Partial Differential Equations[2018]

§4 Integral **T**ransform method

Momentum space 动量空间	$\overset{T}{\underset{T^{-1}}{\leftrightarrows}}$	位形空间 Position space
wave vector $\vec{k}$-space 波矢/$\vec{k}$ 空间	$\leftrightarrows$	坐标空间 coordiante space
frequency-domain rep. 频域	$\leftrightarrows$	时域 time-domain representation
Spectrum 能谱/谱空间	$\leftrightarrows$	物理时空 physical spacetime
score 谱	$\leftrightarrows$	音乐 music
light spectrum 光谱	$\leftrightarrows$	光/电磁波 light

自然的不同表示之间的翻译词典:

Fourier Transform	傅立叶积分变换
Sine/Cosine Transform	傅氏正弦/余弦变换
*Laplace Transform	拉氏变换
◎Hankel/Fourier-Bessel Transform	汉克尔变换
⊞discrete Transforms	有限变换
⋒Gabor/Short-Time Fourier Transform	窗口/短时傅立叶变换
⋒Wavelet Transform	小波变换

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有界 $\xrightleftharpoons[\text{加窗}]{\text{连续化/延拓}}$ 无界

§2 分离变量法/驻波叠加 $\xrightarrow{\text{延拓}}$ §1 行波解/达朗贝尔公式:

$$\Phi(x) = \begin{cases} -\varphi(-x), & x \in [-L, 0] \\ \varphi(x), & x \in [0, L] \end{cases} \qquad \Psi(x) = \begin{cases} -\psi(-x), & [-L, 0] \\ \psi(x), & [0, L] \end{cases}$$

$$\Phi = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi}{L} x, \qquad \Psi(x) = \sum_{n=1}^{\infty} \beta_n \sin \frac{n\pi}{L} x,$$

$$\alpha_n = \frac{1}{L} \int_{-L}^{L} \Phi \sin \frac{n\pi}{L} x dx = \frac{2}{L} \int_0^L \varphi \sin \frac{n\pi}{L} x dx = C_n, \qquad \beta_n = \frac{an\pi}{L} D_n,$$

可以进一步想象延拓

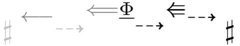
$\cdots [-3L, -2L]$	$[-2L, -L]$	$[-L, 0]$	$[0, L]$	$[L, 2L]$	$[2L, 3L]$	$[3L, 4L]$	$[4L, 5L] \cdots$
$-\varphi[-(x+2L)]$	$\varphi[x+2L]$	$-\varphi[-x]$	$\varphi[x]$	$-\varphi[-(x-2L)]$	$\varphi[x-2L]$	$-\varphi[-(x-4L)]$	$\varphi[x-4L]$
$-\psi[-(x-2L)]$	$\psi[x-2L]$	$-\psi[-x]$	$\psi[x]$	$-\psi[-(x-2L)]$	$\psi[x-2L]$	$-\psi[-(x-4L)]$	$\psi[x-4L]$

分离变量法的叠加解 $u(t,x) = \sum_{n=1}^{\infty} (C_n \cos \frac{an\pi}{L} t + D_n \sin \frac{an\pi}{L} t) \sin \frac{n\pi}{L} x$

积化和差 $= \sum_{n=1}^{\infty} \{ C_n \frac{1}{2} [\sin \frac{n\pi}{L}(x-at) + \sin \frac{n\pi}{L}(x+at)]$
$+ D_n \frac{1}{2} [\cos \frac{n\pi}{L}(x-at) - \cos \frac{n\pi}{L}(x+at)] \}$

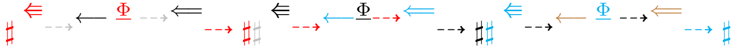
$$= \frac{\sum_{n=1}^{\infty} \alpha_n [\sin \frac{n\pi}{L}(x-at) + \sin \frac{n\pi}{L}(x+at)]}{2} + \frac{1}{2a} \sum_{n=1}^{\infty} \beta_n \frac{L}{n\pi} [\int_{x-at}^{x+at} \sin(\frac{n\pi}{L}\xi) \frac{n\pi}{L} d\xi]$$

融洽于达朗贝尔的行波 $= \frac{1}{2}[\Phi(x-at) + \Phi(x+at)] + \frac{1}{2a} \int_{x-at}^{x+at} \Psi(\xi) d\xi$



t 小时:

t 大时: 每到端点就反射; 或想象延拓区域行波已行至 [0,L] 观察区.



这种 '有界区域 $\xrightarrow[\cdots田田田田\cdots]{}$ 具周期性的无界区域' 的比照可以看透无穷叠加的起源.

有界 $\xrightleftharpoons{\text{连续化}}$ 无界

一般的无界区域呢?

- §2 Fourier 级数的分离变量法, $\xrightarrow[\pm L\to\pm\infty]{\text{连续化}}$ §4 Fourier 积分核的积分变换法, 是同一思想的不同展示.
- F series: $f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos k_n x + B_n \sin k_n x)$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\cos k_n\xi d\xi,\ n \geq 0, \quad B_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\sin k_n\xi d\xi,\ n \geq 1.$$

$$\xrightarrow[C_{-n}=\bar{C}_n=\frac{A_n-iB_n}{2}]{C_0=\frac{A_0}{2},C_n=\frac{A_n+iB_n}{2}} f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{-ik_n x},\ C_n = \frac{1}{2}\frac{1}{L}\int_{-L}^{L} f(\xi)e^{+ik_n\xi}d\xi.$$

$$\boxed{f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{\pi}{L}\int_{-L}^{L} d\xi f(\xi)e^{+ik_n\xi}e^{-ik_n x}}$$

$$\xrightarrow[\text{单位长度上波周期数}k_n=\frac{n2\pi}{2L},\Delta k=\frac{\pi}{L}]{L\to+\infty\ \text{把边界拉到无穷远以连续化}} f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\left[\int_{-\infty}^{\infty} d\xi f(\xi)e^{+ik\xi}\right]e^{-ikx}.$$

- F trans, FT: $\hat{f}(k) \stackrel{\text{def}}{=} \left[\int_{-\infty}^{\infty} dx f(x)e^{+ikx}\right] \stackrel{\text{def}}{=} F[f(x)]$ **傅立叶积分变换**

$F^{-1}T$: $f(x) \stackrel{\text{约定}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\hat{f}(k)e^{-ikx} \stackrel{\text{def}}{=} F^{-1}[\hat{f}(k)]$ **反演/逆变换**

Fourier 变换常用性质

f(x) 在任意有界区逐段光滑[-1], 且在 $(-\infty,+\infty)$ 绝对可积, 则 FT&F^{-1}T 存在.

(1) 线性 $F[c_1 f_1(x) + c_2 f_2(x)] = c_1 F[f_1(x)] + c_2 F[f_2(x)]$

(2) 原函数的微分 若原函数的微分的 $F[f'(x)]$ 存在, 且 $f(\pm\infty) = 0$,
则 $F[f'(x)] = -ikF[f(x)]$.
优势: 时域或位形上 (坐标表示) 的微积分运算,
变成 频域或波数上 (能动量表示) 的代数运算.

证: $\int_{-\infty}^{\infty} f'(x)e^{+ikx}dx = f(x)e^{ikx}|_{-\infty}^{+\infty} - (ik)\int_{\infty}^{\infty} f(x)e^{ikx}dx = 0 - 0 - ikF[f(x)]$

(3) 原函数的积分若 $\int_{-\infty}^{x} f(\xi)d\xi$ 的 FT 存在, 则 $F[\int_{-\infty}^{x} f(\xi)d\xi] = \frac{\hat{f}(k)}{-ik}$.

证: 设 $g(x) = \int_{-\infty}^{x} f(\xi)d\xi$, $g' = f$, $F[g'] = -ikF[g] \to F[g] = \frac{F[g']}{-ik} = \frac{\hat{f}(k)}{-ik}$

(4) 原函数位移/时移 $F[f(t-t_0)] = \int_{-\infty}^{\infty} dt f(t-t_0)e^{+i\omega t}$
$= e^{+i\omega t_0}\int_{-\infty-t_0}^{\infty-t_0} d(t-t_0)f(t-t_0)e^{+i\omega(t-t_0)} = e^{+i\omega t_0}F[f(t)] = e^{+i\omega t_0}\hat{f}(\omega)$

vs. 像函数频移 $f(t)e^{i\omega_0 t} \underset{F^{-1}T}{\overset{FT}{\rightleftharpoons}} \hat{f}(\omega+\omega_0)$

Frequency Modulation 调频实质是把各种信号的频谱搬移, 依附不同频率 (振幅不变, 频率按信息被调制发生频移[4]) 的载波, 各占不相扰频域, 方便接收机按频道分离信息.

[-1] Dirichlet 条件: 仅有限个极值点; 仅有限个第一类间断点 (左右极限存在).

[4] Shifting the carrier's frequency. The difference between the carrier's frequency and its center frequency, is proportional to the modulating signal.

*(5′) 积之像等于像之卷积　(原函数乘积变换为像函数卷积).

(5) 像之积是卷积之像 (像函数乘积对应 ‘原函数卷积’).

$$\boxed{\mathrm{F}[\mathrm{f}]\mathrm{F}[\mathrm{g}] = \mathrm{F}[\mathrm{f} * \mathrm{g}]} \qquad \mathrm{F}^{-1}[\widehat{\mathrm{fg}}] = \mathrm{F}^{-1}[\hat{\mathrm{f}}] * \mathrm{F}^{-1}[\hat{\mathrm{g}}]$$

convolution integral 卷积的定义:

$$f(x) * g(x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi.$$

$$f(t,\vec{x}) * g(t,\vec{x}) \overset{\text{def}}{=} \iiint_{-\infty}^{\infty} f(t,\vec{\xi})g(t,\vec{x}-\vec{\xi})d\vec{\xi},$$

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau, \qquad f(t,x) \star_t g(t,x) \overset{\text{def}}{=} \int_{t_0}^{t_1} f(\tau,x)g(t-\tau,x)d\tau,$$

$$f(t,x) \star_t * g(t,x) \overset{\text{def}}{=} \int_{t_0}^{t_1} f(\tau,x) * g(t-\tau,x)d\tau = \int_{t_0}^{t_1}\int_{-\infty}^{\infty} f(\tau,\xi)g(t-\tau,x-\xi)d\xi d\tau.$$

证: $\boxed{\text{rhs}} \int_{-\infty}^{\infty} dx e^{+ikx}[\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi] \overset{t=x-\xi}{=} \int_{-\infty}^{\infty} dt e^{+ikt}g(t)[\int_{-\infty}^{\infty} e^{+ik\xi}f(\xi)d\xi] \boxed{\text{lhs}}$

*(6) 相似 $F[f(\alpha x)] = \frac{1}{|\alpha|}\hat{f}(\frac{k}{\alpha})$.　* 特例 $\alpha = -1$: 翻转 $F[f(-x)] = \hat{f}(-k)$.

* 共轭 $\overline{f(x)} \rightleftharpoons \overline{\hat{f}(-k)}$.　* 反射 $F[\hat{f}(x)] = 2\pi f(-k)$.

*(7) “Fourier transform is unitary”: 傅里叶变换是幺正的.

Parseval’s theorem　　　　Plancherel theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk. \qquad \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx \sim \int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}dk.$$

能量/单周期积分的信号平均功率, 在不同表示, 相等. 数学上: 保内积不变.

♪* 用 Fourier 变换解无界区域的定解问题

$$
* \text{解} \begin{cases} u_{tt} = a^2 u_{xx}, & t > 0, -\infty < x < +\infty, \quad pde \\ u|_{t=0} = \varphi(x), & u_t|_{t=0} = \psi(x). \quad ic \end{cases}
$$

(一) FT 正变换: $\hat{u}(t,\lambda) = F[u(t,x)] \equiv \int_{-\infty}^{\infty} u(t,x) e^{+i\lambda x} dx$

对 x 空间里 原函数的 *pde* 做积分变换, 任意交换不相干变量的积分求导求极限顺序, 得像函数的 ODE: $\frac{d^2 \hat{u}}{dt^2} = a^2(-i\lambda)^2 \hat{u}$

对 *ic* 做正变换, 得 IC: $\hat{u}|_{t=0} = \hat{\varphi}(\lambda) = F[\varphi(x)]$, $\hat{u}_t|_{t=0} = \hat{\psi}(\lambda) = F[\psi(x)]$.

(二) 解像函数: 观察 λ 空间里像函数的方程, 是关于 t 的微分方程, λ 只是标记一系列不同方程的参数 (类比离散情况下的 n), 不变. 设 $\hat{u}(t,\lambda) = e^{kt}$, 由特征方程 $k^2 = -a^2\lambda^2$, $k = \pm ia\lambda$, 得通解 $\hat{u} = A(\lambda)e^{ia\lambda t} + B(\lambda)e^{-ia\lambda t}$.

用 IC 定系数 (类比离散的 §2 最后一步): $A + B = \hat{\varphi}(\lambda)$, $ia\lambda(A - B) = \hat{\psi}(\lambda)$,

$\rightarrow A = \frac{1}{2}[\hat{\varphi} + \frac{\hat{\psi}}{ai\lambda}], B = \frac{1}{2}[\hat{\varphi} - \frac{\hat{\psi}}{ai\lambda}]$,

得定解 $\hat{u}(t,\lambda) = \frac{1}{2}[\hat{\varphi}(\lambda)e^{iat\lambda} + \hat{\varphi}(\lambda)e^{-iat\lambda}] + [\frac{1}{2a}\frac{\hat{\psi}}{i\lambda}e^{iat\lambda} + \frac{1}{2a}\frac{\hat{\psi}}{-i\lambda}e^{-iat\lambda}]$.

抽象地说, 到这一步问题已得解决, 我们已在另一种表示空间得到定解. 尽管此解不是人们熟悉的时间空间表示, 但满足方程及约束条件的客观存在既被找出, 不依赖人看不看得懂它. 但出于为观众考虑, 为直观, 应多做一步, 变回时间空间表示.

积分变换法
避难趋易
兜圈子
解题

原函数的 pde ic			原函数?
FT ↓			↑ $F^{-1}T$
像函数的 ODE IC	$\xrightarrow{\text{解像}}$		像函数

n+1 自变量pde ic
n维独立变换FT ↓
ODE&IC 代数方程 $\xrightarrow[\text{解代数}]{\text{解常微}}$ 像
FT ↑ ↓ $F^{-1}T$
ode ic 原函数

原函数
↑ $F^{-1}T$

*(三) $F^{-1}T$ 逆变换[1]:

$$F^{-1}[\frac{\hat{\varphi}}{2}e^{iat\lambda}] = \frac{\varphi(x-at)}{2}, \qquad F^{-1}[\frac{\hat{\varphi}}{2}e^{-iat\lambda}] = \frac{\varphi(x+at)}{2},$$

$$F^{-1}[\frac{1}{2a}\frac{\hat{\psi}}{i\lambda}e^{iat\lambda}] = -\frac{1}{2a}\int_{-\infty}^{x-at}\psi(\xi)d\xi,$$

$$F^{-1}[\frac{1}{2a}\frac{\hat{\psi}}{-i\lambda}e^{-iat\lambda}] = +\frac{1}{2a}\int_{-\infty}^{x+at}\psi(\xi)d\xi,$$

$$u(t,x) = F^{-1}[\hat{u}] = [\frac{\varphi(x-at)}{2} + \frac{\varphi(x+at)}{2}] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi.$$

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1 自然的各表示之间的词典

频域/动量表示		时域/空间表示	
频域/动量表示	$\hat{f}(\lambda)e^{+ix_0\lambda}$	$f(x-x_0)$	时域/空间表示
	$\frac{\hat{\psi}}{-i\lambda}$	$\int_{-\infty}^{x}\psi(\xi)d\xi$	
	$e^{-A\lambda^2}$		
	$\hat{f}\hat{g}$	$f*g$	
单色光	$2\pi\delta(\omega-\omega_0)$	$e^{-i\omega_0 t}$	不弥散, 非波包
各频率均布	1	$\delta(t)$	冲击波

用 Fourier 变换解无界区间热传导问题[2]

解例 1 $\begin{cases} u_t = a^2 u_{xx}, & t > 0, -\infty < x < +\infty, \quad pde \\ u|_{t=0} = \varphi(x). & ic \end{cases}$

- (一) FT: $\hat{u}(t,\lambda) = F[u(t,x)] \equiv \int_{-\infty}^{\infty} u(t,x) e^{+i\lambda x} dx$,
  pde$\xrightarrow{FT}$ODE: $\frac{d\hat{u}}{dt} = a^2(-i\lambda)^2 \hat{u}$, ic$\xrightarrow{FT}$IC: $\hat{u}|_{t=0} = \hat{\varphi}(\lambda) = F[\varphi(x)]$.
- (二) 解像: 通解 C(与 t 无关)$e^{-a^2\lambda^2 t}$,
  用 IC 定系数 $Ce^{-a^2\lambda^2 t}|_{t=0} = C = \hat{\varphi}(\lambda)$, 得定解 $\hat{u} = \hat{\varphi}(\lambda) e^{-a^2\lambda^2 t}$.
- (三) $F^{-1}T$ 逆变换: $u(t,x) = F^{-1}[\hat{u}(t,\lambda)] = F^{-1}[\hat{\varphi}(\lambda) e^{-a^2 t\lambda^2}]$
  $= F^{-1}[\hat{\varphi}(\lambda)] * F^{-1}[e^{-(a^2 t)\lambda^2}] \quad = \varphi(x) * \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a^2 t\lambda^2} e^{-ix\lambda} d\lambda$
  $= \varphi(x) * \frac{1}{2\pi}\sqrt{\frac{\pi}{a^2 t}} e^{\frac{(-ix)^2}{4(a^2 t)}} = \varphi(x) * \frac{1}{2a\sqrt{\pi t}} e^{\frac{-x^2}{4a^2 t}} = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{\frac{-(x-\xi)^2}{4a^2 t}} d\xi$.

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[2]物理必备: $\boxed{\int_{-\infty}^{\infty} e^{-A\lambda^2 + B\lambda} d\lambda = \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}}}$

用途广于记 $F^{-1}[e^{-a^2 t\lambda^2}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(a^2 t)\lambda^2} e^{(-ix)\lambda} d\lambda = \frac{1}{2\pi}\sqrt{\frac{\pi}{a^2 t}} e^{\frac{(-ix)^2}{4a^2 t}} = \frac{1}{2a\sqrt{\pi t}} e^{\frac{-x^2}{4a^2 t}}$

* 证明: $\int e^{-(\sqrt{A}\lambda - \frac{B}{2\sqrt{A}})^2 + \frac{B^2}{4A}} \frac{d\sqrt{A}\lambda}{\sqrt{A}}$, 概率积分 $I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy}$

$= \sqrt{\int_0^{\infty} dr \int_0^{2\pi} r d\theta e^{-r^2}} \xrightarrow{\chi = -r^2} \sqrt{2\pi \int_0^{-\infty} \frac{-d\chi}{2} e^{\chi}} = \sqrt{\pi}$

* 非齐次不增原则困难，非线性难解

* 附加题 $\begin{cases} u_t = a^2 u_{xx} + b u_x + cu + d(u_x)^2 + f, & x \in \mathbb{R}, t > 0, \quad pde \\ u|_{t=0} = \varphi(x). & \quad ic \end{cases}$

[d=0]
解 $(1)\mathrm{FT} \to \begin{cases} \hat{u}_t = a^2(-ik)^2\hat{u} + b(-ik)\hat{u} + c\hat{u} + \hat{f}, & \hat{f}(t,k) = F[f(t,x)] \\ \hat{u}|_{t=0} = \hat{\varphi}(k), & \hat{\varphi}(k) = F[\varphi(x)] \end{cases}$

(2) $\xrightarrow[\hat{u}_t^{hom} = -a^2k^2\hat{u}^{hom} - ibk\hat{u}^{hom} + c\hat{u}^{hom}]{\text{先解齐次 (homogeneous)}} \hat{u}^{hom} = Ce^{(-a^2k^2 - ibk + c)t} \xrightarrow[\text{设}\hat{u}^{sp} = A(t)\hat{u}^{hom}]{\text{常数变易法}}$

$A_t(t)\hat{u}^{hom} + A\hat{u}_t^{hom} = A[-a^2k^2\hat{u}^{hom} - ibk\hat{u}^{hom} + c\hat{u}^{hom}] + \hat{f}, \quad A = \int \frac{\hat{f}}{\hat{u}^{hom}} dt,$

$\Longrightarrow \hat{u} = \hat{u}^{hom} + \hat{u}^{sp} = \hat{u}^{hom} + \hat{u}^{hom} \int \frac{\hat{f}}{\hat{u}^{hom}} d\tau = e^{Kt}[C + \int_0^t \hat{f} e^{-K\tau} d\tau].$

定系数 $\hat{u}|_{t=0} = e^{K\cdot 0}[C + \int_0^0 \cdots] = C = \hat{\varphi},$

得定解 $\hat{u} = e^{(-a^2k^2 - ibk + c)t}[\hat{\varphi}(k) + \int_0^t \hat{f}(\tau,k) e^{-(-a^2k^2 - ibk + c)\tau} d\tau].$

$(3)\mathrm{F}^{-1}\mathrm{T}\ u = F^{-1}[e^{-a^2tk^2 - ibtk + ct}] * F^{-1}[\hat{\varphi}] + \int_0^t F^{-1}[\hat{f}] * F^{-1}[e^{(-a^2k^2 - ibk + c)(t-\tau)}] d\tau$

$= \frac{e^{ct}}{2\pi} \int e^{-(a^2t)k^2 - i(x+bt)k} dk * \varphi + \int_0^t f * \{\int e^{-a^2(t-\tau)k^2 - i[x+b(t-\tau)]k} dk\} \frac{e^{c(t-\tau)}}{2\pi} d\tau$

$= e^{ct} \frac{1}{2a\sqrt{\pi t}} e^{\frac{-(x+bt)^2}{4a^2t}} * \varphi + \int_0^t f * \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{\frac{-[x+b(t-\tau)]^2}{4a^2(t-\tau)}} e^{c(t-\tau)} d\tau$

$= \frac{e^{ct}}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-\xi+bt)^2}{4a^2t}} \varphi(\xi) d\xi + \int_0^t \int_{-\infty}^{\infty} f(\tau,\xi) e^{\frac{-[x-\xi+b(t-\tau)]^2}{4a^2(t-\tau)}} \frac{e^{c(t-\tau)}}{2a\sqrt{\pi}\sqrt{t-\tau}} d\xi d\tau.$

化归

- Tricks: 试换自变量.
- Tricks: 试加减特解函数.
- Tricks: 试乘函数, 设 $u(t,x) = v(t,x)e^{\mu x+\epsilon t} = ve^{-\frac{b}{2a^2}x+(c-\frac{b^2}{4a^2})t}$,

$$\begin{cases} v_t = a^2 v_{xx} + F, & F(t,x) = f(t,x)e^{-\mu x-\epsilon t}, \\ v(0,x) = \Phi, & \Phi(x) = \varphi(x)e^{-\mu x}. \end{cases}$$

- Tricks: 试复合函数.

$[d \neq 0]$ * 解 (设 $w = g(u)$), $b=c=f=0$

$$u_t = a^2 u_{xx} + u_x^2 d$$

$$w_x = g'(u)u_x, \quad w_{xx} = (g'u_x)_x = g''(u_x)^2 + g'u_{xx},$$

$$w_t = g'u_t = g'u_{xx}a^2 + g'(u_x)^2 d$$

$$= [w_{xx} - g''u_x^2]a^2 + g'u_x^2 d = w_{xx}a^2 + [g'd - a^2 g'']u_x^2,$$

$[g'd - a^2 g''] = 0$ 就能化“非线性”方程为线性!

不用找通解, 特解够用: $(\ln g')' = \frac{d}{a^2} \to g' = e^{\frac{d}{a^2}u}e^C \to w = g(u) = e^{\frac{d}{a^2}u}$,

$$\begin{cases} w_t = a^2 w_{xx}, & (x \in \mathbb{R}, t > 0) \\ w|_{t=0} = e^{\frac{d}{a^2}u}|_{t=0} = e^{\frac{d}{a^2}\varphi(x)}. \end{cases}$$

换积分核

- $\hat{f}(\lambda) = \int_a^b f(x) \boxed{K(x;\lambda)} dx$, $\boxed{K(x;\lambda)}$ 是积分核，不同表示的译制器.
- Fourier trans FT $\boxed{K(x;k) = e^{+ikx}}$
- Fourier Sine trans F_sT $\boxed{K(x;k) = \sin(kx)}$ $F_s[f(x)] \equiv \int_0^\infty f(x)\sin kx dx$
- Fourier Cosine trans F_cT $\boxed{K(x;k) = \cos(kx)}$ $F_c[f(x)] \equiv \int_0^\infty f(x)\cos kx dx$
- Hankel trans $h_\nu T$ $\boxed{K(r;\omega) = J_\nu(\omega r)}$
- Laplace trans LT $\boxed{K(t;p) = H(t)e^{-pt}}$
- Wavelet trans WT $\boxed{K(t;a,b) = \frac{1}{\sqrt{a}}\bar{\psi}\left(\frac{t-b}{a}\right)}$

- “正交基” 按 pde 和 bc 选，其连续化对应——“积分核” 如何选？
- 自变量变化区间？
- 函数在端点的值，是否足够变换 $f^{(n)}(x)$，变成简明代数运算式？
- 函数在 $\pm\infty$ 或各区间的行为，能保证哪种变换 & 逆变换存在？变换是否幺正？

换积分核，用傅氏正弦/余弦变换解半无界区间定解问题

- F_s series: $f(x)=\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\frac{\pi}{L}\int_0^L f(\xi)\sin k_n\xi d\xi\right]\sin k_n x,$

$$\xrightarrow[\text{波数量子}\Delta k=\frac{\pi}{L}]{L\to+\infty\ \text{连续化}} f(x)=\frac{2}{\pi}\int_0^{+\infty}dk\left[\int_0^{\infty}f(\xi)\sin(k\xi)d\xi\right]\sin kx.$$

- F_s trans: $\hat{f}_s(k)\overset{\text{def}}{=}\int_0^{\infty}f(\xi)\sin k\xi d\xi\overset{\text{def}}{=}F_s[f(\xi)]$ 傅立叶正弦变换

  $F_s^{-1}T$: $f(x)=\frac{2}{\pi}\int_0^{+\infty}\hat{f}_s(k)\sin kx dk\overset{\text{def}}{=}F_s^{-1}[\hat{f}_s(k)]$ 反演

- F_cT: $\hat{f}_c(k)\overset{\text{def}}{=}\int_0^{\infty}f(\xi)\cos k\xi d\xi\overset{\text{def}}{=}F_c[f(\xi)]$ 傅立叶余弦变换

  $F_c^{-1}T$: $f(x)=\frac{2}{\pi}\int_0^{+\infty}\hat{f}_c(k)\cos kx dk\overset{\text{def}}{=}F_c^{-1}[\hat{f}_c(k)]$ 反演

- 原函数的微分怎么变?

$$F_s[f']=\int_0^{\infty}f'(x)\sin kx dx=f(x)\sin kx|_0^{\infty}-k\int_0^{\infty}f(x)\cos kx dx=0-0-k\hat{f}_c(k)$$

$$F_c[f']=-f(0)+k\hat{f}_s(k)$$

$$F_s[f'']=\int_0^{\infty}f''(x)\sin kx dx=f'(\infty)\sin(\infty)-k\int_0^{\infty}f'(x)\cos kx dx$$

$$\xrightarrow{f'(\infty)=0}-k\left[f(x)\cos kx|_0^{\infty}-(-k)\int_0^{\infty}f(x)\sin kx dx\right]\underset{\text{I}}{=}kf(0)-k^2\hat{f}_s(k)$$

$$F_c[f'']=f'(x)\cos kx|_0^{\infty}+k\left[f(x)\sin kx|_0^{\infty}-k\int_0^{\infty}f(x)\cos kx dx\right]\underset{\text{II}}{=}-f'(0)-k^2\hat{f}_c(k)$$

例

$$\text{Q1}\begin{cases} u_{1t}=a^2u_{1xx}, & t>0,x>0 \quad pde \\ u_1|_{x=0}=u_0, & \text{I bc} \\ u_1|_{t=0}=0. & ic \end{cases} \qquad \text{Q2}\begin{cases} u_{2t}=a^2u_{2xx}, & t>\tau \\ u_{2x}|_{x=0}=0, & \text{II bc} \\ u_2|_{t=\tau}=u_1|_{t=\tau}. \end{cases}$$

解 Q1: 分析端点 $x=0$, 发现只知I 类条件 $u_1(t,0)$, 不知 II 类条件 $\frac{\partial u_1(t,0)}{\partial x}$. 只能采用傅立叶正弦变换 (因为余弦变换 u_{1xx} 需 $u_{1x}|_{x=0}$).

(1) F_sT: $\hat{u}_{1s}(t,\lambda)=F_s[u_1(t,x)]=\int_0^\infty u_1(t,x)\sin\lambda x dx$,

pde $\xrightarrow{F_sT}$ ODE: $\frac{d\hat{u}_{1s}}{dt}=a^2F_s[u_{1xx}]=a^2[\lambda u_1(t,0)-\lambda^2\hat{u}_{1s}]\overset{bc}{=}a^2\lambda u_0-a^2\lambda^2\hat{u}_{1s}$,

ic $\xrightarrow{F_sT}$ IC: $\hat{u}_{1s}|_{t=0}=\hat{\varphi}_s(\lambda)=0$.

(2) 解像: 易见特解 $\hat{u}_{1s}^*=\frac{u_0}{\lambda}$, 齐通解 $e^{-a^2\lambda^2t}$, 加出非齐通解 $Ae^{-a^2\lambda^2t}+\frac{u_0}{\lambda}$.

IC 定系数 $Ae^0+\frac{u_0}{\lambda}=0$, 得定解 $-\frac{u_0}{\lambda}e^{-a^2\lambda^2t}+\frac{u_0}{\lambda}$.

(3) $F_s^{-1}T$: $u_1(t,x)=-u_0\frac{2}{\pi}\int_0^\infty e^{-a^2t\lambda^2}\frac{\sin\lambda x}{\lambda}d\lambda+\frac{2}{\pi}\int_0^\infty\frac{u_0}{\lambda}\sin\lambda x d\lambda$

Dirichlet 积分 $\int_0^\infty\frac{\sin\lambda x}{x\lambda}d(x\lambda)=\frac{1}{2}\mathrm{Im}\int_{-\infty}^\infty\frac{e^{iz}}{z}dz=\frac{1}{2}\mathrm{Im}\pi i\operatorname*{Res}_{z=0}\frac{e^{iz}}{z}=\frac{\pi}{2}\lim_{z\to0}z\frac{e^{iz}}{z}=\frac{\pi}{2}$

$\frac{1}{\pi}\int_{-\infty}^\infty d\lambda e^{-a^2t\lambda^2}\frac{\sin\lambda x}{\lambda}=\ldots\frac{\int_0^{\lambda x}\cos\lambda\xi d(\lambda\xi)}{\lambda}=\frac{1}{\pi}\mathrm{Re}\int_0^x d\xi\int_{-\infty}^\infty d\lambda e^{-a^2t\lambda^2}e^{(i\xi)\lambda}$

$=\frac{1}{\pi}\int_0^x d\xi\sqrt{\frac{\pi}{a^2t}}e^{\frac{-\xi^2}{4a^2t}}=\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}}e^{-y^2}dy\equiv\mathrm{erf}(\frac{x}{2a\sqrt{t}})$ 高斯误差函数, 误差落在 [-x,x] 的概率

$u_1=u_0[1-\mathrm{erf}(\frac{x}{2a\sqrt{t}})]=u_0\mathrm{erfc}(\frac{x}{2a\sqrt{t}})$ 余补误差函数

例 3

$$\begin{cases} u_t = a^2 u_{xx}, & t>0, x>0 \quad \text{pde} \\ u|_{x=+\infty} = u_x|_{x=+\infty} = 0, & u_x|_{x=0} = Q, \quad \text{II bc} \\ u|_{t=0} = 0. & \text{ic} \end{cases}$$

解: 分析端点 $x=0$, 发现只知 II 类条件 $\frac{\partial u(t,0)}{\partial x}$, 不知 I 类条件 $u(t,0)$.

$F_s[f''] = \int_0^\infty f''(x)\sin kx dx \underset{\text{I}}{=} kf(0) - k^2\hat{f}_s(k)$

$F_c[f''] \underset{\text{II}}{=} -f'(0) - k^2\hat{f}_c(k)$ 只能采用傅立叶余弦变换 F_c

(1) $F_c T$: $\hat{u}_c(t,\lambda) = F_c[u(t,x)] = \int_0^\infty u(t,x)\cos\lambda x dx$,

pde $\xrightarrow{F_c T}$ ODE: $\frac{d\hat{u}_c}{dt} = a^2 F_c[u_{xx}] = a^2[-u_x(t,0) - \lambda^2\hat{u}_c] \overset{bc}{=} -a^2 Q - a^2\lambda^2\hat{u}_c$,

ic $\xrightarrow{F_c T}$ IC: $\hat{u}_c|_{t=0} = \hat{\varphi}_c(\lambda) = 0$.

(2) 解像: 易见特解 $\hat{u}_c^* = -\frac{Q}{\lambda^2}$, 齐通解 $Ae^{-a^2\lambda^2 t}$, 加出非齐通解 $Ae^{-a^2\lambda^2 t} - \frac{Q}{\lambda^2}$. 用 IC 定系数 $Ae^0 - \frac{Q}{\lambda^2} = 0$, 得定解 $\frac{Q}{\lambda^2}(e^{-a^2\lambda^2 t} - 1)$.

(3) $F_c^{-1} T$: $u(t,x) = -a^2 Q\frac{2}{\pi}\int_0^\infty \int_0^t e^{-a^2\tau\lambda^2} d\tau \cos\lambda x d\lambda$

$$= Re\frac{-2a^2 Q}{\pi}\int_0^t d\tau \frac{1}{2}\int_{-\infty}^\infty e^{-a^2\tau\lambda^2} e^{-ix\lambda} d\lambda = \frac{-a^2 Q}{\pi}\int_0^t d\tau \sqrt{\frac{\pi}{a^2\tau}} e^{\frac{-x^2}{4a^2\tau}}$$

高维推广

各维操作独立

- $\hat{f}(\lambda,\mu,\nu) = F[f(x,y,z)] = \iiint_{-\infty}^{+\infty} f(x,y,z)e^{+i(\lambda x+\mu y+\nu z)}dxdydz$
- $f(x,y,z) = F^{-1}[\hat{f}(\lambda,\mu,\nu)] = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{+\infty} \hat{f}(\lambda,\mu,\nu)e^{-i(\lambda x+\mu y+\nu z)}d\lambda d\mu d\nu$
- $F[\triangle_3 f] = [(-i\lambda)^2+(-i\mu)^2+(-i\nu)^2]F[f] = -(\lambda^2+\mu^2+\nu^2)F[f]$
- $F^{-1}[\hat{f}\hat{g}] = F^{-1}[\hat{f}]*F^{-1}[\hat{g}] = f*g = \iiint_{-\infty}^{+\infty} f(x-\xi,y-\eta,z-\zeta)g(\xi,\eta,\zeta)d\xi d\eta d\zeta$

例 $\begin{cases} u_t = a^2\triangle_3 u, & t>0,\vec{x}\in\mathbb{R}^3 \\ u|_{t=0} = \varphi(x,y,z). \end{cases} \xrightarrow{(1)FT} \begin{cases} \hat{u}_t = a^2(-\lambda^2-\mu^2-\nu^2)\hat{u}, \\ \hat{u}|_{t=0} = \hat{\varphi}(\lambda,\mu,\nu). \end{cases}$

$$\xrightarrow[\rho^2=\lambda^2+\mu^2+\nu^2]{(2)\text{解像}} \hat{u} = \hat{\varphi}e^{-a^2\rho^2 t} \xRightarrow{(3)F^{-1}T} u = F^{-1}[\hat{\varphi}\cdots] = F^{-1}[\hat{\varphi}]*F^{-1}[e^{-a^2t\rho^2}]$$

$$= \varphi * \frac{1}{(2\pi)^3}\iiint_{-\infty}^{+\infty} e^{-a^2t(\lambda^2+\mu^2+\nu^2)}e^{-(ix\lambda+iy\mu+iz\nu)}d\lambda d\mu d\nu = \varphi*\left(\frac{1}{2a\sqrt{\pi t}}\right)^3 e^{-\frac{x^2+y^2+z^2}{4a^2t}}$$

$$= \left(\frac{1}{2a\sqrt{\pi t}}\right)^3 \iiint_{-\infty}^{+\infty} d\xi d\eta d\zeta \varphi(\xi,\eta,\zeta)e^{-\frac{[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]}{4a^2t}}$$

* 换积分核, (回顾复变函数已讲的)Laplace 变换

- 无界积分变换处理 pde: 变无界 x 表示到 k 表示, 或变无界 t 表示到频谱. 半无界正余弦变换处理 pde+bc: 空间端点条件被化用在原函数微分的变换里. 都未涉及对 ic 的化用. **如果换积分核, 对时间半无界的问题做关于时间的某种变换呢? 或许** $\frac{d}{dt}$ 运算的 ode 被化为代数方程, ic 被化用在变微分的分部积分.

- $\bar{f}(p) \stackrel{\text{def}}{=} L[f(t)] \stackrel{\text{def}}{=} \int_0^\infty f(t)H(t)e^{-pt}dt,\ p = \sigma + i\lambda,\ H = \begin{cases} 1, & t \geq 0 \\ 0. & t < 0 \end{cases}$[3]

- LT ↯ FT: $L[f(t)] = \bar{f}(p) = \int_0^\infty f(t)H(t)e^{-\sigma t}e^{-i\lambda t}dt \xleftrightarrow{g(t)=f(t)H(t)e^{-\sigma t}} F[g(t)]$

FT	g 逐段光滑	绝对可积
LT	f 逐段光滑	$|f|_{t\to\infty}$ 不超过 $Me^{\sigma_0 t}$ 指数增长
		$\bar{f}(p)$ 在 $\mathrm{Re}p > \sigma_0$ 半平面解析.

- Mellin 反演公式 $f(t)e^{-\sigma t} = g(t) = F^{-1}[\bar{f}] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\sigma + i\lambda)e^{i\lambda t}d\lambda,$
$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\bar{f}(\sigma + i\lambda)e^{(\sigma+i\lambda)t}d(i\lambda) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\bar{f}(p)e^{pt}dp$
$= \sum \mathrm{Res}[\bar{f}(p)e^{pt}]$

p=iz逆转90°的Jordan引理 孤立奇点

[3]约定 $f(t) = f(t)H(t)$, 只有未来有意义.

*Laplace 变换的性质

(1) 线性 $L[c_1 f_1(t) + c_2 f_2(t)] = c_1 L[f_1(t)] + c_2 L[f_2(t)]$.

(2) 原函数的微分 $L[f^{(n)}(t)] = p^n \overline{f}(p) - p^{n-1} f(+0) - \cdots - p^0 f^{(n-1)}(+0)$.

证: $\int_{-\infty}^{\infty} f'(t) e^{-pt} dt = f(t) e^{-pt}\big|_0^{+\infty} - (-p)\int_0^{\infty} f(t) e^{-pt} dt = pL[f(t)] - f(0)$.

$\int_{-\infty}^{\infty} f^{(n)} e^{-pt} dt = pL[f^{(n-1)}] - f^{(n-1)}(0) = p\{pL[f^{(n-2)}] - f^{(n-2)}(0)\} - f^{(n-1)}(0)$.

$L[f'(t)] = p\overline{f}(p) - f(+0)$, $\quad L[f''(t)] = p^2 \overline{f}(p) - pf(+0) - f'(+0)$.

(3) 原函数的积分 $L[\int_0^t f(\tau) d\tau] = \frac{\overline{f}(p)}{p}$.

像函数的微积分 $L^{-1}[\overline{f}^{(n)}(p)] = (-t)^n f(t)$, $\quad \int_p^{\infty} \overline{f}(p) dp = L[\frac{f(t)}{t}]$.

证: $\int_p^{\infty} dp \int_0^{\infty} f(t) e^{-pt} dt = \int_0^{\infty} f(t) \frac{e^{-pt}}{-t}\Big|_{p=p}^{+\infty} dt = \int_0^{\infty} \frac{f(t)}{t} e^{-pt} dt = L[\frac{f(t)}{t}]$.

(4) 原函数延迟 $L[f(t - t_0)] = L[f(t - t_0) H(t - t_0)] = e^{-pt_0} \overline{f}(p)$. $\boxed{t_0 > 0}$

证: $\int_0^{\infty} dt f(t - t_0) e^{-pt} \xrightarrow[t>t_0]{t<t_0\text{时} f(t-t_0)=0} \int_{t_0}^{\infty} dt f(t - t_0) e^{-pt} \xrightarrow[t=t'+t_0]{t'=t-t_0} e^{-pt_0} \int_0^{\infty} dt' f(t') e^{-pt'}$.

像函数复频移 $\overline{f}(p - p_0) = L[f(t) e^{p_0 t}]$.

(5) 卷积 $\overline{f}(p) \cdot \overline{g}(p) = L[f(t) * g(t)]$.

$f(t) * g(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau \xrightarrow[\tau<0\text{恒为零}]{\tau>t\text{恒为零}} \int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t f(\tau) g(t-\tau) d\tau$.

(6) 相似 对任意 $\alpha > 0$, $L[f(\alpha t)] = \frac{1}{\alpha} \overline{f}(\frac{p}{\alpha})$.

* 用 Laplace 变换解半无界区间定解问题[5]

- LR 电感电阻串联电路 $\begin{cases} L\frac{dj}{dt}+Rj=V(t)=V_0, & ode \\ j|_{t=0}=0. & ic \end{cases}$

解 ode+ic $\xrightarrow[\text{设 } L[j]=J(p)]{(1)LT}$ algebraic eq. $L\{pJ-j(0)\}+RJ=L[V(t)]=V_0\frac{1}{p}$.

$L[1\cdot H(t)]=\int_0^\infty 1\cdot e^{-pt}dt=-\frac{1}{p}e^{-pt}|_0^\infty=\frac{1}{p}$

$\xrightarrow[(2)\text{解像}]{} J=\frac{V_0}{p(pL+R)}=\frac{V_0}{R}\left[\frac{1}{p}-\frac{1}{p+\frac{R}{L}}\right]\xrightarrow[(3)L^{-1}T]{} j=L^{-1}[J]=\frac{V_0}{R}[1-e^{-\frac{R}{L}t}]$.

- 例 2 $\begin{cases} u_t=a^2u_{xx}, & pde \\ u|_{x=0}=f(t), & \text{bc 非齐，变此方向不佳} \\ u|_{t=0}=0. & \text{ic 更好用} \end{cases}\xrightarrow[L[u]=\bar{u}(p,x)]{(1)LT}\begin{cases} p\bar{u}-u|_{t=0}=a^2\bar{u}_{xx} \\ \bar{u}|_{x=0}=\bar{f}(p) \\ u|_{x=\infty}=\bar{u}|_{x=\infty}=0 \end{cases}$

$\xrightarrow{(2)\text{解像}} \bar{u}=Ce^{\frac{\sqrt{p}}{a}x}+De^{-\frac{\sqrt{p}}{a}x}\xrightarrow{BC}$ 定解 $\bar{u}=\bar{f}(p)e^{-\frac{\sqrt{p}}{a}x}$

$\xrightarrow[\text{查表}]{(3)L^{-1}T} u=L^{-1}[\bar{u}]=L^{-1}[\bar{f}]*L^{-1}[e^{-\frac{x}{a}\sqrt{p}}]=f*\frac{\frac{x}{a}e^{-\frac{(x/a)^2}{4t}}}{2\sqrt{\pi t^3}}=\frac{x}{2a\sqrt{\pi}}\int_0^t\frac{f(t-\tau)e^{-\frac{x^2}{4a^2\tau}}}{\sqrt{\tau^3}}d\tau$

[5] LT 表

原	e^{at}	$1\cdot H(t)$	$\cos\omega t$	$\sin\omega t$	t^α	$\frac{1}{\sqrt{t}}$	$\delta(t-b)$	$\frac{be^{-\frac{b^2}{4t}}}{2\sqrt{\pi t^3}}$
像	$\frac{1}{p-a}$	$\frac{1}{p}$	$\frac{p}{p^2+\omega^2}$	$\frac{\omega}{p^2+\omega^2}$	$\frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$	$\sqrt{\frac{\pi}{p}}$	e^{-bp}	$e^{-b\sqrt{p}}$

$$* \text{例} \begin{cases} u_{tt} = a^2 u_{xx}, \\ u_x|_{x=0} = f(t), \\ u|_{t=0} = u_t|_{t=0} = 0. \end{cases}$$

pde
II bc 非齐，FcT 非最佳
ic 齐次，优选 LT

$$\xrightarrow[L[u]=\bar{u}]{(1)LT} \begin{cases} p^2\bar{u} - 0 = a^2 \bar{u}_{xx}, \\ u|_{x=\infty} = \bar{u}|_{x=\infty} = 0, \\ \bar{u}_x|_{x=0} = \bar{f}(p) \end{cases}$$ 唯一力源.

$$\xrightarrow{(2)\text{解像}} \bar{u} = Ce^{\frac{p}{a}x} + De^{-\frac{p}{a}x} \xrightarrow[\text{IIBC}:-Dp/a=\bar{f}]{\text{自然 BC: } C=0} \text{定解 } \bar{u} = \frac{\bar{f}(p)}{-p/a} e^{-\frac{p}{a}x}.$$

$$\xrightarrow[\text{性质 (3)}]{(3)L^{-1}T} L^{-1}[\tfrac{\bar{f}}{p}] = H(t)\int_0^t f(\tau)d\tau \xrightarrow[(4)]{\text{性质}} u = -aL^{-1}[\tfrac{\bar{f}}{p} e^{-\frac{x}{a}p}] = -aH(t-\tfrac{x}{a})\int_0^{t-\frac{x}{a}} f(\tau)d\tau.$$

* 例 3
空间有界
时间半无界

$$\begin{cases} u_{tt} = a^2 u_{xx}, \\ u|_{x=0} = 0, \\ u_x|_{x=l} = A\sin\omega t, \\ u|_{t=0} = u_t|_{t=0} = 0. \end{cases}$$

pde
bc 非齐，分离变量法非最佳
ic 齐，优选 LT

$$\xrightarrow[L[u]=\bar{u}]{(1)LT} \begin{cases} p^2\bar{u} = a^2 \bar{u}_{xx}, \\ \bar{u}|_{x=0} = 0, \\ \bar{u}_x|_{x=l} = \frac{A\omega}{p^2+\omega^2}. \end{cases}$$

$$\xrightarrow{(2)\text{解像}} \bar{u} = C\text{ch}\tfrac{p}{a}x + D\text{sh}\tfrac{p}{a}x \xrightarrow[BC_l:D]{BC_0:C=0} \text{定解 } \bar{u} = \frac{aA\omega}{p(p+i\omega)(p-i\omega)\text{ch}\frac{l}{a}p}\text{sh}\tfrac{x}{a}p.$$

$$\xrightarrow[\text{inverse formula}]{(3)L^{-1}T \text{ Mellin}} u(t,x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \bar{u}(p,x)e^{pt}dp \xrightarrow[\text{residue}]{} \sum \text{Res}[\bar{u}(p,x)e^{pt}]$$ 虚轴和左半复平面孤立奇点

$$(\underset{p=i\omega}{Res} + \underset{p=-i\omega}{Res})[\bar{u}(p,x)e^{pt}] = 2Re\underset{p=i\omega}{Res}[\bar{u}e^{pt}] = 2Re\lim_{p\to i\omega}(p-i\omega)\bar{u}e^{pt} = Re\frac{2aA\omega}{-2\omega^2\cos\frac{l}{a}\omega} i\sin\tfrac{x}{a}\omega e^{i\omega t}$$

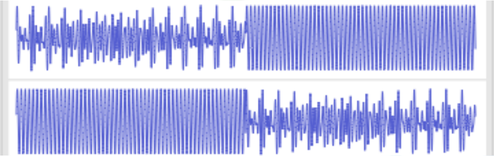
$$ch\tfrac{l}{a}i\omega_k = \cos\tfrac{l}{a}\omega_k = 0 \to \omega_k = \frac{2k-1}{2}\pi\frac{a}{l},\ (\underset{p=i\omega_k}{Res} + \underset{p=-i\omega_k}{Res})[\bar{u}e^{pt}] = 2Re\underset{p=i\omega_k}{Res}\frac{P}{Q} = 2Re\frac{P}{Q'}\Big|_{p=i\omega_k}$$

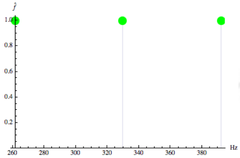
$$= 2Re\frac{aA\omega sh\frac{x}{a}pe^{pt}/[p(p+i\omega)(p-i\omega)]}{\frac{l}{a}sh\frac{l}{a}p}\Big|_{p=i\omega_k}\cdots$$

上述奇点都是单极点；$\bar{u} \sim shp/p \xrightarrow[p\to 0]{} 1 \Rightarrow p=0$ 是可去奇点.

物理意义和积分核的进化

分离变量法 (&FT) 时域 $\xrightarrow{\text{分拆驻波}}$ ♪ 琴频图

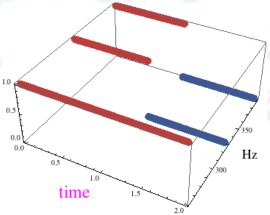




频域看此存在 (C 和弦 &C4) 非常简单.
这正是 Fourier 级数、积分变换
等数理思想的物理意义所在.

问题在于: 不分辨各频率组份的出现时间,
导致左边两种情况的频谱一样, 如上图.

需要既繁杂中反映简明频率、
又反映时间信息的时频图!



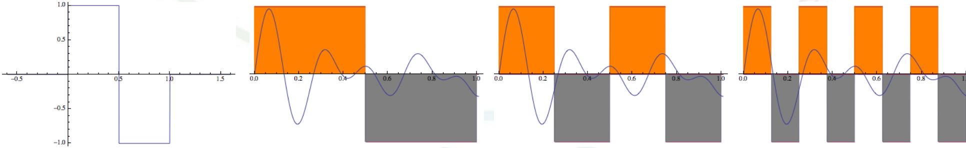
如何实现? 加窗.

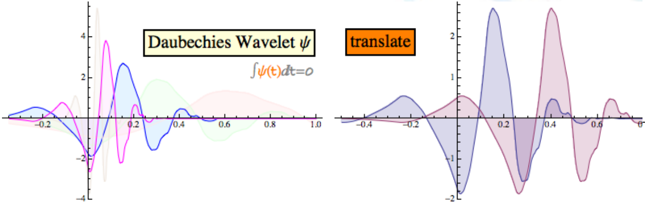
Wavelet - 小波分析

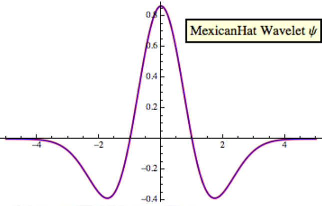
加窗的 Gabor 和新兴的弹性窗 wavelet

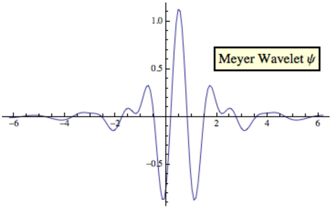
- Short Time FT: $\mathtt{F}(\tau,\omega) = \mathtt{STFT}[\mathtt{f}(\mathtt{t})] = \int_{-\infty}^{\infty} \mathtt{f}(\mathtt{t})\mathtt{G}(\mathtt{t}-\tau)\mathtt{e}^{-\mathtt{i}\omega\mathtt{t}}\mathtt{dt}$
  $G(t-\tau)$ 指把时间分段的 *Gábor* 窗函数 (1946), 种类很多, 例如: 矩形窗; 最优的 Gaussian 窗 $\frac{1}{\sqrt{\pi A}}e^{-\frac{t^2}{4A}}$(*Garbor* 变换); Hann 窗 $\frac{1}{2}[1-\cos\left(\frac{\pi t}{a}\right)]$.
- STFT/GT 是时频分析, 同时获得频率和时间信息. 把整体 FT 局域化.
- 但仍处理不好突变的非平稳信号. 周期反比于频率, 高频部分需窄的时间窗口以提高时轴分辨率, 低频需较宽时间窗确定频率; 而窗口宽度选定后, 仅有平移操作, 是主要缺陷. 另 Gabor 变换的基非正交系, 冗余.
- 需要弹性窗!
- 法国石油公司的 Morlet 提出 wavelet 小波的概念, 并于 1980s 开发连续小波变换用于地质数据处理. 它不只平移 shift b, 还能伸缩 scale a!
  $\mathtt{WT}_{\mathtt{a,b}}[\mathtt{f}(\mathtt{t})] = \int_{-\infty}^{\infty} \mathtt{f}(\mathtt{t})\frac{1}{\sqrt{\mathtt{a}}}\overline{\psi}\left(\frac{\mathtt{t}-\mathtt{b}}{\mathtt{a}}\right)\mathtt{dt}$, $\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$.
- 各种小波 ψ 迅速被构造. *Meyer* 等构造出一种光滑小波, 2^n 倍平移和缩放操作 ψ 及其*scaling*函数 φ 生成正交基. *1988 Mallat*算法统一早前构造小波基的方法, *Daubechies*揭示小波变换和滤波器的联系. 小波变换在声音图象等信号处理领域得迅速广泛应用, 例如降噪, *jpeg2000*的压缩算法, 基于 *Haar* 离散小波的人脸识别, *SpaceX CFD* 小波做湍流分形.

- Haar (1909)　　自适应的数学显微镜: translating 平移 & scaling 缩放分辨率







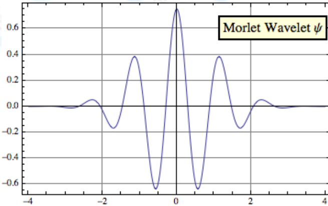


分辨率四

9	5
2	4

分辨率二 $\begin{array}{|c|}\hline 7 \\ \hline 3 \\ \hline\end{array} \pm \begin{array}{c} 2 \\ \leftrightarrow \\ -1 \end{array}$

分辨率一 $\boxed{5} \updownarrow \pm 2 \pm \overset{2}{\underset{-1}{\leftrightarrow}}$



- **Wavelet Trans** 自适应的数学显微镜: translating 平移 & scaling 缩放分辨率

分辨率一 5 $\updownarrow \pm 2 \pm \overset{2}{\underset{-1}{\leftrightarrow}}$

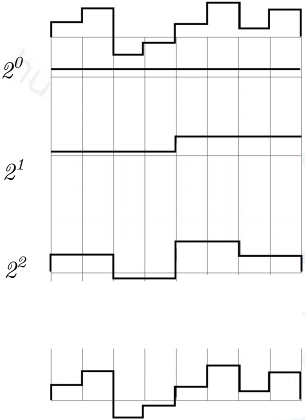
分辨率二

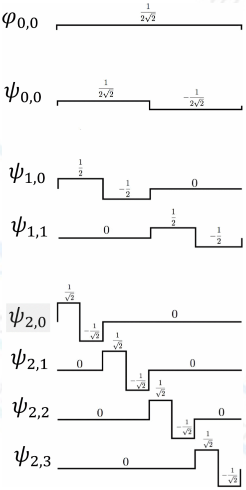
7
3

$\pm \overset{2}{\underset{-1}{\leftrightarrow}}$

分辨率四

9	5
2	4





$$f = \sum_k c_{L_0,k}\varphi_{L_0,k} + \sum_{lev \geq L_0} \sum_k d_{lev,k}\psi_{lev,k}$$