsilon_{\mu\nu}\frac{e}{\hbar}A_\mu A^\nu \tag{37}$$

我们可以得到 SU(2) 规范不变量的另一种表述形式

$$\mathcal{L} = -\frac{1}{2\mu_0}tr[(F^a_{\mu\nu}T^a)^2] = -\frac{1}{4\mu_0}(F^a_{\mu\nu})^2 \tag{38}$$

3.2 推导同位旋守恒 [6]

根据带电粒子和非阿贝尔规范场的耦合公式

$$\mathcal{L} = -\frac{1}{4\mu_0}F^a_{\mu\nu}F^{\mu\nu a} + \hbar c\bar{\psi}(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar})\psi + \hbar\bar{\psi}\gamma^\mu\frac{eA^a_\mu T^a}{\hbar}\psi = -\frac{1}{4\mu_0}F^a_{\mu\nu}F^{\mu\nu a} + \hbar c\bar{\psi}(i\gamma^\mu D_\mu - \frac{mc}{\hbar})\psi \tag{39}$$

由于

$$D_\mu = \partial_\mu - ieA_\mu/\hbar = \partial_\mu - \sum_a ieA^a_\mu T^a/\hbar \tag{40}$$

定义

$$J^{\mu a} = ec\bar{\psi}\gamma^\mu T^a\psi \tag{41}$$

我们将拉格朗日量重新写作

$$\mathcal{L} = -\frac{1}{4\mu_0}F^a_{\mu\nu}F^{\mu\nu a} + \hbar c\bar{\psi}(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar})\psi + J^{\mu a}A^a_\mu \tag{42}$$

根据拉格朗日量的变分原理得到以下两个运动方程

$$(i\gamma^\mu D_\mu - \frac{mc}{\hbar})\psi = 0 \tag{43}$$

$$-\frac{1}{\mu_0}\sum_\mu\partial_\mu F^{\mu\nu a} = J^{\nu a} + \frac{1}{\mu_0}\frac{e}{\hbar}\epsilon_{abc}\sum_\mu A^b_\mu F^{\mu\nu c} \tag{44}$$

根据上式可以导出

$$\sum_\nu\partial_\nu(J^{\nu a} + \frac{1}{\mu_0}\frac{e}{\hbar}\epsilon_{abc}\sum_\mu A^b_\mu F^{\mu\nu c}) = 0 \tag{45}$$

根据上式可定义同位旋电流和同位旋荷，可推导同位旋守恒。定义同位旋为如下形式

$$\mathcal{J}^\nu = J^\nu - \frac{i}{\mu_0}\frac{e}{\hbar}\sum_\mu [A_\mu, F^{\mu\nu}] \tag{46}$$

$$\partial_\nu \mathcal{J}^\nu = 0 \tag{47}$$

类比于电磁场的洛伦兹规范，可引进同位旋的洛伦兹规范

$$\partial_\mu A^{\mu a} = 0 \tag{48}$$

下面我们得到 SU(2) 规范场的量子化 [6]，此和原始文献推导略有出入

$$\Pi_\mu = \frac{\partial L}{\partial(\partial_0 A_\mu)} = -\frac{1}{\mu_0}F^{0\mu} \tag{49}$$

$$[A^a_\mu(x), \Pi^b_\nu(x')] = i\hbar\delta_{ab}\delta_{\mu\nu}\delta^3(\vec{x}-\vec{x}') \tag{50}$$

4 国际单位制下的传播子

根据国际单位制下的 Dirac 方程

$$L = \int d^4x\bar{\psi}\hbar c(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar})\psi \tag{51}$$

根据平面波 $e^{-ik_\mu x^\mu} = e^{-ik\cdot x}$ 因子，傅里叶变换形式为 $f(x) \propto \int \frac{d^4p}{(2\pi)^4}f(p)e^{-ip\cdot x}$，于是动量空间拉格朗日量的形式为

$$S = \int Ldt = \int \frac{d^4p}{(2\pi)^4}\hbar c\bar{\psi}(\gamma\cdot p - \frac{mc}{\hbar})\psi \tag{52}$$

根据场的路径积分公式

$$\langle\bar{\psi}(q)\psi(q)\rangle = \frac{\int D\bar{\psi}_q D\psi_q e^{iS}\bar{\psi}_q\psi_q}{\int D\bar{\psi}_q\psi_q e^{iS}} \tag{53}$$

故可求得传播子形式为

$$\langle\bar{\psi}(q)\psi(q)\rangle = \frac{i}{\hbar c\gamma\cdot p - mc^2} \tag{54}$$

而对于时序格林函数，则需修改为

$$\langle\bar{\psi}(q)\psi(q)\rangle = \frac{1}{\hbar c}\frac{i(\gamma\cdot p + \frac{mc}{\hbar})}{p^2 - (\frac{mc}{\hbar})^2 + i\epsilon} \tag{55}$$

我们定义

$$j^\mu = ce\bar{\psi}\gamma^\mu\psi \tag{56}$$

我们发现此满足电流守恒方程 eq8，我们考虑加上电场后的拉格朗日量，发现恰好满足

$$\begin{aligned} L &= \int d^4x\hbar c\bar{\psi}[i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}]\psi - j^\mu A_\mu \\ &= \int d^4x\hbar c\bar{\psi}[\gamma^\mu(p_\mu - \frac{eA_\mu}{\hbar}) - \frac{mc}{\hbar}]\psi \end{aligned} \tag{57}$$

5 参考文献

参考文献

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