$\mathbb{Z}_4$ parafermions in Weakly Interacting Superconducting Constrictions at the Helical Edge of Quantum Spin Hall Insulators

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Parafermions are generalizations of Majorana fermions that may appear in interacting topological systems. They are known to be powerful building blocks of topological quantum computers. Existing proposals for realizations of parafermions typically rely on strong electronic correlations which are hard to achieve in the laboratory. We identify a novel physical system in which parafermions generically develop. It is based on a quantum constriction formed by the helical edge states of a quantum spin Hall insulator in the vicinity of an ordinary s-wave superconductor. Interestingly, our analysis suggests that $\mathbb{Z}_4$ parafermions are emerging bound states in this setup in the *weakly interacting* regime. Furthermore, we identify a situation in which Majorana fermions and parafermions can coexist.

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Introduction.—During the last decades, topological quantum physics has become one of the most active directions of modern condensed matter research. Especially, the formation of topological boundary excitations, such as Majorana fermions [1,2], has attracted a lot of attention, both theoretically as well as experimentally [3–12]. These robust bound states have been proposed in various host materials, ranging from vortices in $p_x + ip_y$ superconductors [13,14] over ferromagnet-superconductor heterojunctions in quantum spin Hall insulators (QSHIs) [15–21] to spin-orbit coupled quantum wires [3,4]. Due to their non-Abelian statistics [22–24], the interest in those topological bound states is not only fundamental but also practical: they can potentially be used for protocols in topological quantum computation (TQC) [25]. Majorana fermions are the conceptually simplest representatives of non-Abelian particles. However, braiding of Majorana fermions is not able to generate all the operations needed for universal TQC. For this task, more complex anyonic particles, assigned in general to a $\mathbb{Z}_n$ permutation group, are required [23,26]. Due to the high groundstate degeneracy of those $\mathbb{Z}_n$ anyons, electron-electron interactions are essential in the physical realizations thereof. In particular, $\mathbb{Z}_n$ parafermions are concrete examples of topological states that are proposed to emerge in correlated topological systems.

Recently, possible realizations of those exotic bound states have been predicted in different setups, including interacting QSHIs [27–31], fractional quantum Hall insulators [32–36], fractional QSHIs [37,38], quantum wires [39], or lattice systems [40–42]. Typically, s-wave superconductors are placed in proximity to a repulsively interacting region of the electronic system. Then, parafermionic bound states can form at the interface between two distinct regions in space. The experimental realization of parafermions is however an unsolved and undoubtedly challenging task. Difficulties arise as superconductors and strong magnetic fields are, for instance, required at the same time in fractional quantum Hall systems. In QSHIs, where magnetic fields are not essential, many proposals rely on particularly strong repulsive interactions at the corresponding helical edge. Although topological insulators based on InAs/GaSb quantum wells [43–45] have been shown to present a platform for repulsively interacting helical edge states, the magnitude of the interaction strength consistent with experimental data is under debate [46,47]. Moreover, previous proposals for parafermions in the weakly interacting regime [48,49] rely on the controlled use of magnetic impurities at the helical edge, which is definitely difficult to achieve in present-day experiments.

From our point of view, a feasible proposal for the generation of parafermions in the laboratory is still lacking. We argue for closing this gap in this Letter. The system we propose is a quantum constriction (QC) formed at the (weakly) interacting helical edge of a QSHI in proximity to two ordinary s-wave superconductors [see Fig. 1(a) for a schematic]. We are inspired by the investigation of similar setups in the absence of electronic correlations. In particular, the formation of Majorana bound states [50] and the emergence of odd-frequency superconductivity [51] has been theoretically proposed.

In the presence of interactions, the system becomes evidently much richer. Indeed, the QC gives rise to several interaction terms that are relevant in the renormalization group (RG) sense for a wide range of repulsive interactions. Generically, single- and two-particle scattering terms have to be taken into account. For the appearance of parafermions, two-particle scattering has to dominate over single-particle scattering. Surprisingly, we argue below that this

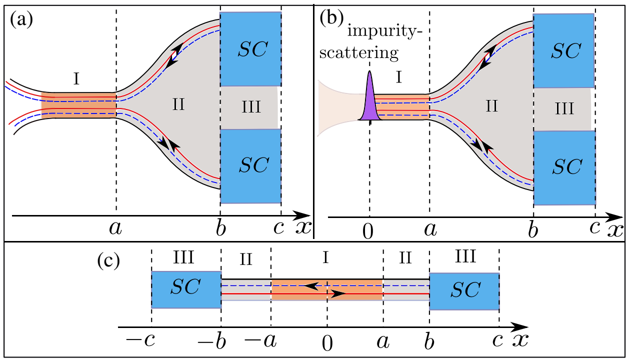

FIG. 1. (a) Schematic of the system: an extended QC in a stripe of a 2D topological insulator is brought in vicinity to two regions with proximity induced superconductivity. (b) The same setup as in (a) with an additional impurity that totally pinches off the QC. (c) Schematic of the unfolded structure after applying appropriate boundary conditions to case (b).

can even happen in the weakly interacting regime in our system. We identify two distinct cases schematically illustrated in Figs. 1(a) and 1(b), respectively. In case (a), a Majorana bound state and a parafermion coexist in region II in a nonlocal fashion, i.e., across the two edges of the QSHI. In case (b), where the QC is totally pinched off, which we illustrate by an impurity in the figure, two local parafermions appear in region II, being spatially separated at the two edges of the QSHI.

Model.—The starting points of our analysis are the two helical edge states formed at the boundary of a QSHI. With $\hbar = 1$, the kinetic energy is then described by the fermionic Hamiltonian

$$H_0 = \int dx \sum_{\substack{l=(R,L)=(+,-)\\ \sigma=\uparrow,\downarrow}} \hat{\psi}^\dagger_{l,\sigma}(x)(-iv_F l\partial_x)\hat{\psi}_{l,\sigma}(x) \quad (1)$$

with the Fermi field operators of the upper ($\hat{\psi}_{R,\uparrow}(x)$, $\hat{\psi}_{L,\downarrow}(x)$) and the lower edge ($\hat{\psi}_{R,\downarrow}(x), \hat{\psi}_{L,\uparrow}(x)$), respectively. Including density-density interactions in the usual way, we can bosonize the theory exploiting the bosonization identity in the charge-spin basis [52], i.e.,

$$\hat{\psi}_{r,\nu}(x) = \frac{\hat{U}_{r,\nu}e^{irk_F x}}{\sqrt{2\pi\alpha}} e^{-\frac{i}{\sqrt{2}}\{r\phi_\rho(x)-\theta_\rho(x)+\nu[r\phi_\sigma(x)-\theta_\sigma(x)]\}}, \quad (2)$$

where $r = R,\ L = +,-$ and $\nu = \uparrow,\ \downarrow = +,-$. $\hat{U}_{r,l}$ are Klein factors lowering the number of fermions by one. In Eq. (2), α denotes a high-energy cutoff. The conjugate bosonic fields $\phi_{\rho/\sigma}(x)$, $\theta_{\rho/\sigma}(x)$ are linear combinations of bosonic fields on the upper and lower edge (designated by the indices 1 and 2): $\phi_\rho = 1/\sqrt{2}[\phi_1(x) + \phi_2(x)]$, $\phi_\sigma = 1/\sqrt{2}[\theta_2(x) - \theta_1(x)]$, $\theta_\rho = 1/\sqrt{2}[\theta_1(x) + \theta_2(x)]$, $\theta_\sigma = 1/\sqrt{2}[\phi_2(x) - \phi_1(x)]$, obeying the commutation relations

$$[\phi_\nu(x), \theta_\mu(y)] = i\pi\theta(y-x)\delta_{\nu\mu}. \quad (3)$$

The interacting extension of the Hamiltonian of Eq. (1) can then be written in the well-known bosonized form

$$H_0 = \frac{1}{2\pi}\int dx \sum_{\nu=\rho,\sigma}\left(\frac{u_\nu}{K_\nu}[\partial_x\phi_\nu(x)]^2 + u_\nu K_\nu[\partial_x\theta_\nu(x)]^2\right) \quad (4)$$

with renormalized velocities u_ν and Luttinger interaction parameters K_ρ and K_σ characterizing the interaction strength. For helical Luttinger liquids, where spin-rotation invariance is strongly broken, $K_\rho < 1$ and $K_\sigma > 1$ for repulsive interactions, likewise, $K_\rho > 1$ and $K_\sigma < 1$ for attractive interactions. For vanishing interedge interaction strength, the interaction parameters of both channels (charge and spin) are coupled to each other by $K_\rho = 1/K_\sigma \equiv K_0$ [53]. This strong coupling of K_ρ and K_σ is, however, lost if interedge interaction is switched on. Hence, in our system, the interaction parameters should obey a spatial dependence when the two helical edges of the QSHI are brought together in the QC. There, we expect to have $K_\rho < K_0$ and $1 \leq K_\sigma < 1/K_0$ provided that intraedge interactions are stronger than interedge interactions.

Apart from density-density interactions, in regions I and III of Fig. 1(a), additional interaction terms, that do not result in a quadratic form after bosonization, have to be taken into account. In region III, we consider superconducting s-wave pairing. This can be incorporated on the basis of a BCS mean field approach by the following fermionic Hamiltonian

$$H_\Delta = \int dx\Delta(x)[\hat{\psi}^\dagger_{R,\uparrow}(x)\hat{\psi}^\dagger_{L,\downarrow}(x) + \hat{\psi}^\dagger_{L,\uparrow}(x)\hat{\psi}^\dagger_{R,\downarrow}(x)] + \text{H.c.}, \quad (5)$$

where $\Delta(x)$ is a spatially dependent pairing potential. Since we do not assume a connection between the two helical edges in region III, justified by a significant width of the QSHI system, the corresponding Hamiltonian is diagonal in the fields of upper and lower edge. Using the bosonization identity [Eq. (2)] neglecting Klein factors [52], the bosonized form of Eq. (5) becomes

$$H_\Delta = \int dx\tilde{\Delta}(x)\{\sin[2\theta_1(x)] + \sin[2\theta_2(x)]\}, \quad (6)$$

with $\tilde{\Delta}(x) = \Delta(x)/(\pi\alpha)$.

For the QC in region I, we consider all possible single- and two-particle scattering terms that (i) preserve time-reversal symmetry, (ii) are able to open a (partial) gap, and (iii) are relevant in the RG sense for a wide range of (weak) repulsive interactions, see Supplemental Material (SM [54]). Those terms are in fermionic representation

$$H_{\mathrm{t}} = \int dx\, t(x)[\hat{\psi}^{\dagger}_{R,\uparrow}\hat{\psi}_{L,\uparrow} + \hat{\psi}^{\dagger}_{L,\downarrow}\hat{\psi}_{R,\downarrow}] + \mathrm{H.c.}, \quad (7)$$

$$H_{\mathrm{um}} = \int dx\, g_{\mathrm{um}}(x)\hat{\psi}^{\dagger}_{R,\uparrow}\hat{\psi}^{\dagger}_{R,\downarrow}\hat{\psi}_{L,\downarrow}\hat{\psi}_{L,\uparrow} + \mathrm{H.c.}, \quad (8)$$

$$H_{\mathrm{pbs}} = \int dx\, g_{\mathrm{pbs}}(x)\hat{\psi}^{\dagger}_{R,\uparrow}\hat{\psi}_{L,\downarrow}\hat{\psi}^{\dagger}_{L,\uparrow}\hat{\psi}_{R,\downarrow} + \mathrm{H.c.}, \quad (9)$$

where we have dropped the explicit spatial dependence of the field operators to save space. Note that the single-particle scattering term [Eq. (7)] [50,55–58] and the Umklapp scattering term [Eq. (8)] [52] are spin-preserving processes, while the pair backscattering term [Eq. (9)] [59–61] requires breaking of axial spin symmetry [62]. Applying the bosonization identity [Eq. (2)] to Eqs. (7)–(9), neglecting Klein factors, we obtain the bosonized Hamiltonians

$$H_{\mathrm{t}} = \int dx \tilde{t}(x) \cos[\sqrt{2}\phi_\rho(x) - \bar{k}_F x] \cos[\sqrt{2}\phi_\sigma(x)], \quad (10)$$

$$H_{\mathrm{um}} = \int dx \tilde{g}_{\mathrm{um}}(x) \cos[2\sqrt{2}\phi_\rho(x) - 2\bar{k}_F x], \quad (11)$$

$$H_{\mathrm{pbs}} = \int dx \tilde{g}_{\mathrm{pbs}}(x) \cos[2\sqrt{2}\theta_\sigma(x) + 2\delta k_F x] \quad (12)$$

with $\tilde{t}(x) = 2t(x)/(\pi\alpha)$, $\tilde{g}_{\mathrm{s}}(x) = g_{\mathrm{s}}(x)/(2\pi^2\alpha^2)$, $\tilde{g}_{\mathrm{pbs}}(x) = g_{\mathrm{pbs}}(x)/(2\pi^2\alpha^2)$, $\bar{k}_F = (k_{F,1} + k_{F,2})$, and $\delta k_F = (k_{F,1} - k_{F,2})$, where $k_{F,1}$, $k_{F,2}$ are the chemical potential in edge 1, 2, respectively.

Note that H_{um} and H_{pbs} commute with each other but both do not commute with H_{t} [63]. Hence, they cannot be ordered simultaneously in the same region of space. Therefore, pinning of the bosonic fields in the strong coupling regime is only possible if either single- or two-particle scattering dominates the physics. For the emergence of parafermions, it is mandatory that at least one of the two 2-particle scattering terms (H_{um} or H_{pbs}) provides the dominant interaction. Only then the required ground state degeneracy is present.

We now consider the relative importance of the terms in Eqs. (10)–(12). At the Dirac point, where $k_{F,1} = k_{F,2} = 0$, the RG equations of the terms [Eqs. (10)–(11)] can be straightforwardly obtained, see SM [54]. All the three terms are RG relevant for a wide range of repulsive interactions [60,64–66]. We can order their relative importance according to their scaling dimension. This ordering yields the following inequalities between K_ρ and K_σ for the regime in parameter space where two-particle terms dominate the low-energy physics

$$K_\sigma > 3K_\rho, \quad (13)$$

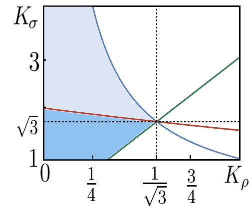

FIG. 2. Illustration of the conditions obtained from the RG analysis of the various terms. The lines correspond to $K_\sigma = 3K_\rho$ (green), $K_\sigma = -K_\rho/2 + \sqrt{K_\rho^2/4 + 4}$ (red), and $K_\sigma = 1/K_\rho$ (blue). The light and dark blue shaded areas mark the parameter regime where two-particle terms are RG dominant.

$$K_\sigma > -K_\rho/2 + \sqrt{K_\rho^2/4 + 4}. \quad (14)$$

These equations emerge from imposing the required hierarchy in the scaling dimensions such that at least one of the two 2-particle backscattering terms [Eqs. (11) or (12)] dominates over the single-particle backscattering term [Eq. (10)], see SM [54]. More precisely, if Eq. (13) [Eq. (14)] is fulfilled, then H_{um} (H_{pbs}) dominates over H_{t} in the RG sense. The conditions are illustrated in Fig. 2. As K_σ cannot exceed $1/K_\rho$ in our model, the shaded areas represent the parameter space for which at least one of the two 2-particle processes is dominant. For the emergence of parafermions, it is indeed sufficient (as we show below) that either H_{pbs} or H_{um} is more relevant than H_{t}. In fact, when H_{um} is the most relevant interaction, we first pin ϕ_ρ to minimize the contribution from the dominating term H_{um}. It turns out that for the pinned values that ϕ_ρ is forced to, H_{t} vanishes exactly. Due to this property, this subsequently also allowed us to pin the bosonic field operators that characterize H_{pbs}. Having H_{pbs} stronger than H_{t}, and consequently pinning θ_σ, this also allows us to neglect H_{t} on the basis of energetics [37]. The lowest energy states, which are degenerate in the absence of H_{t}, are then obtained by pinning H_{um}.

If we look at Fig. 2, it suggests that $K_\rho < 1/\sqrt{3} \approx 0.58$ is needed such that two-particle backscattering dominates over single-particle scattering. This range of K_ρ still corresponds to rather strong Coulomb interactions. In fact, the constraint can be released a little bit if inter- and intraedge forward scattering are more accurately taken into account, see SM [54]. Nevertheless, there is no substantial gain by these refined arguments in terms of a reduction of the required interaction strength. Fortunately, there are two physical scenarios that lead to a suppression of single-particle backscattering compared to the two-particle processes (even in the regime $1/\sqrt{3} < K_\rho < 1$). (i) We can take into account the different origin of the coupling constants of single- and two-particle scattering. The amplitude of single-particle scattering is related to the overlap of the wave functions of the edge states. Thus, it decays exponentially with the width of the QC. In contrast, the amplitude of two-particle scattering is related to Coulomb interactions following a power-law decay. We hence expect the existence of a geometric regime of the QC where single particle backscattering can be neglected compared to two-particle backscattering [67,68]. In this scenario, we require the chemical potential to be sufficiently close to the Dirac point in both edges such that $\bar{k}_F \ll a^{-1}$ and $\delta k_F \ll a^{-1}$ with a being the length of the QC. (ii) For narrow QCs, where the electrostatical environment is approximately the same for both edges, we can assume $\delta k_F = 0$. Thus, a finite chemical potential mainly affects H_{t} and H_{um}, but leaves H_{pbs} invariant. If the flow is analyzed at a fixed chemical potential, we find a finite chemical potential range in which H_{t} and H_{um} are still relevant [52]. However, for $\bar{k}_F a \gtrsim \pi$ the energy contribution of these terms is reduced at least by $1/(\bar{k}_F a)$ with a being the length of the QC. Typical values in experiments based on Hg(Cd)Te quantum wells are given by $a \sim 300$ nm and $k_F^{\max} \sim 2 \times 10^7$ m^{-1} [18]. This reduction can then lead to a crossover for which H_{pbs} becomes the dominant scattering process.

Under the assumption that two-particle scattering dominates the physics in the QC at low energies, we drop the single-particle scattering completely for the subsequent discussion of parafermions. As the relative weight of $\tilde{g}_{\mathrm{um}}$ and $\tilde{g}_{\mathrm{pbs}}$ does not influences the groundstate degeneracy, we choose for simplicity $g_{\mathrm{pbs}}(x) = g_{\mathrm{um}}(x)$ and reorganize Eqs. (11) and (12) into

$$H_{2\mathrm{p}} = H_{\mathrm{um}} + H_{\mathrm{pbs}} = \int dx 2\tilde{g}_{\mathrm{pbs}}(x)\cos[2\phi_1(x)]\cos[2\phi_2(x)]. \tag{15}$$

This Hamiltonian constitutes the basis for the following construction of parafermionic operators.

Parafermions.—Non-Abelian exchange statistics requires ground-state degeneracy. In the thermodynamic limit for the gapped phase of a sine-Gordon theory, it makes sense to assume that, deep inside the gapped area, the fields are pinned such that the cosine potential is minimized. The corresponding fields $\phi_{1/2}(x), \theta_{1/2}(x)$ can take values $[0, 2\pi[$ (modulo 2π) [27]. Within this range, several minima of the assigned cosine or sine potentials can be reached, which implies a degenerate ground state. For the superconducting section [region III of Fig. 1(a)], the fields $\theta_1(x)$ and $\theta_2(x)$ are pinned independently. To properly formalize this, we introduce phase fields $\hat{\theta}_1$ and $\hat{\theta}_2$, where the corresponding eigenvalues are designated to the pinned values. The gap induced by the QC [region I of Fig. 1(a)], however, involves both edges, implying a correlation between $\phi_1(x)$ and $\phi_2(x)$. Indeed, minimization of Eq. (15) is achieved whenever one of the two cosines is maximized and the other is minimized. This constraint forces a relation between the assigned phase fields $\hat{\phi}_1$ and $\hat{\phi}_2$

$$\hat{\phi}_2 = -\hat{\phi}_1 - \pi/2 + \pi\hat{l}, \tag{16}$$

where $\hat{\phi}_1$ takes the eigenvalues $\phi_1 \in \{0, \pi/2, \pi, 3/2\pi\}$ and the integer valued operator $\hat{l}$ with eigenvalues $l \in \{1, 2\}$ (modulo 2) relates $\hat{\phi}_1$ and $\hat{\phi}_2$. The representation of $\hat{\phi}_2$ in terms of $\hat{\phi}_1$, together with Eq. (3) for the system shown in Fig. 1(a), implies the following commutation relations

$$[\hat{l}, \hat{\theta}_1] = [\hat{l}, \hat{\theta}_2] = i, \quad [\hat{\phi}_1, \hat{\theta}_1] = i\pi, \quad [\hat{\phi}_1, \hat{\theta}_2] = 0. \tag{17}$$

With Eq. (16), the two-particle scattering [Eq. (15)] in the QC can be written as

$$H_{2p} = -\tilde{g}_{\mathrm{pbs}} a(\cos[4\hat{\phi}_1 - 2\pi\hat{l}] + \cos[2\pi\hat{l}]), \tag{18}$$

where we assumed the length of each section to be a for simplicity.

With the (quasi)conjugate variables $\hat{\phi}_1$, $\hat{l}$, $\hat{\theta}_1$, $\hat{\theta}_2$ and the different sections being disjoint, we are able to construct parafermionic bound states at the interface between two neighboring sections, in a similar way as in Refs. [28,32,33]. The major difference in our case (as compared to previous work) stems from the presence of the operator $\hat{l}$. It turns out that this operator can change the ground-state manifold and lead to *nonlocal* bound state operators. This nonlocality arises as any parafermionic operator, applied to a certain ground state, cannot add energy to the system but rather projects the system onto another (degenerate) ground state. This implies that the parafermionic creation operator necessarily needs to commute with the Hamiltonian. For our present case, however, it is not possible to write a purely local operator that obeys this constraint. This is indeed a direct consequence of the presence of the operator $\hat{l}$ that couples $\hat{\phi}_1$ and $\hat{\phi}_2$.

We find that the set of operators $\{e^{i(\pi/2)\hat{S}}, e^{i\pi\hat{l}}\}$, where $\pi\hat{S} = \hat{\theta}_1 - \hat{\theta}_2$, commute with the Hamiltonian and among themselves. They induce the degenerate set of states $|s, l\rangle$, each of which satisfying $e^{i(\pi/2)\hat{S}}|s, l\rangle = e^{i(\pi/2)s}|s, l\rangle$ and $e^{i\pi\hat{l}}|s, l\rangle = e^{i\pi l}|s, l\rangle$ with distinct eigenvalues $s \in \{0, 1, 2, 3\}$ (modulo 4) and $l \in \{1, 2\}$ (modulo 2). Moreover, it is easy to demonstrate that the operators

$$\hat{\chi}_s = e^{i\pi/4} e^{i\pi\hat{\phi}_1} e^{i/2(\hat{\theta}_1 - \hat{\theta}_2)}, \quad \hat{\chi}_l = e^{i\pi/2} e^{i/2(\hat{\theta}_1 + \hat{\theta}_2)} e^{i\pi\hat{l}} \tag{19}$$

commute with the Hamiltonian and describe creation operators of the quantum numbers s and l, respectively,

$$\hat{\chi}_s|s, l\rangle = e^{i(\pi/2)s}|s+1, l\rangle, \quad \hat{\chi}_l|s, l\rangle = e^{i\pi l}|s, l+1\rangle. \tag{20}$$

From Eqs. (19) and (20), we obtain the relations

$$\hat{\chi}_s\hat{\chi}_l = e^{-i\pi/2}\hat{\chi}_l\hat{\chi}_s, \qquad \hat{\chi}_s^4 = 1, \qquad \hat{\chi}_l^2 = 1. \tag{21}$$

These relations imply the simultaneous presence of a $\mathbb{Z}_4$ parafermion ($\hat{\chi}_s$) as well as a Majorana zero mode ($\hat{\chi}_l$).

To the best of our knowledge, this combination has not been predicted before. It should be noted that the fields $\hat{\theta}_1$ and $\hat{\theta}_2$ are necessarily contained within each operator. Hence, they describe nonlocal bound states delocalized across the upper and lower edge of region II in Fig. 1(a).

Interestingly, we can slightly modify our setup according to Fig. 1(b) to obtain local parafermions. The physical situation corresponds to a pinched off QC, which is either realized by a strong impurity or a physical boundary of the structure. We can model it by adding the impurity Hamiltonian [65]

$$H_{\mathrm{imp}} = V[\hat{\psi}^{\dagger}_{R,\uparrow}(0)\hat{\psi}_{L,\uparrow}(0) + \hat{\psi}^{\dagger}_{L,\downarrow}(0)\hat{\psi}_{R,\downarrow}(0)] + \mathrm{H.c.} \quad (22)$$

The corresponding (hard-wall) boundary conditions are derived on the fermionic level [69–72] (see SM [54]). In bosonic language, it forces the following (nonlocal) relations between bosonic field operators

$$\phi_2(x) = -\phi_1(-x) - \pi/2, \qquad \theta_2(x) = \theta_1(-x). \quad (23)$$

Although Eq. (23) contains two different points x and $-x$ in space, the similarity to Eq. (16) is apparent, where now the operator $\hat{l}$ (that induced the nonlocality of the resulting bound state operators) is absent since the pinning of the fields due to interactions has not been performed yet. Starting from the physical setup depicted in Fig. 1(b), taking into account Eq. (23), we unfold the system to arrive at an equivalent (sine-Gordon) model illustrated in Fig. 1(c). Note that the unfolding is a pure mathematical procedure with no physical meaning. The resulting sine-Gordon model contains the following mass terms

$$H_{\Delta} = H_{\Delta-} + H_{\Delta+} = \tilde{\Delta}\left[\int_{-c}^{-b} + \int_{b}^{c}\right] dx \sin[2\theta_1(x)], \quad (24)$$

$$\tilde{H}_{2\mathrm{p}} = -\tilde{g}_{\mathrm{pbs}} \int_{-a}^{a} dx \cos[2\phi_1(x)]\cos[2\phi_1(-x)], \quad (25)$$

which includes nonlocal interaction terms coupling the fields at position x and $-x$ [69]. The minimization of Eq. (25) requires a constant field $\phi_1(x) = \phi_1(-x) = \hat{\phi}$, since any modulation with space adds energy $\propto \tilde{g}_{\mathrm{pbs}}$. With the introduction of phase fields $\hat{\phi}_1$ and $\hat{\theta}_{\pm}$ [where $\pm$ refers to the superconductor right $(+)$ and left $(-)$ of the origin in Fig. 1(c)], we obtain effective Hamiltonians

$$H_{\Delta\pm} = \tilde{\Delta}(c-b)\sin[2\hat{\theta}_{\pm}], \quad (26)$$

$$H_{2\mathrm{p},j} = -\tilde{g}_{\mathrm{pbs}} a(\cos[4\hat{\phi}_1] + 1). \quad (27)$$

The relevant bosonic field operators obey the following commutation relations $[\hat{\phi}_1, \hat{\theta}_-] = 0$, $[\hat{\phi}_1, \hat{\theta}_+] = i\pi$, which imply that

$$\hat{\xi}_- = e^{i\hat{\phi}_1}e^{(i/2)\hat{\theta}_-}, \qquad \hat{\xi}_+ = e^{i\hat{\phi}_1}e^{(i/2)\hat{\theta}_+} \quad (28)$$

commute with the Hamiltonian and obey parafermionic exchange relations

$$\hat{\xi}_-\hat{\xi}_+ = e^{-i\pi/2}\hat{\xi}_+\hat{\xi}_-, \qquad \hat{\xi}^4_{\pm} = 1. \quad (29)$$

The fourfold degenerate groundstate manifold is formed by eigenstates of the operator $\hat{\xi}^{\dagger}_-\hat{\xi}_+ = e^{(i/2)(\hat{\theta}_+ - \hat{\theta}_-)}$, measuring the spin trapped in between the two superconducting regions in space. $\hat{\xi}_{\pm}$ are purely local operators, bound in region II of Fig. 1(b), where each helical edge is occupied by a single parafermion. Including small overlaps between the two parafermionic bound states, this would result in a 8π-periodic Josephson current, when a superconducting phase shift is applied between the superconductors of upper and lower edge [27].

To summarize, we have proposed a system composed of a QC and proximity induced s-wave superconductivity in a QSHI that can host $\mathbb{Z}_4$ parafermionic bound states even in the weakly interacting regime. This finding is based on the competition between different coupling terms in the QC. We discuss their relative importance and construct explicit operators for the bound states. Our predictions should be observable by tunneling spectroscopy or the Josephson effect.

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