Functional integral bosonization for an impurity in a Luttinger liquid

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We use a functional integral formalism developed earlier for the pure Luttinger liquid (LL) to find an exact representation for the electron Green function of the LL in the presence of a single backscattering impurity in the low-temperature limit. This allows us to reproduce results (well known from the bosonization techniques) for the suppression of the electron local density of states (LDOS) at the position of the impurity and for the Friedel oscillations at finite temperature. In addition, we have extracted from the exact representation an analytic dependence of LDOS on the distance from the impurity and shown how it crosses over to that for the pure LL.

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I. INTRODUCTION

One of the first exactly soluble models in the problem of strongly correlated electrons was formulated in one dimension in the seminal papers of Tomonaga[1] and Luttinger[2] and solved by Mattis and Lieb.[3] Considerable further contributions to understanding of generic properties of the one-dimensional (1D) electron liquid have been made in papers.[4–6] In particular, Haldane[6] has coined the notion of a Luttinger liquid (LL), stressing the generic properties of the Luttinger model for 1D Fermi systems, and has formulated fundamentals of a modern bosonization technique as one of the most elegant ways for solving the problem. In this approach, the Fermi creation and annihilation operators are explicitly represented in terms of Bose operators and a four-fermionic Hamiltonian is eventually diagonalized in the bosonic representation.

There exists an alternative way to bosonize the problem sometimes called ''functional bosonization,'' which was elaborated in Refs. 7–10. In this paper, we will apply such a functional method in the form developed earlier by one of us[10] for the treatment of a single-impurity problem in the Luttinger model. The essence of the method is in the decoupling of the four-fermion interaction by the standard Hubbard-Stratonovich transformation, typical for higher-dimensional problems, and a subsequent elimination of a mixed fermion-boson term in the action by a gauge transformation which is exact for the pure 1D Luttinger model and gives a convenient starting point for including a single backscattering impurity.

The problem of a single impurity in the LL has been actively investigated by many authors.[11–18] One of the main results of these considerations[11,12,16–18] was the suppression[19] at low temperatures of the local density of states (LDOS) *at the impurity site* and the related suppression of the conductance,[11,12] and the x-ray edge singularity.[16–18] Another prominent result was the dependence of the Friedel oscillations[9,13,14] on the distance from the impurity.

In the present paper, we use the functional bosonization approach to build up an exact representation of the electron Green function for this problem in the low-temperature limit. First we demonstrate this approach to be workable by extracting from this representation both the LDOS at the impurity site and the Friedel oscillations in a unified way. Then we show that it allows us to go further to obtain an analytic expression for the LDOS at any distance from the impurity, which turns out to be a universal power-law dependence on this distance. Thus we have described in full a crossover from the impurity-dominated behavior to that for the pure LL at large distances from the impurity.

II. EXACT FUNCTIONAL REPRESENTATION FOR GREEN FUNCTION

In this section, starting from the standard fermionic action, we introduce intermediate boson variables via the Hubbard-Stratonovich transformation and integrate over both sets of variables, arriving at a formally exact representation of the electron Green function as infinite series, Eq. (17). Then we represent this series as a functional integral with a new bosonic action. This derivation is similar in spirit, if not in letter, to that of Fernández *et al.*;[9] however, the functional representation obtained is more general, as described at the end of this section.

The Hamiltonian of Luttinger liquid with one backscattering impurity can be written as

$$\hat{H} = -i\eta v_{\mathrm{F}} \int dx \hat{\psi}^{\dagger}_{\eta}(x) \frac{\partial}{\partial x} \hat{\psi}_{\eta}(x) + \frac{1}{2} \int dx dx' \hat{\psi}^{\dagger}_{\eta}(x) \hat{\psi}^{\dagger}_{\eta'}(x') V_0(x-x') \hat{\psi}_{\eta'}(x') \hat{\psi}_{\eta}(x) + v_{\mathrm{F}} \int dx \lambda(x) [\hat{\psi}^{\dagger}_{+}(x) \hat{\psi}_{-}(x) + \hat{\psi}^{\dagger}_{-}(x) \hat{\psi}_{+}(x)]. \quad (1)$$

Here $\psi^{\dagger}_{\pm}$ and $\psi_{\pm}$ are the standard creation and annihilation operators for left- and right-moving electrons ($\psi = \psi_{+} e^{ip_{\mathrm{F}}x} + \psi_{-} e^{-ip_{\mathrm{F}}x}$), $\eta = \pm$ and the summation over repeated indices is implied; V_0 is a bare electron-electron interaction. The impurity potential is given by $v_{\mathrm{F}}\lambda(x) = v_{\mathrm{F}}\lambda u(x)$, where $u(x)$ is some form factor of the impurity and $\lambda \ll 1$ is its strength.

The temperature Green functions of the Hamiltonian (1) can be represented by the functional integral

$$\mathcal{G}_{\eta\eta'}(\xi;\xi') = Z_{\lambda}^{-1} \int \psi_{\eta}(\xi) \psi^{*}_{\eta'}(\xi') e^{-S[\psi]} \mathcal{D}\psi, \quad (2)$$

where $\xi \equiv (x,\tau)$, $Z_\lambda = \int e^{-S[\psi]} \mathcal{D}\psi$, and the action

$$S[\psi] = \int d\xi [\psi_\eta^*(\xi)\partial_\tau \psi_\eta(\xi) + H(\psi^*,\psi)]. \tag{3}$$

Here the integration over the imaginary time τ goes from 0 to $\beta = 1/T$, and the "classical" Hamiltonian H is obtained from $\hat{H}$, Eq. (1), by substituting the Grassmann fields $\psi^*(\xi)$ and $\psi(\xi)$ for $\hat{\psi}^\dagger$ and $\hat{\psi}$.

The Green function of real electrons is given as a sum of all $\mathcal{G}_{\eta\eta'}$ taken with appropriate phase factors,

$$\mathcal{G}(\xi,\xi') = \sum_{\eta,\eta'} \mathcal{G}_{\eta\eta'}(\xi,\xi') e^{ip_F(\eta x - \eta' x')}. \tag{4}$$

The functional bosonization is achieved via the standard Hubbard-Stratonovich transformation decoupling the four-fermion term in the action. Introducing a new classical (bosonic) field ϕ we arrive at the action

$$\begin{aligned} S[\phi,\psi] = &\int d\xi \psi_\eta^*(\xi)(\partial_\eta - i\phi)\psi_\eta(\xi) \\ &+ \frac{1}{2}\int d\xi d\xi' \phi(\xi) V_0^{-1}(\xi-\xi')\phi(\xi') \\ &- v_F \int d\xi \lambda(x)[\psi_+^*(\xi)\psi_-(\xi) + \psi_-^*(\xi)\psi_+(\xi)]. \end{aligned} \tag{5}$$

Here V_0^{-1} is a function inverse to V_0 in the operator sense, whose τ dependence is just $\delta(\tau-\tau')$, and the "chiral derivatives" ∂_η are defined by

$$\partial_+ \equiv 2\partial_z = \partial_\tau - iv_F\partial_x, \quad \partial_- \equiv 2\partial_{\bar{z}} = \partial_\tau + iv_F\partial_x.$$

In order to eliminate the mixed term in the chiral derivative in Eq. (5), we apply the gauge transformation

$$\psi_\eta \mapsto \psi_\eta e^{i\theta_\eta}, \quad \psi_\eta^* \mapsto \psi_\eta^* e^{-i\theta_\eta}, \quad \partial_\eta \theta_\eta = \phi, \tag{6}$$

where θ_η is a complex function which depends on the field ϕ. Since ϕ is real,

$$\theta_- = \theta_+^* \equiv \theta \equiv \theta_1 + i\theta_2.$$

This transformation produces[10] the Jacobian J, derived explicitly in the Appendix,

$$\ln J = -\frac{1}{2}\int d\xi d\xi' \phi(\xi)\Pi(\xi-\xi')\phi(\xi'), \tag{7}$$

where Π is the polarization operator whose Fourier transform is given by

$$\Pi(q,\Omega) = \frac{1}{\pi v_F}\frac{v_F^2 q^2}{\Omega^2 + v_F^2 q^2}, \tag{8}$$

where $\Omega = 2\pi nT$ is a bosonic Matsubara frequency.

Therefore, after the transformation (6) the interaction in the action (5) becomes random-phase-approximation (RPA) screened as expected, $V_0^{-1} \mapsto V^{-1} = V_0^{-1} + \Pi$, and can be split into the sum $S[\psi,\phi] = S_f[\psi] + S_b[\phi] + S_{imp}[\psi,\phi]$:

$$S_b = \frac{1}{2}\int d\xi d\xi' \phi(\xi) V^{-1}(\xi-\xi')\phi(\xi'),$$

$$S_f = \int d\xi \psi_\eta^*(\xi)\partial_\eta \psi_\eta(\xi),$$

$$S_{imp} = -v_F \int d\xi \lambda [e^{2\theta_2}\psi_+^*(\xi)\psi_-(\xi) + e^{-2\theta_2}\psi_-^*(\xi)\psi_+(\xi)]. \tag{9}$$

The Green function (2) can be represented as the functional average over the fermionic and bosonic fields with the weight $S_b + S_f$:

$$\mathcal{G}_{\eta\eta'}(\xi;\xi') = \frac{\langle\langle e^{i\theta_\eta(\xi) - i\theta_{\eta'}(\xi')}\psi_\eta(\xi)\psi_{\eta'}^*(\xi')e^{-S_{imp}}\rangle\rangle}{\langle\langle e^{-S_{imp}}\rangle\rangle},$$

$$\langle\langle \mathcal{O}[\phi,\psi]\rangle\rangle \equiv \frac{\int \mathcal{D}\phi\mathcal{D}\psi \mathcal{O}[\phi,\psi] e^{-S_b[\phi]-S_f[\psi]}}{\int \mathcal{D}\phi\mathcal{D}\psi e^{-S_b[\phi]-S_f[\psi]}}. \tag{10}$$

The bosonic field ϕ enters the preexponential factor in Eq. (10) only implicitly, via $\theta(\phi)$. Before proceeding further it is convenient to work out correlation functions of θ which follow straightforwardly from $\langle \phi(\xi)\phi(\xi')\rangle_\phi = V(\xi-\xi')$ and Eq. (6):

$$\langle \theta_1(\xi)\theta_1(\xi')\rangle_\phi = \frac{1}{2}\ln\frac{|\sin(z_F - z_F')|}{|\sin(z-z')|^{1/g}},$$

$$\langle \theta_2(\xi)\theta_2(\xi')\rangle_\phi = \frac{1}{2}\ln\frac{|\sin(z-z')|^g}{|\sin(z_F - z_F')|},$$

$$\langle \theta_1(\xi)\theta_2(\xi')\rangle_\phi = \frac{1}{2}\arg\frac{\sin(z-z')}{\sin(z_F - z_F')}, \tag{11}$$

where

$$z_F = \pi T(\tau + ix/v_F), \quad z = \pi T(\tau + ix/v),$$

$$v = v_F\left(1 + \frac{V(q=0)}{\pi v_F}\right)^{1/2}, \quad g = \frac{v_F}{v}. \tag{12}$$

Here we assumed that the Fourier transform of the forward-scattering pair interaction only weakly depends on momentum, i.e., $V(q \ll 2p_F) \approx V(q=0)$.

For the purpose of this paper, the representation given by Eqs. (9) and (10) is an intermediate step for the bosonization. However, in the case of the weak interaction $(1-g \ll 1)$ the bosonic part of the action (9) can be treated within the perturbative renormalization-group approach, thus reproducing the results of Ref. 12.

Now we reduce the partition function $Z_\lambda = \langle\langle e^{-S_{imp}}\rangle\rangle$ in Eq. (10) to the product of fermionic and bosonic integrals. This can be done for an arbitrary scattering potential $\lambda(x)$. On expanding $e^{-S_{imp}}$ and keeping only the terms with equal numbers of ψ_+^* and ψ_+ (as well as of ψ_-^* and ψ_-), we obtain

$$Z_\lambda=\sum_{n=0}^{\infty}\frac{v_F^{2n}}{(n!)^2}\prod_{k=1}^{n}\int d\xi_k d\xi_k'\lambda(x_k)\lambda(x_k') \times\left\langle\prod_{k=1}^{n}\psi_+^*(\xi_k)\psi_-(\xi_k)\psi_-^*(\xi_k')\psi_+(\xi_k')\right\rangle_\psi \times\left\langle\exp\left[2\sum_{k=1}^{n}[\theta_2(\xi_k)-\theta_2(\xi_k')]\right]\right\rangle_\phi. \quad (13)$$

Carrying out the bosonic average with the help of formulas (11), we find

$$\left\langle\exp\left[2\sum_{k=1}^{n}[\theta_2(\xi_k)-\theta_2(\xi_k')]\right]\right\rangle_\phi=\frac{\alpha^{2gn}|P_n(z)|^{2g}}{\alpha^{2n}|P_n(z_F)|^2}, \quad (14)$$

where

$$\alpha\equiv\pi T/\varepsilon_F\ll 1$$

is a cutoff parameter, and $P_n(z)$ is given by

$$P_n(z)=\frac{\prod_{i<j}^{n}\sin(z_i-z_j)\sin(z_i'-z_j')}{\prod_{i,j=1}^{n}\sin(z_i-z_j')}. \quad (15)$$

The parameters z and z_F, entering with the appropriate indices, Eq. (14), are defined by Eq. (12).

The fermionic average in Eq. (13) is independent for the left- and right-moving electrons and yields

$$\left\langle\prod_{k=1}^{n}\psi_-(\xi_k)\psi_-^*(\xi_k')\right\rangle=\left(\frac{T}{2v_F}\right)^n\det\frac{1}{\sin(z_{Fi}-z_{Fj}')}. \quad (16)$$

Applying the Cauchy formula,[20]

$$\det\frac{1}{\sin(z_i-z_j')}=(-1)^{n(n-1)/2}P_n(z),$$

one sees that the fermionic average (16) cancels $|P_n(z_F)|^2$ in the denominator of Eq. (14) so that

$$Z_\lambda=\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{T}{2\alpha^{1-g}}\right)^{2n} \times\prod_{k=1}^{n}\int d\xi_k d\xi_k'\lambda(x_k)\lambda^*(x_k')|P_n(z)|^{2g}. \quad (17)$$

The result above has been obtained by formally calculating both fermionic and bosonic integrals for the partition function in Eq. (10). Now we ''rebosonize'' this expression by presenting it as a result of the integration over a new bosonic field Θ,

$$Z_\lambda=\langle e^{-S_\lambda[\Theta]}\rangle_0, \quad (18)$$

where

$$S_\lambda[\Theta]=-\frac{T}{\alpha}\int d\xi\lambda(x)\cos\Theta(\xi), \quad (19)$$

and $\langle\cdots\rangle_0$ average is defined with the action S_0,

$$S_0[\Theta]=\frac{1}{8\pi gv}\int d\xi[(\partial_\tau\Theta)^2+v^2(\partial_x\Theta)^2]. \quad (20)$$

To verify the validity of the representation (18), (19) one needs to expand the exponent in Eq. (18) using the fact that the pair-correlation function of Θ with the action S_0 is given (with a proper regularization) by

$$G_0(\xi,\xi')\equiv\langle\Theta(\xi)\Theta(\xi')\rangle_0=-2g\ln|\sin(z-z')|. \quad (21)$$

The sum resulting from such an expansion coincides with that in Eq. (17). We remind that here ξ stands as a shorthand for x,τ with the appropriate indices, while $z=\pi T(\tau+ix/v)$, Eq. (12).

On repeating the steps outlined in Eqs. (13)–(17), we obtain the following Θ representation for the Green function of Eq. (10):

$$\mathcal{G}_{\eta\eta'}(\xi,\xi')=\frac{T}{2v_F\alpha^{1-1/2g}}\frac{s_{\eta\eta'}(z-z')}{|\sin(z-z')|^{1/2g}}\tilde{\mathcal{G}}_{\eta\eta'}(\xi,\xi'), \quad (22)$$

where we introduced an auxiliary function $\tilde{\mathcal{G}}_{\eta\eta'}(\xi,\xi')$,

$$\tilde{\mathcal{G}}_{\eta\eta'}(\xi,\xi')=Z_\lambda^{-1}\langle e^{(ia/2)\Theta(\xi)-(ia'/2)\Theta(\xi')}e^{-S_\lambda[\Theta-\chi]}\rangle_0. \quad (23)$$

The $\cos\Theta$ term in action S_λ, Eq. (19), is now shifted by the phase factor $\chi(\xi_1)$,

$$S_\lambda[\Theta-\chi]=-\frac{T}{\alpha}\int d\xi_1\lambda(x_1)\cos\{\Theta(\xi_1)-\chi(\xi_1)\}, \quad (24)$$

where $\chi(\xi_1)$ parametrically depends on the arguments of the Green function (22),

$$\chi(x_1,\tau_1)\equiv\chi(z_1)=\arg\frac{\sin(z_1-z)}{\sin(z_1-z')}. \quad (25)$$

Finally, the sign factor $s_{\eta\eta'}$ in Eq. (22) is defined (with $\eta=\pm1$) by

$$s_{\eta\eta'}(z-z')=\exp[\tfrac{1}{2}i(\eta+\eta')\arg\sin(z-z')].$$

In one particular case, $x=x'$ and $\tau=0$ (i.e., when describing the electron density distribution at any distance from impurity), the representation (22)–(25) coincides with that obtained by Fernández *et al.*[9] In another particular case, for $x=x'=0$ and arbitrary τ (corresponding to the LDOS energy dependence at the impurity site), by integrating out (like in Ref. 11) $\Theta(x\neq0)$, this representation reduces to that obtained by Oreg and Finkel'stein.[15]

The full action (22)–(25) is both exact and most general. Its distinctive feature is the shift of the bosonic field Θ in the standard cosine term, Eq. (24), by the nonlocal phase, χ, Eq. (25). This action allows one to consider a wider set of problems, including the energy dependence of the LDOS at an arbitrary distance from the impurity addressed below.

III. SPATIAL DEPENDENCE OF LDOS

The representation (22), (23) is exact and further calculations are only possible after some approximations which we

will make below. Note that a relatively compact representation for the electron Green function at any distance from the impurity, which was claimed to be exact, has been obtained in Ref. 21. Although the Green function[21] leads to well-known correct limits for the nonoscillating part of the LDOS at the impurity ($x\to 0$) and in the bulk ($x\to\infty$), it does not explicitly contain the impurity strength which, as we will show here, defines the intermediate scale where a crossover from the impurity-dominated to the bulk behavior takes place.

In this section we aim at obtaining asymptotic expressions for the Green function at small, intermediate (crossover), and large distances from the impurity. First, we assume the impurity to be pointlike, $\lambda(x)=\lambda\delta(x)$, and weak, i.e., $\lambda\ll 1$. This ensures the spectrum linearization to be valid in the presence of the impurity and all the relevant energy scales to be small compared to the Fermi energy.

It is well known that even such a ''weak'' impurity leads to strong changes to the single-electron density of states in its vicinity. Its influence is perturbative only in the high-temperature limit, $\lambda\ll\alpha\sim T/\varepsilon_F$. In the low-temperature regime, $\lambda\gg\alpha$, a nonperturbative approach is required. In the present context, it can be developed within the so-called self-consistent harmonic approximation (SCHA).[22,23] It is based on the fact that for $\alpha\ll\lambda\ll 1$, the deviation of Θ from $\chi(\tau_1)$ in the action (23) is prohibitive so that their difference can be presented as small quadratic fluctuations around one of the minima of $\cos(\Theta-\chi)$. By minimizing the difference between actual cosine-shaped potential and its quadratic fit (the Feynmann-Vernon variational principle), one substitutes the exact cosine potential by the harmonic one, thus reducing the action (24) to the following one:

$$S_\Lambda[\Theta-\chi]=\frac{\Lambda T}{2\alpha}\int d\tau_1[\Theta(0,\tau_1)-\chi(0,\tau_1)]^2, \qquad (26)$$

where the new ''impurity strength'' Λ is chosen to provide the best fit to the real potential. As a result, one obtains[22] the renormalized ''self-consistent'' impurity strength Λ as $\Lambda=\lambda^{1/(1-g)}$.

Now the action is quadratic in Θ,

$$S[\Theta]=S_0[\Theta]+\frac{\Lambda T}{2\alpha}\int d\tau\Theta^2(0,\tau), \qquad (27)$$

so the integral (23) for $\tilde{\mathcal{G}}_{\eta\eta'}(\xi,\xi')$ is reduced to calculating the averages with the action (27) of linear in Θ terms in the exponent, using the standard formulas of the type

$$\langle e^{b\Theta}\rangle_\Theta=\exp\left[\frac{b^2}{2}\langle\Theta^2\rangle_\Theta\right],$$

where $\langle\cdots\rangle_\Theta$ stand for the functional averaging with the action (27). The integration thus yields

$$-\ln\tilde{\mathcal{G}}_{\eta\eta'}(\xi,\xi')=\frac{1}{8}G(\xi,\xi)+\frac{1}{8}G(\xi',\xi')-\frac{\eta\eta'}{4}G(\xi,\xi')$$
$$-i\Phi_{\eta\eta'}(\xi,\xi')+\Xi(\xi,\xi'). \qquad (28)$$

All the terms above can be expressed via the pair-correlation function of the auxiliary bosons Θ defined by

$$G(\xi,\xi')=\langle\Theta(\xi)\Theta(\xi')\rangle_\Theta. \qquad (29)$$

In the absence of the Λ term in action (27), G reduces to the standard bosonic Green function G_0, Eq. (21). The full Green function G in the presence of impurity is straightforward to find in the Matsubara frequency representation,

$$G(\xi;\xi')=T\sum_\omega G(x,x';\omega)e^{-i\omega(\tau-\tau')},$$

where it is expressed via $G_0(x,x';\omega)$ as follows:

$$G(x,x';\omega)=G_0(x,x';\omega)-\frac{\Lambda T}{\alpha}\frac{G_0(x,0;\omega)G_0(0,x';\omega)}{1+\frac{\Lambda T}{\alpha}G_0(0,0;\omega)}$$
$$=\frac{2\pi g}{|\omega|}\left[e^{-(|\omega|/v)|x-x'|}-\frac{e^{-(|\omega|/v)(|x|+|x'|)}}{\frac{\alpha|\omega|}{2\pi g\Lambda T}+1}\right]. \qquad (30)$$

Exponentiating the denominator of the second term above by $1/D=2\int ds e^{-2Ds}$, we obtain G in the x,τ representation,

$$G(\xi,\xi')-G_0(\xi,\xi')=4g\int_0^\infty ds e^{-2s}\ln\left|\sin\left[\pi T\left(\tau-\tau'+i\frac{|x|+|x'|}{v}\right)+i\frac{\alpha s}{g\Lambda}\right]\right|. \qquad (31)$$

The impurity-induced terms in Eq. (28), Φ and Ξ, which are, respectively, linear and quadratic in χ, Eq. (25), result from the averaging of the first- and zeroth-order terms in Θ arising from Eq. (26). Noting that for the pointlike impurity under considerations in this section $\chi(z_1)\to\chi(\tau_1)$, we find these terms as follows:

$$\Xi=\frac{\Lambda T}{2\alpha}\int d\tau_1 d\tau_2\left[\frac{\Lambda T}{\alpha}G(0,\tau_1;0,\tau_2)-\delta(\tau_1-\tau_2)\right]\times\chi(\tau_1)\chi(\tau_2), \qquad (32)$$

$$\Phi_{\eta\eta'}=\frac{\Lambda T}{2\alpha}\int d\tau_1[\eta G(\xi;0,\tau_1)-\eta' G(\xi';0,\tau_1)]\chi(\tau_1). \qquad (33)$$

It should be stressed that both Ξ and Φ depend on the ''observation points'' ξ and ξ' via the appropriate dependence of the parameter χ, Eq. (25). All these functions can be calculated for arbitrary ξ and ξ'. However, since we are only interested in the local density of states (at an arbitrary distance from the impurity) and Friedel oscillations, it is sufficient to consider $x=x'$ case only; τ' for convenience is set to zero so that from now on we use $\xi=(x,\tau)$ and $\xi'=(x,0)$. Introducing the shorthand notations $\Xi(x,\tau;x,0)\equiv\Xi(x;\tau)$, etc., we obtain

$$\Xi(x;\tau)=\frac{\Lambda T^2}{2\alpha}\sum_\omega|\chi(0,\omega)|^2\left[1-\frac{\Lambda T}{\alpha}G(0,0;\omega)\right], \qquad (34)$$

$$\Phi_{\eta\eta'}(x;\tau)=\frac{\Lambda T^2}{2\alpha}\sum_{\omega}\chi(0,\omega)[\eta G(x,0;\omega)e^{-i\omega\tau} - \eta' G(0,x;\omega)]. \tag{35}$$

Substituting the Fourier transform of Eq. (25),

$$\chi(0,\omega)=\frac{i\pi}{\omega}\mathrm{sgn}\, x e^{-|\omega|(|x|/v)}[e^{i\omega\tau}-1], \tag{36}$$

and Eq. (30) into Eqs. (34) and (35), we carry out the Matsubara summation to obtain Ξ and Φ in the same representation as follows:

$$\Xi(x;\tau)=\frac{1}{2g}\int_0^\infty ds e^{-2s}\ln\left[1+\frac{\sin^2\pi T\tau}{\sinh^2\left(\frac{\alpha s}{g\Lambda}+2\pi T\frac{|x|}{v}\right)}\right],$$

$$\Phi(x;\tau)=\mathrm{sgn}\, x\left[\frac{\pi}{2}-\mathrm{Im}\ln\sin\pi T\left(\tau+2i\frac{|x|}{v}\right)\right], \tag{37}$$

where $\Phi\equiv\Phi_{+-}=-\Phi_{-+}$, while $\Phi_{++}=\Phi_{--}=0$.

Now we have all the ingredients to find the Green function (22). Using Eq. (28), we express $\mathcal{G}(\xi)$ in terms of G, Φ, and Ξ as follows:

$$\mathcal{G}(x;\tau)=\frac{T}{v_F}\frac{\alpha^{1/2g-1}}{|\sin\pi T\tau|^{1/2g}}e^{-\Xi(\xi)}e^{-(1/4)G(x,x;\tau=0)} \times\{\mathrm{sgn}\,\tau e^{(1/4)G(\xi)}-e^{-(1/4)G(\xi)}\cos[2p_F x+\Phi(\xi)]\}. \tag{38}$$

Equation (38) combined with the expressions for G, Eq. (31), and Φ and Ξ, Eq. (37), gives a formal representation for the electron Green function in the presence of a single impurity which is asymptotically exact when $T\to 0$. To extract from this manageable expressions for physical quantities in different regions, we need to get the appropriate asymptotic behavior of G, Φ, and Ξ. Using the dimensionless notations $\tilde{x}\equiv gxp_F$ and $\tilde{\tau}\equiv\varepsilon_F\tau$, we find

$$G(x;\tau)=\begin{cases}2g\ln\dfrac{\alpha}{\Lambda\sin\alpha\tilde{\tau}}, & |\tilde{\tau}+i\tilde{x}|\ll\Lambda^{-1}\\ 2g\ln\dfrac{\sin\alpha|\tilde{\tau}+i\tilde{x}|}{\sin\alpha\tilde{\tau}}, & |\tilde{\tau}+i\tilde{x}|\gg\Lambda^{-1},\end{cases}$$

$$\Xi(x,\tau)=\begin{cases}\dfrac{1}{4g}\ln\left[1+\dfrac{\sin^2(\alpha\tilde{\tau})}{\sinh^2(\alpha\tilde{x})}\right], & \Lambda^{-1}\ll\tilde{x}\\ \dfrac{1}{2g}\ln\dfrac{\Lambda\sin\alpha\tilde{\tau}}{\alpha}, & \tilde{x}\ll\Lambda^{-1}\ll\tilde{\tau}\\ g(\Lambda\tilde{\tau})^2\ln(\Lambda\tilde{\tau})^{-1}, & \tilde{x},\tilde{\tau}\ll\Lambda^{-1}.\end{cases}$$

For simplicity, we keep here only positive $\tilde{x}$ and $\tilde{\tau}$, the latter changing between 1 and π/α. We do not write asymptotics for $\Phi(x,\tau)$ explicitly since we will not use it in the present considerations. The above expressions enable us to find the Green function $\mathcal{G}(x,\tau;x,0)\equiv\mathcal{G}(x;\tau)$, Eq. (38), at any distance x from the impurity:

$$\mathcal{G}(x;\tau)=\frac{p_F}{2\pi}\times\begin{cases}\dfrac{\alpha^{1/g}}{(\sin\alpha\tilde{\tau})^{1/g}}[\max(\Lambda^{-1},\tilde{x})]^{(1/2)(1/g-g)}[1-\cos(2p_F x+\Phi)], & \max(\tilde{x},\Lambda^{-1})\ll\tilde{\tau}\\ \dfrac{\alpha^{(1/2)(1/g+g)}}{(\sin\alpha\tilde{\tau})^{(1/2)(1/g+g)}}\left[1-\left(\dfrac{\sin\alpha\tilde{\tau}}{\sinh\alpha\tilde{x}}\right)^g\cos(2p_F x+\Phi)\right], & \min(\tilde{x},\Lambda^{-1})\gg\tilde{\tau}.\end{cases} \tag{39}$$

In the first line of the above expression we have restricted $\tilde{x}$ to the region $\tilde{x}\ll\alpha^{-1}$ which is equivalent to $x\ll\ell_T$ ($\ell_T\equiv v_F/\pi T$ is the thermal dephasing length). The reason is that we do not need to consider larger x in this region of energy (or τ) as the influence of the impurity on the LDOS is suppressed at much shorter distances. In the second line we have kept all $\tilde{x}$ as this allows us to extract a well-known result for the Friedel oscillations.[9,13,14] By putting $\tau=0$, we obtain in the region $\tilde{x}\gg\Lambda^{-1}$,

$$\Delta\rho(x)=\frac{p_F\alpha^g}{2\pi}\frac{1}{\left|\sinh\frac{2gx}{\ell_T}\right|^g}\cos(2p_F x+\Phi). \tag{40}$$

For $x\ll\ell_T$, the amplitude of the Friedel oscillations decreases $\propto|x|^{-g}$ while for $x\gg\ell_T/g$ it is exponentially suppressed.

The local density of states at a distance x from the impurity is defined via an appropriate analytical continuation of the Fourier transform of $\mathcal{G}$,

$$\nu(x,\varepsilon)=-\frac{1}{\pi}\mathrm{Im}\int d\tau e^{-i\epsilon\tau}\mathcal{G}(\xi)|_{i\epsilon=\varepsilon}.$$

Using asymptotics for the Green function (39), we find for $g\geqslant 1/2$ explicit expressions for the LDOS smoothed over the length scale much larger than p_F^{-1} in three different regions:

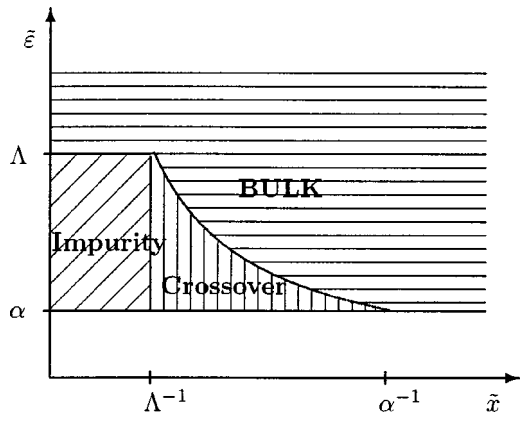

FIG. 1. Regions with different behavior of local density of states $\nu(x,\varepsilon)$; here $\tilde{x}\equiv g p_F|x|$, $\tilde{\varepsilon}\equiv\varepsilon/\varepsilon_F$, $\alpha\equiv\pi T/\varepsilon_F$, and $\Lambda=\lambda^{1/(1-g)}$ is the renormalized impurity strength.

$$\nu(x,\epsilon) \sim \begin{cases} \tilde{\varepsilon}^{1/g-1}\Lambda^{-(1/2)(1/g-g)}, & \tilde{x}\ll\Lambda^{-1}\ll\tilde{\varepsilon}^{-1} \quad (41a) \\ \tilde{\varepsilon}^{1/g-1}\tilde{x}^{(1/2)(1/g-g)}, & \Lambda^{-1}\ll\tilde{x}\ll\tilde{\varepsilon}^{-1} \quad (41b) \\ \tilde{\varepsilon}^{(1/2)(1/g+g)-1}, & \min(\tilde{x},\Lambda^{-1})\gg\tilde{\varepsilon}^{-1}. \quad (41c) \end{cases}$$

These three regions with different behavior of LDOS are sketched in Fig. 1. Equation (41a) describes LDOS in the vicinity of impurity, in full correspondence with the original results of Kane and Fisher[11] obtained for the LDOS at $x=0$, i.e., exactly at the impurity. In addition, we have established here the LDOS dependence on the impurity strength $\Lambda\equiv\lambda^{1/(1-g)}$. The region of applicability of Eq. (41a) corresponds to the diagonally hatched region in Fig. 1.

Equation (41c) gives the LDOS at very large distances from the impurity. As expected (and has already been noted, e.g., in Refs. 17 and 21), it coincides with a well-known result for the LDOS in the homogenous Luttinger liquid (see for reviews Ref. 24). Its region of applicability is horizontally hatched in Fig. 1.

In the intermediate region, vertically hatched in Fig. 1, the LDOS depends both on the energy and the distance from the impurity. This analytic dependence given by Eq. (41b) describes the crossover from the impurity-induced dip in the LDOS to the bulk behavior.

Finally, the unhatched region for $\tilde{\varepsilon}<\alpha$ corresponds to small energies, $\varepsilon\lesssim T$, where the energy dependence saturates (by $\varepsilon\to T$) in all the three lines of Eqs. (41a)–(41c).

In conclusion, we have demonstrated that the formalism developed here allowed us to obtain in a unified way the known results for the Friedel oscillations and the LDOS both in the vicinity of the impurity and in the bulk. This formalism has also allowed us to obtain not only the limiting cases described above but a full analytic description of the crossover between them. We have shown that, as a function of x at a fixed energy ε, $\nu(x)$ remains constant for $\tilde{x}\lesssim\Lambda^{-1}$, then it increases until $\tilde{x}$ becomes of order $\tilde{\varepsilon}^{-1}$ (which happens before x overtakes the thermal dephasing length, ℓ_T), where the LDOS reaches its bulk (x-independent) value, given by Eq. (41c). The crossover between the impurity-dominated and bulk values is governed by a universal power law of Eq. (41b), $\nu(x)\sim\tilde{x}^{(1/2)(1/g-g)}$.

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APPENDIX

The Jacobian of the gauge transformation, Eq. (6), can be written as

$$\ln J[\phi]=\sum_{\eta=\pm}\mathrm{Tr}\ln\left|\frac{\partial_\eta-i\phi}{\partial_\eta}\right|=-\sum_{\eta=\pm}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}(i\phi g_\eta)^n,$$

where the Green function of noninteracting right- or left-moving electrons, obeying $\partial_\eta g_\eta=1$, are given by

$$g_-(\xi,\xi')=g_+^*(\xi,\xi')=\frac{T}{2v_F}\frac{1}{\sin(z_F-z_F')}, \qquad (A1)$$

where z_F is given by Eq. (12). The nth order term in ϕ is the sum of two vertices made of the loops Γ_n^+ and Γ_n^- with n external lines corresponding to ϕ's, each loop being built of the n Green functions $g_\pm$, respectively,

$$\mathrm{Tr}(g_\eta\phi)^n=\int\prod_{k=1}^{n}dx_k d\tau_k\Gamma_n^{(a)}(z_{F_1};\ldots;z_{Fn})\prod_{i=1}^{n}\phi(x_i,\tau_i),$$

$$\Gamma_n^{(a)}(z_{F_1};\ldots;z_{Fn})=\prod_{i=1}^{n}g_\eta(z_{Fi}-z_{Fi+1}) \qquad (A2)$$

with the boundary condition $z_{Fn+1}=z_{F_1}$. Substituting g_η from Eq. (A1), one finds

$$\Gamma_n^+(z_{F_1};\ldots;z_{Fn})\propto\prod_{i=1}^{n}\frac{s_i}{s_i-s_{i+1}}, \quad s_i=e^{2iz_{Fi}}.$$

One can rewrite the symmetric part of this vertex, which contributes into the integral in Eq. (A2), as follows:

$$\Gamma_n^+(z_{F_1};\ldots;z_{Fn})\propto\frac{\mathcal{A}_n(s_1,\ldots,s_n)}{\prod_{i<j}^{n}(s_i-s_j)}\prod_{k=1}^{n}s_k,$$

where $\mathcal{A}_n$ is an absolutely antisymmetric polynomial built on n variables s_i. By power counting, its order should be $n(n-3)/2$. On the other hand, the minimal possible order of an absolutely antisymmetric polynomial of n variables is $n(n+1)/2$, as follows from the fact that the powers of different variables should be different for any monomial in order the entire polynomial to be antisymmetric. The two inequalities can only be satisfied for $n\leqslant 2$ so that $\mathcal{A}_{n>2}=0$. Therefore, all loops containing more than two external lines are zero.[4]

Therefore, we are left with the contributions from the

loops with one or two external lines. The loop with one external line is proportional to the zero-momentum mode of the Coulomb interaction and is canceled, as always, due to electroneutrality. The loop with two external lines is just the standard polarization operator in the random-phase approximation (exact for the LL), given in (q,Ω) representation by Eq. (8) so that the Jacobian is reduced to that in Eq. (7) in the main text.

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