Spin-1 ladder: A bosonization study

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We construct a field-theoretic description of two coupled spin-1 Heisenberg chains, starting with the known representation of a single spin-1 chain in terms of Majorana fermions (or Ising models). After reexamining the bosonization rules for two Ising models, taking particular care of order and disorder operators, we obtain a bosonic description of the spin-1 ladder. From renormalization-group and mean-field arguments, we conclude that, for a small interchain coupling, the spin-1 ladder is approximately described by three decoupled, two-frequency sine-Gordon models. We then predict that, starting with decoupled chains, the spin gap decreases linearly with interchain coupling, in both the ferromagnetic and antiferromagnetic directions. Finally, we discuss the possibility of an incommensurate phase in the spin-1 zigzag chain.

I. INTRODUCTION

Among the properties of spin ladders, the best known is the reduction of order as we go from a single spin-$\frac{1}{2}$ Heisenberg chain to two coupled chains: the single spin-$\frac{1}{2}$ chain is critical (its correlation length is infinite) whereas the spin-$\frac{1}{2}$ ladder has finite-range correlations and an excitation gap, growing linearly with interchain coupling $J_\perp$, at least for small $J_\perp$ (for a review and further references, see Ref. 1). This may seem paradoxical because one would naively expect that coupling two quasiordered chains would only increase the tendency to order, but a critical system like the spin-$\frac{1}{2}$ Heisenberg chain is easily sent off-criticality by a perturbation such as ladder coupling $J_\perp$. In this paper we will study the corresponding spin-1 ladder (two coupled spin-1 chains), which is already disordered and has a finite gap at $J_\perp=0$. On the contrary, we will argue that the spin gap decreases as $J_\perp$ increases from zero, and does so for both antiferromagnetic and ferromagnetic interchain couplings, thus giving the gap $\Delta(J_\perp)$ a nonanalytic behavior (a cusp) at zero (cf. Fig. 4 below). We will arrive at this conclusion after obtaining a field-theoretic description of the spin-1 ladder in terms of six quantum Ising models or, alternately in terms of three boson fields. The motivation for using bosonization is that it offers a safer description of the system at weak $J_\perp$, valid for both positive and negative $J_\perp$, and allows at the same time for a description of the spin-1 zigzag chain, in which frustration plays a role. Thus, at small $J_\perp$, this method is more general and reliable than a σ model description. For a small antiferromagnetic interchain coupling, the drop in the gap as a function of $J_\perp$ was already noticed in Monte Carlo simulations and accounted for with a nonlinear σ model description of the spin-1 ladder.[2]

We will consider the spin-1 ladder as a perturbed critical model, so that the low-energy description of the system will be a perturbed conformal field theory. The critical model used as a starting point is a pair of decoupled biquadratic spin chains, with Hamiltonian

$$H_0=\sum_{\alpha,i}\{\mathbf{S}_{\alpha,i}\cdot\mathbf{S}_{\alpha,i+1}-(\mathbf{S}_{\alpha,i}\cdot\mathbf{S}_{\alpha,i+1})^2\}, \tag{1}$$

where $\mathbf{S}_{\alpha,i}$ is a spin-1 operator at site i on chain $\alpha(\alpha=1,2)$. At this critical point the two chains are decoupled, each chain being described by an integrable model[3,4] which is equivalent in the continuum limit to a level-2 su(2) Wess-Zumino-Witten (WZW) model.[5] We then need to consider the following perturbation (see Fig. 1):

$$H_I=(1+\eta)\sum_{i,\alpha}(\mathbf{S}_{\alpha,i}\cdot\mathbf{S}_{\alpha,i+1})^2 + \frac{1}{2}J_\perp\sum_i \mathbf{S}_{1,i}\cdot[(1+\delta)\mathbf{S}_{2,i}+(1-\delta)\mathbf{S}_{2,i+1}]. \tag{2}$$

When $(1+\eta)>0$, the first term brings us back to the Heisenberg point $(\eta=0)$. The interchain interaction, of strength $J_\perp$, is that of a ladder $(\delta=\pm1)$ or of a zigzag chain $(\delta=0)$. We will proceed by (i) constructing a continuum description of the interaction in term of WZW models and (ii) finding out the behavior of this perturbed WZW model by field-theoretic methods, mainly through representations in terms of Ising models (fermionization) and sine-Gordon models (bosonization).

This paper is organized as follows. In Sec. II, we review the field-theoretic description of a single spin-1 chain, in particular its representation in terms of three Majorana fermions (or Ising models). In Sec. III, we write down a representation of two coupled spin-1 chains in terms of three bosons, using the bosonization formulas for pairs of Ising models given in Appendix A. In Sec. IV, the behavior of the spin gap as a function of interchain coupling is inferred from this bosonized description. In Sec. V, the spin-1 zigzag chain is considered instead, and a weak interchain coupling is ar-

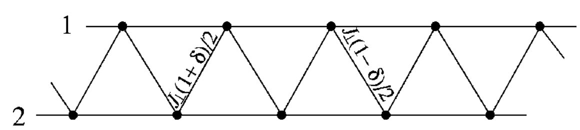

FIG. 1. Schematic illustration of the coupled spin chains with the various couplings, normalized to the intrachain coupling.

gued to cause a short-distance incommensurability, i.e., a displacement of the minimum of the one-magnon spectrum from $q=\pi$.

II. CONTINUUM DESCRIPTION OF THE SINGLE SPIN-1 CHAIN

A. Phase diagram

Let us first review the phase diagram of the biquadratic spin-1 chain:[6]

$$H=\sum_i \{\mathbf{S}_i\cdot\mathbf{S}_{i+1}+\eta(\mathbf{S}_i\cdot\mathbf{S}_{i+1})^2\}. \tag{3}$$

At $\eta=-1$, the Hamiltonian is integrable and has gapless modes at $k=0$ and $k=\pi$. It is also integrable at $\eta=1$ and has then gapless modes at $k=0$ and $k=\pm 2\pi/3$. If $\eta<-1$, we have a dimerized phase characterized by two degenerate ground states with a finite gap. On the other hand, in the interval $\eta\in(-1,1)$ the spectrum has a singlet ground state with a finite gap. This is the so-called Haldane phase, characterized by the spontaneous breakdown of a $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry.[7,8] This breakdown implies a fourfold-degenerate ground state in an open chain, but these different ground states differ only by the spins at the ends of the chain, and in this sense they are equivalent in the thermodynamic limit. The excitations are solitons switching from one ground state at $x\to-\infty$ to another ground state at $x\to\infty$. Related to the symmetry breaking is a dilute antiferromagnetic order; schematically,

$$+0\cdots 0-0\cdots 0+0\cdots 0-. \tag{4}$$

This order is defined by an alternation of sites with $S_z=1$ and $S_z=-1$, with some $S_z=0$ sites in between. It can be measured by the so-called string order parameter[7]

$$\mathcal{O}^z=\lim_{m-n\to\infty}\left\langle S_n^z \exp\left(i\pi\sum_{k=n+1}^{m-1} S_k^z\right)S_m^z\right\rangle. \tag{5}$$

This order parameter and the gap are maximal at $\eta=1/3$, where the valence-bond-solid-like ground state is exactly known.[9] The gap grows monotonically from $\eta=-1$ to $\eta=0$ without a phase transition, and thus we may consider the Heisenberg point ($\eta=0$) as a perturbation of the critical point ($\eta=-1$).

Note also that incommensurability develops starting at $\eta\approx 0.4$: the peak in the spin-spin correlation function moving from $k=\pi$ to $k=2\pi/3$ at $\eta=1$.[10] This last transition point is described by an SU(3) generalization of the Kosterlitz-Thouless phase transition.[11]

B. Field-theoretic description

The critical point ($\eta=-1$) is equivalent, in the low-energy limit, to a conformal field theory: the su(2) Wess-Zumino-Witten model at level $k=2$, plus a marginally irrelevant perturbation.[5] This WZW model contains two scaling fields: a spin doublet $g_{mn}(m,n\in\{-\frac{1}{2},\frac{1}{2}\})$ with left and right conformal dimensions $(\frac{3}{16},\frac{3}{16})$ and a spin triplet $\Phi_{mn}(m,n\in\{-1,0,1\})$ with dimensions $(\frac{1}{2},\frac{1}{2})$. They are, respectively, 2×2 and 3×3 matrix fields. The link between the spin chain and the WZW model is given by the following representation of the spin operators in the continuum limit:[5,12]

$$\frac{1}{a_0}S^a(x)=\frac{1}{2}[J^a(x)+\overline{J^a}(x)]+(-1)^{x/a_0}\Theta g^a(x), \tag{6}$$

where a_0 is the lattice constant, Θ a nonuniversal constant, J^a and $\overline{J^a}$ are the right and left su(2) currents, and g^a is defined in terms of Pauli matrices as

$$g^a=\frac{1}{\sqrt{2}}\mathrm{Tr}(\sigma^a g)=\frac{1}{\sqrt{2}}\sum_{m,n}\sigma^a_{mn}g_{nm}. \tag{7}$$

The currents (J^a, $\overline{J^a}$) and the field g^a correspond to the soft modes of the spin chain near $k=0$ and π, respectively.

For $1+\eta$ not too large, the spin chain may be described by the above WZW model, plus the following perturbation:[5]

$$\mathcal{L}_1=m\,\mathrm{Tr}\,\Phi-\lambda_1 J^a\overline{J^a}, \tag{8}$$

where a summation over repeated indices is implicit and $-m$ is proportional to $(1+\eta)$ (m is negative in the Haldane phase). The second term is the marginally irrelevant perturbation alluded to above (if $\lambda_1>0$). On the other hand, the first term ($\mathrm{Tr}\,\Phi$) is relevant, with scaling dimension 1, and leads to a gap proportional to $|m|\propto|1+\eta|$.

There is an interesting equivalence between the $k=2$ su(2) WZW model and three quantum Ising models,[13] and so we will not have to deal with the WZW model directly. This equivalence is defined by the following relations:

$$J^a=\frac{-i}{\sqrt{2}}\epsilon_{abc}\psi_b\psi_c,\quad \overline{J^a}=\frac{-i}{\sqrt{2}}\epsilon_{abc}\overline{\psi_b}\,\overline{\psi_c}, \tag{9}$$

$$\Phi_1=\frac{\zeta}{\sqrt{2}}(-\psi_1+i\psi_2),\quad \Phi_0=\zeta\psi_3,\quad \Phi_{-1}=\frac{\zeta}{\sqrt{2}}(\psi_1+i\psi_2), \tag{10}$$

$$\overline{\Phi}_1=\frac{\overline{\zeta}}{\sqrt{2}}(-\overline{\psi_1}-i\overline{\psi_2})\quad \overline{\Phi}_0=\overline{\zeta\psi_3},\quad \overline{\Phi}_{-1}=\frac{\overline{\zeta}}{\sqrt{2}}(\overline{\psi_1}-i\overline{\psi_2}), \tag{11}$$

$$g^0=\sqrt{2}\sigma_1\sigma_2\sigma_3,\quad g^a=-\sqrt{2}\sigma_a\mu_{a+1}\mu_{a+2}, \tag{12}$$

where the latin index goes from 1 to 3; ψ_a and $\overline{\psi_a}$ are respectively, the right and left fermions associated with each Ising model (see the Appendix). σ_a and μ_a are the order and disorder fields of each Ising model. The 3×3 matrix field Φ_{nm} is here factorized as $\Phi_{nm}\equiv\Phi_n\overline{\Phi}_m$. The constants ζ and $\overline{\zeta}$ are such that their product is $\zeta\overline{\zeta}=i$. Note that our relations differ slightly from those given by Fateev and Zamolodchikov.[13] The action of the WZW model in imaginary time becomes simply that of free Majorana fermions:

$$S_{\mathrm{WZW}}=\frac{1}{2\pi}\int dx\,d\tau(\psi_a\overline{\partial}\psi_a+\overline{\psi_a}\partial\overline{\psi_a}), \tag{13}$$

where $\partial=(\partial_\tau-i\partial_x)/2$ and $\overline{\partial}=(\partial_\tau+i\partial_x)/2$ (in order to lighten the notation, the characteristic velocity v of the WZW model has been set to unity). The perturbation (8) becomes

$$\mathcal{L}_1 = m\psi_a\overline{\psi}_a - \lambda_1\psi_a\overline{\psi}_a\psi_b\overline{\psi}_b\,. \tag{14}$$

Except for the marginally irrelevant term, the spin chain is thus equivalent to three Majorana fermions of mass m. This description of the spin-1 chain has been proposed by Tsvelik[14] who used it to study the effect of a magnetic field on the low-energy spectrum. It was also used in a field-theoretic treatment of the spin-$\frac{1}{2}$ ladder[39] and of the spin-$\frac{1}{2}$ zigzag chain.[16]

The representation (9)–(12) of the WZW fields is invariant under the following changes (for $a=1,2,3$ simultaneously):

$$\psi_a \to -\psi_a\,, \quad \overline{\psi}_a \to -\overline{\psi}_a\,, \quad \mu_a \to -\mu_a\,, \quad \sigma_a \to \sigma_a\,. \tag{15}$$

This is related to the absence of fermionic field in the WZW model. This ''gauge'' symmetry accounts for the expected degeneracy of the ground state near the critical point in open chains. Specifically, recall that $m<0$ in the Haldane phase. In our formulation, this corresponds to the disordered phase of the Ising models (see the Appendix) and the expectation value of the disordered operators is nonzero: $\langle \mu_a \rangle \neq 0$. In this phase each Ising model has a doubly degenerate ground state, associated with different spin configurations at the ends of the open Ising chain. The two ground states differ in the sign of $\langle \mu_a \rangle$. For the spin chain, this degeneracy implies an apparent eightfold ($8=2^3$) degeneracy, but the gauge invariance (15) reduces this to a physical fourfold degeneracy. These different ground states come from the breakdown of the hidden $\mathbb{Z}_2\times\mathbb{Z}_2$ nonlocal symmetry alluded to above, and are physically equivalent in the thermodynamic limit. In this Ising model description of the spin chain, the elementary excitations are kinks switching from one value of $\langle \mu_a \rangle$ at $x\to-\infty$ to its opposite at $x\to\infty$. On the other hand, Fáth and Sólyom[6] have shown that the excitations of the Heisenberg model are solitons connecting the ground states with different values of the string parameter (5). We are thus led to identify these solitons with the kinks of the Ising model.

We can do the same exercise for $m>0$ (or $\eta<-1$). We are now in the ordered phase of the Ising models: $\langle \sigma_a \rangle \neq 0$. Such an expectation value is already invariant under the gauge change (15) and therefore there are really eight physically different ground states for the open chain. A hidden $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry breaking is again expected and so these eight different ground states will be locally equivalent to two distinct ground states in the thermodynamic limit, corresponding to the expected dimerized state.

III. BOSONIZATION

Using the continuum description (6) of the spin operators, we obtain the following Lagrangian density from the Hamiltonian (1),(2), in terms of WZW fields:

$$\begin{aligned}\mathcal{L} = {} & \mathcal{L}_{\rm WZW}[g] + \mathcal{L}_{\rm WZW}[g'] + m\,{\rm Tr}\,\Phi + m\,{\rm Tr}\,\Phi' \\ & - \lambda_1(J^a\overline{J}^a + J'^a\overline{J}'^a) + \lambda_2(J^aJ'^a + \overline{J}^a\overline{J}'^a) \\ & + \lambda_3(J^a\overline{J}'^a + \overline{J}^aJ'^a) + \lambda g^a g'^a \\ & + \rho[g^a\partial_x g'^a - (\partial_x g^a)g'^a].\end{aligned} \tag{16}$$

The unprimed fields correspond to the first chain and the primed fields to second chain. The first three terms represent the intrachain interaction and the last four terms the interchain coupling. The interchain couplings $\lambda_2,\lambda_3,\lambda$, and ρ are, respectively, proportional to $J_\perp, J_\perp, J_\perp\delta$, and $J_\perp(1-\delta^2)$ at high energy, but they renormalize differently towards low energy. The last term, neglected in previous field-theoretic studies of the spin-$\frac{1}{2}$ zigzag chain,[15,16] has been considered by Nersesyan *et al.*[17] The particularity of this perturbation is its nonzero conformal spin, which makes the study of its relevance nontrivial. Nersesyan *et al.* have shown that, for the spin-1/2 *XX* zigzag chain, this perturbation (called the *twist term*) leads to a critical incommensurable phase. Finally note that ρ must be zero for the spin ladder, whereas λ vanishes for the pure zigzag chain. In the following we will consider the Haldane phase only so that m is negative.

The Lagrangian (16) is difficult to study in terms of WZW fields. The simplest information we may extract from it is the scaling dimension of the various perturbations, from those of the various WZW fields. Thus, the interchain couplings $\lambda_2,\lambda_3,\lambda$, and ρ, respectively, have scaling dimension 2, 2, $\frac{3}{4}$, and $\frac{7}{4}$. Moreover, the couplings λ_2 and ρ have conformal spin. By itself, a relevant coupling g of scaling dimension $\gamma<2$ and zero conformal spin is expected to produce a gap of order $\Delta\sim g^{1/(2-\gamma)}$. Thus, at the in-chain critical point ($\eta=-1$), the interchain coupling λ would open a gap of order

$$\Delta(\lambda)\sim\lambda^{4/5} \tag{17}$$

in the spin-1 ladder.

However, far from the critical point, the WZW model is of little help in predicting the behavior of the gap and the fermionic language seems more appropriate. Using the representation (9)–(12), we can express the Lagrangian density (16) in terms of Majorana fermions, order and disorder fields. Unfortunately, the resulting expression is not easy to study since it contains a mixture of fields that are mutually nonlocal (the order and disorder operators).

An interesting way to deal with the Lagrangian (16) is bosonization. The two-dimensional (2D) Ising models may be bosonized by pairing them (see the Appendix). The natural way to bosonize the ladder is to pair an Ising model describing one chain with its twin on the other chain. Using the relations (9)–(12), (16), and (A11), we obtain the following Lagrangian density for two coupled spin-1 chains:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_\rho + \mathcal{L}_\lambda\,,$$

$$\mathcal{L}_0 = \sum_{a=1,2,3}\left[\frac{1}{8\pi}[(\partial_\tau\varphi_a)^2 + (\partial_x\varphi_a)^2] - 2m\cos\varphi_a\right],$$

$$\mathcal{L}_1 = 16\lambda_1\sum_{a=1,2,3}(\cos\varphi_{a+1}\cos\varphi_{a+2} + \cos\theta_{a+1}\cos\theta_{a+2}),$$

$$\mathcal{L}_2 = 4\lambda_2\sum_{a=1,2,3}(\partial_\tau\varphi_{a+1}\partial_\tau\varphi_{a+2} - \partial_x\varphi_{a+1}\partial_x\varphi_{a+2}),$$

$$\mathcal{L}_3 = -8\lambda_3\sum_{a=1,2,3}(\sin\varphi_{a+1}\sin\varphi_{a+2} + \sin\theta_{a+1}\sin\theta_{a+2}),$$

$$\mathcal{L}_\lambda = 4\sqrt{2}\lambda \sum_{a=1,2,3} \cos\frac{\varphi_a}{2}\sin\frac{\varphi_{a+1}}{2}\sin\frac{\varphi_{a+2}}{2},$$

$$\mathcal{L}_\rho = -4\sqrt{2}\rho \sum_{a=1,2,3} \cos\theta_a\left[\cos\frac{\varphi_a}{2}\sin\frac{\varphi_{a+1}}{2}\sin\frac{\varphi_{a+2}}{2} - \sin\frac{\varphi_a}{2}\sin\left(\frac{\varphi_{a+1}+\varphi_{a+2}}{2}\right)\right], \quad (18)$$

where θ_a is the boson dual to φ_a. To shorten the expression, we have adopted a periodic condition on the index a, i.e., $a+3\equiv a$. The twist term $\mathcal{L}_\rho$, the trickiest to bosonized, has been inferred from the representation (A12) of the stress-energy tensor for each Ising model, plus the usual operator product expansion (OPE) between the energy-momentum tensor and a conformal field.

Thus, we have transformed the problem into a system of three perturbed sine-Gordon models, although the simultaneous presence of the bosons φ_a and of their dual fields θ_a makes some perturbations nonlocal. However, as we will see, the most relevant perturbation is local and makes the problem tractable in this language. Note that our normalization is such that $\cos(\beta\varphi_a)$ is marginal for $\beta=\sqrt{2}$, and thus bound states appear in the sine-Gordon model for $\beta<1$. Also, for $\beta<\sqrt{2}$, the $\varphi_a \to \varphi_a+2\pi$ symmetry is spontaneously broken and we have to consider fluctuations around one of the minima of the potential. However, it is important to keep in mind that our bosonization procedure is from the start invariant under the translation $\varphi_a\to\varphi_a+4\pi$, and this 4π-periodicity property must be regarded as a constitutive constraint imposed on the sine-Gordon models. Thus, each sine-Gordon model in $\mathcal{L}_0$ has *two* inequivalent ground states, associated with the minima $\varphi_a=\pm\pi$ of the potential (for $m<0$). The spontaneous breakdown of the symmetry $\varphi_a \to \varphi_a+2\pi$ implies a nonzero expectation value for the operators $\sin\varphi_a/2$ (the disorder operators μ_a) and $\cos\varphi_a$. Moreover, this breakdown becomes explicit if the perturbation $\mathcal{L}_\lambda$ or $\mathcal{L}_\rho$ is added. This symmetry breaking of the three sine-Gordon models corresponds in fact to the hidden symmetry breaking in the spin-1 chain (cf. Sec. II A). Therefore the different choices of the ground state ($\varphi_a=\pm\pi$) are equivalent, since the different ground states of the spin-1 chain are equivalent in the thermodynamic limit. Finally, let us recall that the elementary excitations of each sine-Gordon model have finite mass and correspond to the kink and antikink connecting the two different ground states. The charge conjugation changing kink into antikink corresponds to the following transformation:

$$\varphi(x)\to 2\pi-\varphi(x) \mod 4\pi. \quad (19)$$

The presence of nonzero expectation values for the operators $\sin\varphi_a/2$ and $\cos\varphi_a$ implies that more relevant terms may be generated from the perturbations (18). Let

$$\alpha=\left\langle \sin\frac{\varphi}{2}\right\rangle, \quad \alpha_1=\langle\cos\varphi\rangle, \quad (20)$$

the expectation values being taken in the $\varphi=\pi$ ground state. These have been calculated by Lukyanov and Zamolodchikov[18,19] and are proportional to $|m|^{1/4}$ and m, respectively. Then

$$\cos\frac{\varphi_a}{2}\sin\frac{\varphi_{a+1}}{2}\sin\frac{\varphi_{a+2}}{2} = \pm\alpha^2\cos\frac{\varphi_a}{2} + \text{fluctuations}. \quad (21)$$

The sign depends on the relative choice of the ground state for φ_{a+1} and φ_{a+2}. Keeping only the most relevant terms and neglecting the fluctuations of $\sin\varphi/2$ and $\cos\varphi$ around these expectation values, we find the following effective Lagrangian:

$$\mathcal{L}_{\text{eff}} = \sum_a \left\{ \frac{1}{8\pi}[(\partial_\tau\varphi_a)^2+(\partial_x\varphi_a)^2] - (2m-16\lambda_1\alpha_1)\cos\varphi_a \pm 4\sqrt{2}\lambda\alpha^2\cos\frac{\varphi_a}{2} \mp 4\sqrt{2}\rho\alpha\cos\theta_a\left[\cos\frac{\varphi_a}{2}-\cos\frac{\varphi_{a+1}}{2}-\cos\frac{\varphi_{a+2}}{2}\right]\right\}. \quad (22)$$

At this level of approximation, we have three perturbed sine-Gordon models—mutually coupled only if $\rho\neq 0$—and the sign of the interchain coupling can be incorporated in the choice of ground state. Thus, a ferromagnetic or antiferromagnetic interchain coupling would have the same effect. Note that the couplings λ and ρ break the charge conjugation symmetry (19).

IV. BEHAVIOR OF THE GAP IN THE SPIN LADDER

Let us first consider the spin-1 ladder, which corresponds to $\rho=0$. The effective Lagrangian (22) then reduces to three decoupled, two-frequency sine-Gordon models:

$$\mathcal{L}_{\text{lad}} = \sum_a \left\{ \frac{1}{8\pi}[(\partial_\tau\varphi_a)^2+(\partial_x\varphi_a)^2] - M\cos\varphi_a \pm \Lambda\cos\frac{\varphi_a}{2}\right\}, \quad (23)$$

where $M=(2m-16\lambda_1\alpha_1)$ and $\Lambda=4\sqrt{2}\lambda\alpha^2$. The two-frequency sine-Gordon model has been studied by Delfino and Mussardo.[20]

A. Consistency at the mean-field level

From our point of view, the Lagrangian (23) is a mean-field approximation, whose parameters M and Λ must be determined, as functions of λ and m, by solving Eq. (20) self-consistently. This is impossible to do exactly within the two-frequency sine-Gordon model, and we will proceed approximately. To simplify matters, let us neglect $\lambda_1\alpha_1$ and simply set $M=2m$. We then concentrate on calculating $\alpha(\Lambda)$. For $\Lambda=0$, α can be determined exactly,[18] with the result $\alpha(0)\approx 0.4909|m|^{1/4}$. However, no such exact result exists for $\Lambda\neq 0$. The crudest way to estimate $\alpha(\Lambda)$ is classical: we simply neglect all fluctuations and set $\langle\sin(\varphi/2)\rangle \approx \sin(\langle\varphi\rangle/2)$, where $\langle\varphi\rangle=\varphi_0$ is the location of the minimum of the potential $M\cos\varphi-\Lambda\cos(\varphi/2)$, such that

$$\cos\varphi_0=\frac{\Lambda}{4M}. \quad (24)$$

The self-consistent relation is then

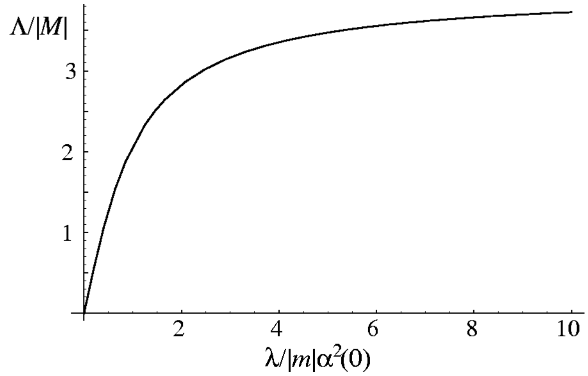

FIG. 2. Self-consistent value of the parameter Λ/M as a function of the coupling constant λ and m.

$$\frac{\Lambda}{M}=2\sqrt{2}\,\frac{\lambda}{m}\,\alpha(\Lambda)^2=2\sqrt{2}\,\frac{\lambda}{m}\left(1-\frac{\Lambda^2}{16M^2}\right), \quad (25)$$

from which Λ/M can be extracted.

A more refined calculation, which takes quantum fluctuations into account, consists in defining a new sine-Gordon field $\tilde{\varphi}$, such that $\varphi=\tilde{\varphi}+\varphi_0-\pi$. Then

$$\left\langle \sin\frac{\varphi}{2}\right\rangle=\left\langle \sin\frac{\tilde{\varphi}}{2}\right\rangle \sin\frac{\varphi_0}{2}+\left\langle \cos\frac{\tilde{\varphi}}{2}\right\rangle \cos\frac{\varphi_0}{2}. \quad (26)$$

We treat $\tilde{\varphi}$ as a single sine-Gordon field, with the usual potential $M\cos\beta\tilde{\varphi}$ with a minimum at $\tilde{\varphi}=\pi$ and a value of β that can be inferred from the second derivative of the potential at the minimum φ_0. This is obviously an approximation, but fares better than the above semiclassical calculation. Simple matching of the second derivative at the minimum yields $\beta=\sqrt{1-(\Lambda/4M)^2}$. For Λ not too large, one may neglect the second term on the right-hand side (RHS) of Eq. (26), since it behaves like Λ^3. Keeping only the first term, one ends up with the following self-consistent equation for Λ:

$$\Lambda=4\sqrt{2}\lambda\left\langle \sin\frac{\tilde{\varphi}}{2}\right\rangle^2\left(1-\frac{\Lambda^2}{16M^2}\right). \quad (27)$$

Having neglected the second term of Eq. (26), we may set $\beta=1$ in the above, and therefore $\langle\sin(\tilde{\varphi}/2)\rangle=\alpha(0)$. We end up with the approximate self-consistent equation

$$\frac{\Lambda}{M}=2\sqrt{2}\,\frac{\lambda}{m}\,\alpha(0)^2\left(1-\frac{\Lambda^2}{16M^2}\right), \quad (28)$$

which differs from Eq. (25) simply by a renormalization $\lambda\to\lambda\alpha^2(0)$. Solving for Λ/M, we find

$$\frac{1}{2\sqrt{2}}\,\frac{\Lambda}{M}=-\frac{m}{\lambda\alpha^2(0)}+\sqrt{2+\frac{m^2}{\lambda^2\alpha^4(0)}}. \quad (29)$$

The dependence of Λ/M on m/λ is shown in Fig. 2. We see that when the interchain coupling becomes large, the ratio Λ/M reaches a maximum.

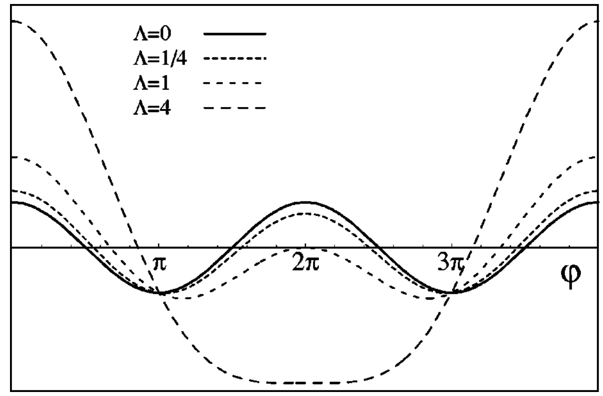

FIG. 3. Evolution of the sine-Gordon potential $\cos\varphi+\Lambda\cos(\varphi/2)$ for $\Lambda=0$, 1/4, 1, and 4. The mass of the lowest-energy kink decreases linearly with Λ, and vanishes classically at $\Lambda=4$.

B. Evolution of the gap with interchain coupling

The potential in the two-frequency sine-Gordon model is illustrated on Fig. 3 for some values of Λ/M. It is intuitively clear that, as Λ increases from zero, one of the kinks becomes more massive, whereas the other one becomes less massive:[20] the soliton having to bridge the potential barrier from $\varphi\sim\pi$ to $\varphi\sim 3\pi$ (towards the right) has a lower energy than the soliton going from $\varphi\sim\pi$ to $\varphi\sim-\pi\equiv 3\pi$ (towards the left). Which kink sees its mass decrease depends on the sign of the perturbation, but the net result is the same whatever this sign is.

With the help of sine-Gordon form factors,[18,19] we can ascertain how the kink mass varies with Λ. At first order, the variation of the mass squared is[20]

$$\delta m_a^2\approx|\Lambda|F_{a\bar{a}}(i\pi), \quad (30)$$

where the form factor F is

$$F_{a\bar{a}}(\eta)\equiv\langle 0|\sin\frac{\varphi}{2}|a(\eta_1)\overline{a(\eta_2)}\rangle, \quad (31)$$

where a and $\bar{a}$ represent the kink and antikink and $\eta_{1,2}$ are the associated rapidities ($\eta=\eta_1-\eta_2$). From Ref. 18, we extract the following expression:

$$\begin{aligned}F_{a\bar{a}}(\eta)&=-\langle e^{i\varphi/2}\rangle e^{\eta/2}-\langle e^{-i\varphi/2}\rangle e^{-\eta/2}\\&=-\left(\frac{1}{2}m_a\right)^{1/4}2^{1/6}A^3e^{-1/4}2\cosh\frac{\eta}{2},\end{aligned} \quad (32)$$

where m_a is the mass of the kink and $A\approx 1.282\,427$ is the Glaisher constant. From this result, we see that δm_a^2 vanishes at first order. We thus expect it to be proportional to Λ^2. This is compatible with the semiclassical result that the variation of the mass of the kink is proportional to the variation of the height of the potential. We thus conclude that

$$\delta m_a\propto\Lambda. \quad (33)$$

The most striking feature of the two-frequency sine-Gordon model is the existence of a critical point at a finite value of Λ. Classically, this critical point occurs when the two minima illustrated on Fig. 3 coalesce pairwise, at Λ

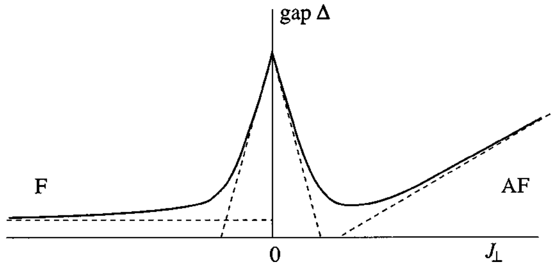

FIG. 4. Conjectured dependence of the spin gap Δ upon the interchain coupling $J_\perp$ in the spin-1 ladder.

$=4M$. In Ref. 20, it is shown that this fixed point has Ising character, with central charge $c=\frac{1}{2}$. Since the scaling dimensions of M and Λ at the Gaussian fixed point ($\Lambda=0,M=0$) are, respectively, $2-1=1$ and $2-\frac{1}{4}=\frac{7}{4}$, the ratio $\zeta=\Lambda/M^{7/4}$ is invariant under renormalization-group (RG) flow and is in fact a control parameter which tells us how far we are from the Ising fixed point, characterized by a critical value ζ_c. At this value, i.e., at $\Lambda=\zeta_c M^{7/4}$, the light kinks have exactly zero mass. If we return to an Ising-model description of the system, we can understand intuitively how this flow happens: The effective Lagrangian (23) corresponds to six 2D Ising models coupled pairwise by the following interaction:

$$\mathcal{L}_{\text{Ising}}=-\frac{\Lambda}{\sqrt{2}}\sigma\sigma'. \tag{34}$$

Thus, the excitation such that $\sigma(x)$ is parallel to $\sigma'(x)$ will have a lower mass if $\Lambda>0$ (a similar reasoning holds when $\Lambda<0$, by changing the sign of σ'). When Λ is large enough, σ must be parallel to σ' and this parallel configuration defines a new Ising model, whose critical point occurs at some value of the ratio $\Lambda/M^{7/4}$.

Our approximate self-consistent solution (Fig. 2) shows that this critical point will not be reached even for a very large interchain coupling. Of course, it is dangerous to extrapolate the above calculation to large values of Λ/M, in view of the approximations leading to Eq. (28). However, this conclusion is robust for the following reason: At the classical critical point ($\Lambda=4M$), the potential has an absolute minimum at $\varphi=2\pi$ and therefore $\langle\sin(\varphi/2)\rangle$ must vanish, by symmetry. Then, Eq. (24) has no solution, except in the limit $\lambda\rightarrow\infty$. Thus, the dependence of Λ/M on interchain coupling λ illustrated in Fig. 2 is qualitatively correct, even beyond the approximations made above.

To conclude our analysis, we expect that the gap of the spin-1 ladder should decrease linearly with a weak interchain coupling [Eq. (34)], both on the ferromagnetic and antiferromagnetic sides (with the same slope). The gap $\Delta(J_\perp)$ is then conjectured to have a cusplike maximum at $J_\perp=0$, a peculiar nonanalytic feature, as illustrated schematically in Fig. 4. This is to be compared with the Monte Carlo data of Fig. 3 of Ref. 2, which illustrates this drop in the gap, for an antiferromagnetic interchain coupling only. We emphasize again that the sign of λ is immaterial, being determined by the minima picked by the three sine-Gordon fields φ_a. This explains the symmetry between weak ferromagnetic and antiferromagnetic couplings. On the other hand, the sign of λ_2 and λ_3, associated with marginal terms neglected in this section, is important. Being marginal, these terms will have an impact at larger interchain coupling, but only on one side, corresponding to the antiferromagnetic case: eventually the gap must increase linearly at large positive $J_\perp$, since the lowest-lying excitations are then rung triplets, costing an energy $J_\perp$. On the ferromagnetic side, we can expect the gap to decrease like Λ/M in a wider domain. Translating this Λ dependence into a λ dependence with the help of Fig. 2, one conjectures a coupling dependence of the gap as illustrated in Fig. 4. That the gap drops on the ferromagnetic side is not surprising, considering that (i) the ladder becomes equivalent to a spin-2 chain at large ferromagnetic coupling and (ii) the gap of an antiferromagnetic Heisenberg chain with integer spin s decreases with s.

V. ZIGZAG SPIN CHAIN

The zigzag spin-1 chain corresponds to $\delta=0$, and thus $\lambda=0,\rho\neq 0$. The effective Lagrangian is then

$$\begin{aligned}\mathcal{L}_{\text{zigzag}}=\sum_a \Bigg\{&\frac{1}{8\pi}[(\partial_\tau\varphi_a)^2+(\partial_x\varphi)^2]\\&-(2m-16\lambda_1\alpha_1)\cos\varphi_a\mp 4\sqrt{2}\rho\alpha\cos(\theta_a)\\&\times\left[\cos\frac{\varphi_a}{2}-\cos\frac{\varphi_{a+1}}{2}-\cos\frac{\varphi_{a+2}}{2}\right]\Bigg\}.\end{aligned}\tag{35}$$

This Lagrangian is not easily analyzed. Let us go back to the fermionic representation of the twist term by order and disorder fields:

$$\mathcal{L}_\rho=2\rho\sum_a \sigma_a\mu_{a+1}\mu_{a+2}\partial_x(\sigma'_a\mu'_{a+1}\mu'_{a+2}). \tag{36}$$

With $\langle\mu_a\mu'_a\rangle=-\sqrt{2}i\alpha$ [cf. Eq. (A11)] the most relevant term will be

$$\mathcal{L}_\rho\approx-4\alpha^2\rho\sigma_a\partial_x\sigma'_a. \tag{37}$$

We will now study the effect of this approximate representation of the twist term by considering the corresponding lattice model (see the Appendix). Let us map the order fields in the following way:

$$\sigma_a(x)\rightarrow\sigma_a^x(n)\qquad\sigma'_a(x)\rightarrow\sigma_a^x\left(n+\frac{1}{2}\right). \tag{38}$$

With the representation (37) for the twist term, the system is described by the following Hamiltonian:

$$\begin{aligned}H=\sum_{n,a}&\{-\sigma_a^z(n/2)-\kappa\sigma_a^x(n/2)\sigma_a^x(n/2+1)\}\\&-4\rho\alpha^2\sum_{n,a}\sigma_a^x(n)[\sigma_a^x(n+1/2)-\sigma_a^x(n-1/2)],\end{aligned}\tag{39}$$

where κ is related to the constant m (i.e., $-1-\eta$) by the relation $\kappa=1+a_0m$, where a_0 is the lattice constant. Thus $\kappa=1$ for $m=0(\eta=-1)$ and tends to 0 when η grows. To bring this Hamiltonian to a more familiar form, we perform a

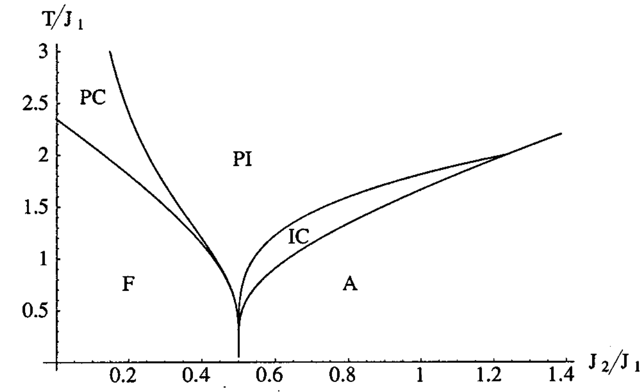

FIG. 5. Phase diagram of the classical ANNNI model. T is the temperature, and J_1 and J_2 are the nearest-neighbor and next-nearest-neighbor Ising couplings, respectively.

rotation of π around the z axis of the spin operator at every other site, on each chain. This changes the sign of σ_a^x on those sites and gives the Hamiltonian a slightly different form:

$$H=\sum_{n,a} \{-\sigma_a^z(n/2)+\kappa\sigma_a^x(n/2)\sigma_a^x(n/2+1)\}$$
$$+4\rho\alpha^2\sum_{n,a} \sigma_a^x(n)[\sigma_a^x(n+1/2)+\sigma_a^x(n-1/2)]. \tag{40}$$

The Hamiltonian (40) defines the quantum axial next-nearest-neighbor Ising (ANNNI) model. Together with its two-dimensional, classical counterpart (cf. Refs. 21–23), it has been extensively studied by a variety of methods: mean-field theory,[24] Monte Carlo simulations,[25–28] Muller-Hartmann-Zittartz approximation,[25] perturbative expansions,[29–31] free fermion approximation,[21,22,32] and exact diagonalizations.[32,33] The phase diagram for the classical model is shown in Fig. 5. In the scaling limit, the temperature T of the classical model model is related to the mass m by $T=(1-a_0m)T_c=(2-\kappa)T_c$. The nearest-neighbor coupling is proportionnal to the interchain coupling ρ and J_2 to κ. Thus, the case of small zigzag interaction corresponds to the limit of small J_1. The different phases are the following: ferromagnetic (F), paramagnetic commensurate (PC), paramagnetic incommensurate (PI), incommensurate critical phase (IC, also called ''floating phase''), and antiphase (A) of alternating pairs $(++--++\cdots)$. A disorder line found by Peschel and Emery[34] divides the PC and the PI phase.

We conclude from this phase diagram that incommensurability will arise in the spin-1 zigzag chain as soon as the interchain coupling is nonzero [the model (40) is then on the far right of the PI phase]. One premise for this deduction is that the incommensurability of the Ising spins (σ_a) is reflected in the correlation of the spins of the quantum chain; this comes from the relation (12). Note that increasing ρ brings us from infinity on the phase diagram 5 towards the origin, along a straight line. One could expect such a line to go through other phases (like the IC phase) at some point. However, we should note that the omission of the fluctuation of μ_a in Eq. (37) is valid only when $T-T_c$ is large compare to ρ.

Moreover, we can have an idea of how the incommensurability develops as a function of ρ. A recent analysis using a high-temperature expansion and bosonization[35] shows that in the limit of very strong next-nearest-neighbor interaction in the ANNNI model, the incommensurability is proportional to ρ/κ. Explicitly, in the high-temperature limit, the incommensurate wave vector is given by

$$q_0=\pm\frac{2\alpha^2\rho(1+\kappa+\kappa^2+\cdots)}{2\kappa(1-\kappa)}, \tag{41}$$

where the ellipsis stands for higher powers of κ. The $\pm$ sign are, respectively, associated with the correlation function of the combination $\pm\sigma_a^z(n)+\sigma_a^z(n\mp1/2)$. This result that the incommensurability is linear with the interchain coupling confirms the one obtained by a semiclassical analysis.[36]

ACKNOWLEDGMENTS

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APPENDIX: THE 2D ISING MODEL

In this appendix, we review briefly the correspondence of the Ising model with fermions, the conformal structure of the model and we indicate a set of careful bosonization formulas for a pair of Ising models.

1. Definitions

As is well known, the 2D statistical Ising model is equivalent to a quantum Ising chain in a transverse field, with Hamiltonian

$$H=-\lambda\sum_i \sigma_i^z-\sum_i \sigma_i^x\sigma_{i+1}^x, \tag{A1}$$

where $\sigma^{1,2,3}$ are the Pauli matrices. The Hamiltonian (A1) can be diagonalized through a Jordan-Wigner transformation followed by a Bogolubov-Valatin transformation. The solution shows that $\langle\sigma_i^x\rangle\neq0$ if $\lambda<1$ and $\langle\sigma_i^x\rangle=0$ otherwise. Thus $\lambda=1$ is the critical point. A peculiarity of this model is the existence of a duality transformation mapping the ordered phase to the disordered phase and vice versa. Under this transformation the spin operators σ_i^a are mapped to the so-called disorder operators, defined on links (dual lattice) by the following relations:

$$\mu_{i+1/2}^z=\sigma_i^x\sigma_{i+1}^x,$$
$$\mu_{i+1/2}^x=\prod_{j\leq i} \sigma_j^z. \tag{A2}$$

Let us apply the Jordan-Wigner transformation on the dual lattice. The fermion creation and annihilation operators are defined as

$$c_{j+1/2}=\mu^{-}_{j+1/2}\exp\left(\frac{i\pi}{2}\sum_{k<j}(\mu^z_{k+1/2}-1)\right),$$

$$c^\dagger_{j+1/2}=\mu^{+}_{j+1/2}\exp\left(\frac{-i\pi}{2}\sum_{k<j}(\mu^z_{k+1/2}-1)\right), \tag{A3}$$

where $\mu^{\pm}=(\mu^x\pm i\mu^y)/2$. The fermions correspond to the kinks in the original formulation. Indeed the fermion number on link $j+1/2$ is

$$c^\dagger_{j+1/2}c_{j+1/2}=\frac{1-\sigma^x_j\sigma^x_{j+1}}{2}; \tag{A4}$$

i.e., there is no fermion on the link if the spins on j and $j+1$ are parallel and one if they are antiparallel. Note that the order parameter σ^x has a bosonic character, whereas the disorder parameter μ^x is fermionic. This is easily seen from the following equivalence:

$$\mu^x_{j+1/2}=c^\dagger_{j+1/2}\exp\left(-i\pi\sum_{k<j}c^\dagger_{k+1/2}c_{k+1/2}\right)+c_{j+1/2}\exp\left(i\pi\sum_{k<j}c^\dagger_{k+1/2}c_{k+1/2}\right),$$

$$\sigma^x_j=\sigma^x_{-N}\exp\left(\pm i\pi\sum_{k<j}c^\dagger_{k+1/2}c_{k+1/2}\right). \tag{A5}$$

2. Continuum limit

The critical point of the 2D Ising model is equivalent, in the continuum limit, to a free, massless Majorana fermion: a conformal field theory with central charge $c=1/2$ and three conformal families: the identity operator, the energy operator ϵ, and the the spin density (or order) operator σ. The use of complex coordinates $z=\tau+ix$ and $\bar{z}=\tau-ix$ is standard, along with the complex derivatives $\partial=\partial_z=(\partial_\tau-i\partial_x)/2$ and $\bar{\partial}=\partial_{\bar{z}}=(\partial_\tau+i\partial_x)/2$ (the notation used is that of Ref. 37). The energy density operator may be expressed in terms of the chiral components of the Majorana fermion as $\epsilon=i\psi\bar{\psi}$. The order field σ is the continuum limit of the spin operator σ^x_i, and a fermionic disorder field μ may be introduced as the continuum limit of the disorder operator $\mu^x_{i+1/2}$. The field μ has the same scaling properties as the field σ, but is nonlocal with respect to σ. The conformal transformations are generated by the energy-momentum tensor, whose chiral components are $T=-\frac{1}{2}\psi\partial\psi$ and $\bar{T}=-\frac{1}{2}\overline{\psi\partial\psi}$. All these fields have the following short-distance products or OPE:

$$\psi(z)\psi(w)\sim\frac{1}{z-w}+2(z-w)T(w),$$

$$\bar{\psi}(\bar{z})\bar{\psi}(\bar{w})\sim\frac{1}{\bar{z}-\bar{w}}+2(\bar{z}-\bar{w})\bar{T}(\bar{w}),$$

$$\sigma(z,\bar{z})\sigma(w,\bar{w})\sim\frac{1}{|z-w|^{1/4}}+\frac{1}{2}|z-w|^{3/4}\epsilon(w,\bar{w}),$$

$$\mu(z,\bar{z})\mu(w,\bar{w})\sim\frac{1}{|z-w|^{1/4}}-\frac{1}{2}|z-w|^{3/4}\epsilon(w,\bar{w}),$$

$$\sigma(z,\bar{z})\mu(w,\bar{w})\sim\frac{\gamma(z-w)^{1/2}\psi(w)+\gamma^*(\bar{z}-\bar{w})^{1/2}\bar{\psi}(\bar{w})}{\sqrt{2}|z-w|^{1/4}},$$

$$\mu(z,\bar{z})\sigma(w,\bar{w})\sim\frac{\gamma^*(z-w)^{1/2}\psi(w)+\gamma(\bar{z}-\bar{w})^{1/2}\bar{\psi}(\bar{w})}{\sqrt{2}|z-w|^{1/4}},$$

$$\psi(z)\sigma(w,\bar{w})\sim\frac{\gamma}{\sqrt{2}(z-w)^{1/2}}\mu(w,\bar{w}),$$

$$\psi(z)\mu(w,\bar{w})\sim\frac{\gamma^*}{\sqrt{2}(z-w)^{1/2}}\sigma(w,\bar{w}),$$

$$\bar{\psi}(\bar{z})\sigma(w,\bar{w})\sim\frac{\gamma^*}{\sqrt{2}(\bar{z}-\bar{w})^{1/2}}\mu(w,\bar{w}),$$

$$\bar{\psi}(\bar{z})\mu(w,\bar{w})\sim\frac{\gamma}{\sqrt{2}(\bar{z}-\bar{w})^{1/2}}\sigma(w,\bar{w}), \tag{A6}$$

where $\gamma=\exp i\pi/4$ (or, equivalently, $\exp -i\pi/4$).

3. Bosonization

Two Ising models form a $c=1/2+1/2=1$ conformal theory. We therefore expect a representation of the different fields in terms of a free boson φ, defined by the action

$$S=\frac{1}{8\pi}\int dx\,d\tau[(\partial_\tau\varphi)^2+(\partial_x\varphi)^2]. \tag{A7}$$

Our choice of normalization $(1/8\pi)$ simplifies the exponentials and circular functions appearing in the sine-Gordon theory. The OPE's of the boson field and of its (normal-ordered) exponentials are

$$\partial\varphi(z)\partial\varphi(w)\sim\frac{1}{(z-w)^2},$$

$$e^{i\alpha\varphi(z)}e^{i\beta\varphi(w)}\sim(z-w)^{\alpha\beta}e^{i(\alpha+\beta)\varphi(w)}+\cdots. \tag{A8}$$

The boson field can be separated into chiral components: $\varphi(x,\tau)=\phi(z)+\bar{\phi}(\bar{z})$. These fields have the following mode expansion in radial quantization:[37]

$$\phi(z)=q-ip\ln z+i\sum_{k\neq 0}\frac{1}{k}a_k z^{-k},$$

$$\bar{\phi}(\bar{z})=\bar{q}-i\bar{p}\ln\bar{z}+i\sum_{k\neq 0}\frac{1}{k}\bar{a}_k\bar{z}^{-k}, \tag{A9}$$

where the operators p,q,a_n satisfy the commutation relations

$$[q,p]=i,\quad [a_n,a_m]=n\delta_{n,m}, \tag{A10}$$

with similar relations for the left-moving (barred) operators.

A faithful representation of the Ising fields is then given by the following relations:[38]

$$\psi = \frac{1}{\sqrt{2}} e^{i\pi\bar{p}} e^{i\phi} + \frac{1}{\sqrt{2}} e^{-i\pi\bar{p}} e^{-i\phi},$$

$$\bar{\psi} = -\frac{i}{\sqrt{2}} e^{-i\pi\bar{p}} e^{-i\bar{\phi}} - \frac{i}{\sqrt{2}} e^{i\pi\bar{p}} e^{i\bar{\phi}},$$

$$\psi' = -\frac{i}{\sqrt{2}} e^{i\pi\bar{p}} e^{i\phi} + \frac{i}{\sqrt{2}} e^{-i\pi\bar{p}} e^{-i\phi},$$

$$\bar{\psi}' = -\frac{1}{\sqrt{2}} e^{-i\pi\bar{p}} e^{-i\bar{\phi}} + \frac{1}{\sqrt{2}} e^{i\pi\bar{p}} e^{i\bar{\phi}},$$

$$\sigma\sigma' = \sqrt{2}\cos\frac{\varphi}{2},$$

$$\sigma\mu' = -i\sqrt{2}\sin\frac{\theta}{2},$$

$$\mu\sigma' = -i\sqrt{2}\cos\frac{\theta}{2},$$

$$\mu\mu' = -\sqrt{2}i\sin\frac{\varphi}{2}, \tag{A11}$$

where $\theta = \phi - \bar{\phi} + 2\pi\bar{p}$ is the field dual to φ (the operator $2\pi\bar{p}$ is added to ensure proper anticommuation properties). This representation leads to the correct OPE (A6) between the Ising fields. The phase factor $e^{i\pi\bar{p}}$ is similar to the phase factor in the Jordan-Wigner transformation, $\bar{p}$ being to the number of left fermions. Only its odd or even character matters. We note the natural periodicity property $\varphi \to \varphi + 4\pi$ and $\theta \to \theta + 4\pi$ of this representation.

The energy-momentum tensors T and T' of the two Ising models, along with their antiholomorphic counterparts, are bosonized as follows:

$$T(z) - T'(z) = 4\sqrt{2}e^{2\pi i\bar{p}}\cos[2\phi(z)],$$

$$\bar{T}(\bar{z}) - \bar{T}'(\bar{z}) = -4\sqrt{2}e^{2\pi i\bar{p}}\cos[2\bar{\phi}(\bar{z})]. \tag{A12}$$

This relation is useful when bosonizing the twist term [the last term of Eq. (16)].

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