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Hamiltonian Formalism of de-Sitter Invariant Special Relativity*

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Abstract *The Lagrangian of Einstein's special relativity with universal parameter c ($\mathcal{SR}_c$) is invariant under Poincaré transformation, which preserves Lorentz metric $\eta_{\mu\nu}$. The $\mathcal{SR}_c$ has been extended to be one which is invariant under de Sitter transformation that preserves so-called Beltrami metric $B_{\mu\nu}$. There are two universal parameters, c and R, in this Special Relativity (denoted as $\mathcal{SR}_{cR}$). The Lagrangian-Hamiltonian formulism of $\mathcal{SR}_{cR}$ is formulated in this paper. The canonic energy, canonic momenta, and 10 Noether charges corresponding to the space-time's de Sitter symmetry are derived. The canonical quantization of the mechanics for $\mathcal{SR}_{cR}$-free particle is performed. The physics related to it is discussed.*

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1 Introduction

Einstein's special relativity is the cornerstone of physics. The theory indicates that the space-time metric is $\eta_{\mu\nu} = \mathrm{diag}\,\{+,-,-,-\}$. The most general transformation to preserve metric $\eta_{\mu\nu}$ is Poincaré group (or inhomogeneous Lorentz group ISO(1,3)). It is well known that the Poincaré group is the limit of the de Sitter group with sphere radius $R \to \infty$. Thus a natural question arising from this fact is whether there exists or not another type of de Sitter transformation with $R \to$ finite that also leads to a special relativity theory. In 1970's, K.H. Look (Qi-Keng Lu) and his collaborators pursued this question and obtained a positive answer.[1,2] In the recent years, some interesting studies on Lu's theory in Refs. [3] and [4] were stimulated by the recent observations which show that there should be a positive cosmological constant.[5,6] In Refs. [3] and [4], the length parameter R in Lu's theory has been identified as $\sqrt{3/\Lambda}$, where Λ is the cosmological constant. In the present paper, we try to study and reexamine Lu's theory in Lagrangian–Hamiltonian formalism. Lu's theory will be called as the de-Sitter Invariant Special Relativity hereafter.

Inertial motion law for free particles is the foundation of mechanics. This law states that in the inertial reference frames the free particle (i.e., without any force acting on it) will move along straight line and with constant coordinate velocities. The Newtonian mechanics is the first mechanical theory built on this foundation and without any universal parameters. The Lagrangian for free particle is

$$L_{\mathrm{Newton}} = \frac{1}{2} m_0 v^2\,, \tag{1}$$

where m_0 is the mass of the particle, $\boldsymbol{v} = \dot{\boldsymbol{x}}$ is the velocity, and $v^2 = \boldsymbol{v}^2$. We may regard it as a parameter-free realization of the inertial motion law. The second mechanic theory realizing this inertial motion law is the Einstein's Special Relativity with one universal parameter c (the velocity of light). Denoting it as $\mathcal{SR}_c$, the Lagrangian of free particle is (e.g., see Ref. [7])

$$L_c = -m_0 c \frac{\mathrm{d}s}{\mathrm{d}t} = -m_0 c \frac{\sqrt{\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}}{\mathrm{d}t} = -m_0 c^2 \sqrt{1 + \frac{\eta_{ij}\dot{x}^i\dot{x}^j}{c^2}} = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}}\,, \tag{2}$$

where Lorentz metric $\eta_{\mu\nu} = \mathrm{diag}\,\{+,-,-,-\}$, $\mathrm{d}x^\mu = \{\mathrm{d}(ct), \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3\}$ and $i, j = 1, 2, 3$. By means of the Lagrange–Hamilton mechanics formulation, the particle's momentum and Hamiltonian reads

$$p_i = \frac{\partial L_c}{\partial \dot{x}^i} = \frac{-m_0 \dot{x}^j \eta_{ij}}{\sqrt{1 - (v^2/c^2)}}\,, \tag{3}$$

$$H = p_i \dot{x}^i - L_c = c\sqrt{-\eta^{ij} p_i p_j + m_0^2 c^2}\,. \tag{4}$$

It is easy to check that when $c \to \infty$ the Special Relativity goes back to the Newtonian mechanics, i.e.,

$$L_c|_{c\to\infty} = L_{\mathrm{Newton}} + \mathrm{constant}\,. \tag{5}$$

An interesting and challenging question is whether a mechanical realization of the inertial motion law with *two universal parameters* can be formulated or not. Surprisingly, the answer to it is confirmative, and actually such a theory has already existed in literature even though it is still less known so far. About thirty five years ago, K.H. Look (Qi-Keng Lu) found out that the velocity of motion of the free particle along the geodesic line in the de Sitter (dS)-space with Beltrami metric is constant, and the geodesic is straight line.[1,2] This theory is just the de Sitter invariant special relativity mentioned above. In Lu's

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theory, there are two universal parameters: the light velocity c and the de Sitter sphere radius R (or original notation $\lambda = 1/R^2$ used in Refs. [1] and [2]. The coordinate-transformation to preserve the Beltrami metric has also been derived in Refs. [1] and [2]. This means that the realization of the inertial motion law with *two universal parameters* has been formulated. The theory will be shortly denoted as $\mathcal{SR}_{cR}$ due to the existence of two universal parameters c and R in the theory. In the present paper, we try to provide a Lagrangian–Hamiltonian formulation to illustrate the free-particle mechanics in the de-Sitter invariant special relativity.

It is well known that the Lagrangian–Hamiltonian formulation in the mechanics theory provides a sound foundation to discuss the particle's motion, to deduce the particle's canonical (or conjugate) momenta and the canonical energy (or Hamiltonian), to derive the Noether's charges corresponding to the symmetries, and to over the classical mechanics for constructing the quantum mechanics, and so on. In the previous works on $\mathcal{SR}_{cR}$,[1–4] the free-particle-motion in the space-time with Beltrami metric was discussed by means of solving the geodesic equation, and it has been found that the velocity of the particle is a constant. This remarkable claim should be reconfirmed in Lagrangian–Hamiltonian formulation. Especially, because any reliable quantization procedures of a classical mechanics theory rely upon the theory's Lagrangian–Hamiltonian formulation, it is a basic task to determine the system's canonical momenta and the Hamiltonian. To $\mathcal{SR}_{cR}$, the particle's canonical momenta and Hamiltonian are unusual and somewhat subtle, which have to be derived. Furthermore, the Noether's charges in $\mathcal{SR}_{cR}$, which are the quantities in physics, should also be derived in this formulation. For all these purposes, a systematic and careful study on the Lagrangian–Hamiltonian formulation for $\mathcal{SR}_{cR}$ is necessary.

Equation (2) shows the Lagrangian of free particle in $\mathcal{SR}_c$ (i.e., ordinary Einstein's special relativity) is time- and coordinate-independent (or x^i are cyclic coordinates). So, both Hamiltonian and canonical momenta are motion of constants. Furthermore, the most general space-time transformation preserving $\eta_{\mu\nu}$ in $\mathcal{SR}_c$ is simply the Poincaré transformation group (or inhomogeneous Lorentz group) ISO(1,3). Therefore conserved Noether charges are just its Hamiltonian, canonic momenta, the angular momenta (and plus three Lorentz boost charges). All of these are well known. To $\mathcal{SR}_{cR}$, however, the situation is much more complicated than in $\mathcal{SR}_c$. Because the Beltrami metric is time- and coordinate-dependent, we face a mechanical system with time-dependent Hamiltonian and without any cyclic coordinates. The space-time transformation preserving Beltrami metric is a sort of de-Sitter transformation. In this case, a careful enough revisiting to the classical mechanics with time- and coordinate-dependent Lagrangian is necessary for getting convincible conclusions. It will be found out that the Hamiltonian (or canonical energy), canonical momenta are different from the conserved Noether charges corresponding to the external space-time symmetry of $\mathcal{SR}_{cR}$. The latter are energy and momenta in physics, and the former are the canonical quantities which are also useful for mechanics, especially for the quantization of the system.

Following $\mathcal{SR}_c$, in the framework of $\mathcal{SR}_{cR}$, the wave equation of relativistical quantum mechanics is derived in this paper by means of the standard canonic quantization procedure: i) The Hamiltonian mechanics leads to quantum canonic equations, then Hamiltonian operator $\hat{H}$ and canonical momentum operators $\hat{\pi}_i$ are defined; ii) By the mechanics again, the dispersion relation between $\hat{H}$ and $\hat{\pi}_i$ is obtained, and hence we achieve the wave equation for the $\mathcal{SR}_{cR}$ quantum mechanics. Due to existence of $x^i\hat{\pi}_i$-terms in the time-dependent Hamiltonian $\hat{H}$, the operator ordering has to be taken care. In our quantization scheme a generalized Weyl ordering is taken, in which the external space-time symmetry of $\mathcal{SR}_{cR}$ is preserved. This indicates that $\mathcal{SR}_{cR}$ is consistent with the principle of quantum mechanics.

The contents of this paper are organized as follows: In Sec. 2, we show explicitly that the Euler–Lagrangian equations are equivalent to the geodesic equations for generic metric $g_{\mu\nu}$. In Sec. 3, we construct the Lagrangian for $\mathcal{SR}_{cR}$ by means of the Beltrami metrics, and solve the equation of motions of free particle. The Hamiltonian and the canonical momenta are also derived. Section 4 is devoted to calculating the Noether charges corresponding to external space-time symmetry of $\mathcal{SR}_{cR}$. In Sec. 5 we discuss the quantization of the system. Finally, we summarize our results briefly. In the Appendix, we show how to derive space-time transformation to preserve Beltrami metric following Refs. [1] and [2].

2 Equation of Motion for Free Particle in Space-Time with Metric $g_{\mu\nu}(x)$

The motion of a free material particle is determined in the special theories of relativity from the principle of least action,

$$\delta S \equiv \delta \int L(t, x^i, \dot{x}^i)\,\mathrm{d}t = -m_0 c\delta \int \mathrm{d}s = 0\,, \qquad (6)$$

where S is the action integral, $\mathrm{d}s = \sqrt{g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}$ is the space-time interval. $g_{\mu\nu} = \eta_{\mu\nu}$ for $\mathcal{SR}_c$, but for $\mathcal{SR}_{cR}$, $g_{\mu\nu}$ should be Beltrami metric. Generally, from Eq. (6), we have

$$L(t, x^\mu, \dot{x}^\mu) = -m_0 c\frac{\mathrm{d}s}{\mathrm{d}t} = -m_0 c\sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}\,. \qquad (7)$$

By variation of the action with respect to x^μ we get a four-dimensional Euler–Lagrangian equation, where variables x^0 and $\dot{x}^0$ emerge as independent variables,

$$\frac{\partial L}{\partial x^\lambda} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^\lambda} = 0\,, \qquad (8)$$

where t serves as a parameter rather than the physical coordinate time, $\dot{x}^\lambda = \mathrm{d}x^\lambda/\mathrm{d}t$, and λ runs over all the space-time indices including $\lambda = 0$. Obviously, they are equations of motion, but not the standard Euler–Lagrangian equations in the Hamiltonian formalism of mechanics because here t is independent of x^0. At this stage, therefore, we cannot derive the canonical momentum and Hamiltonian by means of L, x^i and $\dot{x}^i$. If we choose the parameter t such that

$$\mathrm{d}s = c\mathrm{d}t\,, \tag{9}$$

and substitute Eqs. (7) into Eqs. (8), we get the standard geodesic equation,

$$g_{\lambda\mu}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \Gamma_{\lambda,\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0\,, \tag{10}$$

where

$$\Gamma_{\lambda,\mu\nu} = g_{\lambda\rho}\Gamma^\rho_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})\,. \tag{11}$$

In order to derive the equations of motion in the Hamiltonian framework, we have to fix the parameter

$$t = \frac{x^0}{c} \quad \text{or} \quad x^0 = ct\,, \tag{12}$$

and write

$$S = \int L(x^i, \dot{x}^i, t)\mathrm{d}t\,. \tag{13}$$

From Eq. (7), it is obvious that

$$L(x^i, \dot{x}^i, t) = -m_0c^2\sqrt{g_{00}(x^i,t) + 2g_{0j}(x^i,t)\frac{1}{c}\dot{x}^j + g_{jk}(x^i,t)\frac{1}{c^2}\dot{x}^j\dot{x}^k}\,, \tag{14}$$

where i only runs over the space indices. Then, by variation of the action with respect to both x^i and t, we have the desired Euler–Lagrangian equations as follows:

$$\frac{\partial L}{\partial x^i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^i}\,, \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[L - \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i\Big] = \frac{\partial L}{\partial t}\,. \tag{16}$$

These equations are the equations of motion in the time-dependent Lagrangian–Hamiltonian framework, and the corresponding Lagrangian can be used to deduce the momentum and energies of the system. It is easy to check that under $x^0 = ct$, equations (15) and (16) are consistent with four-dimensional Euler–Lagrangian equation (8) (or the geodesic equation (10)).

The equivalence of the two sets of equations comes from the fact that the original action has a reparametrization symmetry of t and so the space and time coordinates are mixed together. That is to say,

$$L = \frac{\partial L}{\partial \dot{x}^\lambda}\dot{x}^\lambda\,,$$

$$\lambda \text{ runs over all the space and time indices}\,, \tag{17}$$

which states that L is homogeneous of degree 1 as a function of $\dot{x}^\lambda$. It is a special property of Eq. (7) but also a general requirement for the action to have parametrization symmetry of t before x^0 is set to be ct. From this we know that the above discussion for the equivalence of two sets of Euler–Lagrangian equations does not apply to the general Hamiltonian system but a special nice relation for the free particle moving in the space and time described by theories of special relativity.

3 Lagrangian, Canonic Momentum and Hamiltonian of Free Particle in de-Sitter Invariant Special Relativity

According to the discussions in previous sections, similar to L_c (see Eq. (3)), the Lagrangian for free particle in $\mathcal{SR}_{cR}$ is

$$L_{cR} = -m_0c\frac{\mathrm{d}s}{\mathrm{d}t} = -m_0c\frac{\sqrt{B_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu}}{\mathrm{d}t}$$

$$= -m_0c\sqrt{B_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}\,, \tag{18}$$

where $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}t$, $B_{\mu\nu}(x)$ is Beltrami metric,[1–4]

$$B_{\mu\nu}(x) = \frac{\eta_{\mu\nu}}{\sigma(x)} + \frac{1}{R^2\sigma(x)^2}\eta_{\mu\lambda}\eta_{\nu\rho}x^\lambda x^\rho$$

$$\text{with}\ \ \sigma(x) \equiv 1 - \frac{1}{R^2}\eta_{\mu\nu}x^\mu x^\nu\,, \tag{19}$$

where the constant R is the radius of the pseudo-sphere in dS-space, and it can be related to cosmological constant via $R = \sqrt{3/\Lambda}$.[3,4] Setting up the time $t = x^0/c$, $B_{\mu\nu}(x)$ can be rewritten as follows:

$$\begin{aligned}\mathrm{d}s^2 &= B_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu \\ &= \tilde{g}_{00}\,\mathrm{d}(ct)^2 + \tilde{g}_{ij}[(\mathrm{d}x^i + N^i\,\mathrm{d}(ct))(\mathrm{d}x^j + N^j\,\mathrm{d}(ct))] \\ &= c^2(\mathrm{d}t)^2\Big[\tilde{g}_{00} + \tilde{g}_{ij}\Big(\frac{1}{c}\dot{x}^i + N^i\Big)\Big(\frac{1}{c}\dot{x}^j + N^j\Big)\Big]\,,\end{aligned} \tag{20}$$

where

$$\tilde{g}_{00} = \frac{R^2}{\sigma(x)(R^2 - c^2t^2)}\,, \tag{21}$$

$$\tilde{g}_{ij} = \frac{\eta_{ij}}{\sigma(x)} + \frac{1}{R^2\sigma(x)^2}\eta_{il}\eta_{jm}x^lx^m\,, \tag{22}$$

$$N^i = \frac{ctx^i}{R^2 - c^2t^2}\,. \tag{23}$$

Substituting Eqs. (19) $\sim$ (23) into Eq. (18), we obtain the Lagrangian for free particle in $\mathcal{SR}_{cR}$,

$$L_{cR} = -m_0c^2\sqrt{\tilde{g}_{00} + \tilde{g}_{ij}\Big(\frac{1}{c}\dot{x}^i + N^i\Big)\Big(\frac{1}{c}\dot{x}^j + N^j\Big)}\,. \tag{24}$$

By means of the explicit expressions of Eqs. (21) $\sim$ (24) and doing straightforward calculations, we can prove the following equation:

$$\frac{\partial L_{cR}}{\partial x^i} = \frac{\partial^2 L_{cR}}{\partial t\partial \dot{x}^i} + \frac{\partial^2 L_{cR}}{\partial x^j\partial \dot{x}^i}\dot{x}^j\,. \tag{25}$$

Substituting Eq. (24) into the Euler–Lagrangian equation (15) and using identity (25), we have

$$\frac{\partial^2 L_{cR}}{\partial \dot{x}^i\partial \dot{x}^j}\ddot{x}^j = -\frac{m_0^4c^6R^4}{L_{cR}^3R^6c^2\sigma^3(x)}M_{ij}\ddot{x}^j = 0\,, \tag{26}$$

where M_{ij} is a matrix that satisfies $\det[M_{ij}] \neq 0$. We conclude

$$\ddot{x}^j = 0\,, \quad \dot{x}^j = \text{constant}\,. \tag{27}$$

This result indicates that the free particle in the Beltrami space-time $\mathcal{B} \equiv \{x^\mu, g_{\mu\nu}(x) = B_{\mu\nu}(x)\}$ moves along straight line and with constant coordinate velocities. Namely the inertial motion law for free particles holds true in the space-time $\mathcal{B}$, and hence the inertial reference frame can be set in $\mathcal{B}$. Thus, by means of solving Euler–Lagrangian equations in the Lagrangian–Hamiltonian formulation, we have reconfirmed the claim in Refs. [1] and [2] on the velocity of motion of free-particles based on solving geodesic equation originally.

As an essential advantage in the Lagrangian–Hamiltonian formulation over other formulism, both canonical momentum π_i conjugating to the Beltrami-coordinate x^i and canonical energy H_{cR} (or Hamiltonian) conjugating to the Beltrami-time t for free particles in the inertial reference frame can be determined rationally by the mechanism principle. By Eq. (24), the canonical momentum and the canonical energy (or Hamiltonian) reads

$$\pi_i = \frac{\partial L_{cR}}{\partial \dot{x}^i} = -m_0\sigma(x)\Gamma B_{i\mu}\dot{x}^\mu\,, \tag{28}$$

$$H_{cR} = \sum_{i=1}^{3}\frac{\partial L_{cR}}{\partial \dot{x}^i}\dot{x}^i - L_{cR} = m_0 c\sigma(x)\Gamma B_{0\mu}\dot{x}^\mu\,, \tag{29}$$

where

$$\Gamma^{-1} = \sigma(x)\frac{\mathrm{d}s}{c\mathrm{d}t} = \frac{1}{R}\sqrt{(R^2 - \eta_{ij}x^ix^j)\Big(1 + \frac{\eta_{ij}\dot{x}^i\dot{x}^j}{c^2}\Big) + 2t\eta_{ij}x^i\dot{x}^j - \eta_{ij}\dot{x}^i\dot{x}^jt^2 + \frac{(\eta_{ij}x^i\dot{x}^j)^2}{c^2}}\,. \tag{30}$$

Under the motion equation (27), we have the following relation,

$$\dot{\Gamma}|_{\ddot{x}^i=0} = 0\,, \tag{31}$$

whose corresponding one in $\mathcal{SR}_c$ is

$$\dot{\gamma}|_{\ddot{x}^i=0} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\sqrt{1-v^2/c^2}}\Big)\Big|_{v=\text{constant}} = 0\,. \tag{32}$$

It is easy to check that

$$\lim_{R\to\infty}\Gamma = \lim_{x^i\to 0}\Gamma = \gamma \equiv \frac{1}{\sqrt{1-(v^2/c^2)}}\,. \tag{33}$$

And, in the $R\to\infty$ limit, π_i and H_{cR} go back to the standard Einstein Special Relativity's expressions,

$$\pi_i|_{R\to\infty} = \frac{m_0 v_i}{\sqrt{1-(v^2/c^2)}}\,,$$

$$H_{cR}|_{R\to\infty} = \frac{m_0c^2}{\sqrt{1-(v^2/c^2)}}\,, \tag{34}$$

where $v_i = -\eta_{ij}\dot{x}^j$. Furthermore, at the original point of space-time coordinates $t = x^i = 0$, but R = finite, we have also expressions like Eq. (34),

$$\pi_i|_{t=x^i=0} = \frac{m_0 v_i}{\sqrt{1-(v^2/c^2)}}\,,$$

$$H_{cR}|_{t=x^i=0} = \frac{m_0c^2}{\sqrt{1-(v^2/c^2)}}\,. \tag{35}$$

In Table 1, we list some results of Lagrange formalism both in the ordinary special relativity $\mathcal{SR}_c$ and in the de Sitter invariant special relativity $\mathcal{SR}_{cR}$. Comparing the results in $\mathcal{SR}_{cR}$ with ones in well-known $\mathcal{SR}_c$, we learn that as an extending theory of $\mathcal{SR}_c$, $\mathcal{SR}_{cR}$ can simply be formulated by a variable alternating in $\mathcal{SR}_c$: i) $\eta_{\mu\nu} \Rightarrow B_{\mu\nu}$; ii) $\gamma \Rightarrow \sigma\Gamma$. This is a natural and nice feature for the Lagrangian formalism of $\mathcal{SR}_{cR}$.

Table 1 Metric, Lagrangian, equation of motions, canonic momenta, and Hamiltonian in the special relativity, $\mathcal{SR}_c$, and in the de Sitter special relativity, $\mathcal{SR}_{cR}$. The quantities $\gamma^{-1} = \sqrt{1+(\eta_{ij}\dot{x}^i\dot{x}^j/c^2)}$ and $\Gamma^{-1} = R^{-1}\sqrt{(R^2-\eta_{ij}x^ix^j)[1+(\eta_{ij}\dot{x}^i\dot{x}^j/c^2)] + 2t\eta_{ij}x^i\dot{x}^j - \eta_{ij}\dot{x}^i\dot{x}^jt^2 + (\eta_{ij}x^i\dot{x}^j)^2/c^2}$ (see Eq. (30)).

	$\mathcal{SR}_c$	$\mathcal{SR}_{cR}$
Space-time metric	$\eta_{\mu\nu}$	$B_{\mu\nu}(x)$, (Eq. (19))
Lagrangian	$L_c = -m_0c^2\gamma^{-1}$	$L_{cR} = -m_0c^2\sigma^{-1}\Gamma^{-1}$
Equation of motion	$v^i = \dot{x}^i$ = constant, (or $\dot{\gamma} = 0$)	$v^i = \dot{x}^i$ = constant, (or $\dot{\Gamma} = 0$)
Canonic momenta	$\pi_i = -m_0\gamma\eta_{i\mu}\dot{x}^\mu$	$\pi_i = -m_0\sigma\Gamma B_{i\mu}\dot{x}^\mu$
Hamiltonian	$H_c = m_0c\gamma\eta_{0\mu}\dot{x}^\mu$	$H_{cR} = m_0c\sigma\Gamma B_{0\mu}\dot{x}^\mu$

Combining Eq. (28) with Eq. (29), the covariant four-momentum in $\mathcal{B}$ is

$$\begin{aligned}\pi_\mu \equiv (\pi_0, \pi_i) = \Big(-\frac{H_{cR}}{c}, \pi_i\Big) &= -m_0\sigma\Gamma B_{\mu\nu}\dot{x}^\nu \\ &= -m_0cB_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\,,\end{aligned} \tag{36}$$

and

$$B^{\mu\nu}\pi_\mu\pi_\nu = m_0^2c^2\,. \tag{37}$$

From Eqs. (24), (28), (29), and (37), we have

$$H_{cR} = \sqrt{\tilde{g}_{00}}\sqrt{m_0^2c^4 - c^2\tilde{g}^{ij}\pi_i\pi_j} - c\pi_iN^i\,, \tag{38}$$

where $\tilde{g}_{00}$ and N^i have been shown in Eqs. (21) and (23), and $\tilde{g}^{ij} = \sigma(x)[\eta^{ij} - x^i x^j/(R^2 - c^2t^2)]$ from Eq. (22). It is straightforward to get the following canonical equations:

$$\dot{x}^i = \frac{\partial H_{cR}}{\partial \pi_i} = \{H_{cR}, x^i\}_{\rm PB}\,,$$
$$\dot{\pi}_i = -\frac{\partial H_{cR}}{\partial x^i} = \{H_{cR}, \pi_i\}_{\rm PB}\,, \tag{39}$$

where the Poisson brackets

$$\{x^i, \pi_j\}_{\rm PB} = \delta^i_j\,, \quad \{x^i, x^j\}_{\rm PB} = 0\,,$$
$$\{\pi_i, \pi_j\}_{\rm PB} = 0\,, \tag{40}$$

are as usual. It is also straightforward to check $\dot{x}^i =$ constant by Eq. (39).

Finally, we would like to mention that generally, the canonical momenta π_i and the Hamiltonian H_{cR} are not the physical momentum and the energy of the particle respectively.

4 Space-Time Symmetry of de-Sitter Invariant Special Relativity and Noether Charges

The space-time transformations preserving the Beltrami metric were discovered about 30 years ago by Lu, Zou, and Guo (LZG)[1,2] (see also Appendix). When we transform from one initial Beltrami frame x^μ to another Beltrami frame $\tilde{x}^\mu$, and when the origin of the new frame is a^μ in the original frame, the transformations between them with ten parameters are as follows:

$$x^\mu \xrightarrow{\rm LZG} \tilde{x}^\mu = \pm\sigma(a)^{1/2}\sigma(a,x)^{-1}(x^\nu - a^\nu)D^\mu_\nu\,, \quad D^\mu_\nu = L^\mu_\nu + R^{-2}\eta_{\nu\rho}a^\rho a^\lambda(\sigma(a) + \sigma^{1/2}(a))^{-1}L^\mu_\lambda\,,$$
$$L := (L^\mu_\nu) \in {\rm SO}(1,3)\,, \quad \sigma(x) = 1 - \frac{1}{R^2}\eta_{\mu\nu}x^\mu x^\nu\,, \quad \sigma(a,x) = 1 - \frac{1}{R^2}\eta_{\mu\nu}a^\mu x^\nu\,. \tag{41}$$

It will be called as LZG-transformation hereafter, and we prove it in the Appendix by means of the method in Ref. [2]. Under LZG-transformation, the $B_{\mu\nu}(x)$ and the action of $\mathcal{SR}_{cR}$ transfer respectively as follows:

$$B_{\mu\nu}(x) \xrightarrow{\rm LZG} \tilde{B}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\lambda}{\partial \tilde{x}^\mu}\frac{\partial x^\rho}{\partial \tilde{x}^\nu}B_{\lambda\rho}(x) = B_{\mu\nu}(\tilde{x})\,, \tag{42}$$

$$S_{cR} \equiv \int {\rm d}t L_{cR}(t) = -m_0 c\int {\rm d}t\frac{\sqrt{B_{\mu\nu}(x){\rm d}x^\mu {\rm d}x^\nu}}{{\rm d}t} \xrightarrow{\rm LZG} \tilde{S}_{cR} = S_{cR}\,. \tag{43}$$

By the mechanics principle, this action invariance indicates that there are ten conserved Noether charges in $\mathcal{SR}_{cR}$ like the $\mathcal{SR}_c$ case. For $\mathcal{SR}_c$ the Noether charges are (e.g., see pp. 581–586 and Part 9 in Ref. [8]):

$$\text{Noether charges for Lorentz boost}: \quad K^i_c = m_0\gamma c(x^i - t\dot{x}^i)$$
$$\text{Charges for space-transitions (momenta)}: \quad P^i_c = m_0\gamma\dot{x}^i\,,$$
$$\text{Charge for time-transition (energy)}: \quad E_c = m_0c^2\gamma$$
$$\text{Charges for rotations in space (angular momenta)}: \quad L^i_c = \epsilon^i_{jk}x^jP^k\,. \tag{44}$$

Here $\gamma = 1/\sqrt{1-(v^2/c^2)}$. Note the Noether charges here are the same as the corresponding canonical quantities, because the Lagrangian for $\mathcal{SR}_c$ is time-independent and all the coordinates are cyclic, while in $\mathcal{SR}_{cR}$ there are no cyclic coordinates and the Lagrangian is space-time dependent.

When space rotations are neglected temporarily for simplify, the LZG-transformation both due to a Lorentz-like boost and a space-transition in the x^1 direction with parameters $\beta = \dot{x}^1/c$ and a^1 respectively and due to a time transition with parameter a^0 can be explicitly written as follows:

$$t \to \tilde{t} = \frac{\sqrt{\sigma(a)}}{c\sigma(a,x)}\gamma\Big[ct - \beta x^1 - a^0 + \beta a^1 + \frac{a^0 - \beta a^1}{R^2}\frac{a^0ct - a^1x^1 - (a^0)^2 + (a^1)^2}{\sigma(a) + \sqrt{\sigma(a)}}\Big]\,,$$
$$x^1 \to \tilde{x}^1 = \frac{\sqrt{\sigma(a)}}{\sigma(a,x)}\gamma\Big[x^1 - \beta ct + \beta a^0 - a^1 + \frac{a^1 - \beta a^0}{R^2}\frac{a^0ct - a^1x^1 - (a^0)^2 + (a^1)^2}{\sigma(a) + \sqrt{\sigma(a)}}\Big]\,,$$
$$x^2 \to \tilde{x}^2 = \frac{\sqrt{\sigma(a)}}{\sigma(a,x)}x^2\,, \quad x^3 \to \tilde{x}^3 = \frac{\sqrt{\sigma(a)}}{\sigma(a,x)}x^3\,. \tag{45}$$

It is easy to check when $R \to \infty$ the above transformation goes back to Poincaré transformation. Notice that in the LZG-transformation there are three boost parameters $\beta^i = \dot{x}^i/c = v^i/c$, four spacetime transition parameters (a^0, a^1, a^2, a^3) and three rotation parameters $\theta^i = 0$. Here (a^0, a^1, a^2, a^3) is the origin of the resulting Beltrami initial frame in the original Beltrami frame.

In terms of the standard procedure (e.g., see Ref. [8] pp. 581–586), the Noether charges corresponding to the LZG transformation (Eq. (41)) invariance can be derived.

(i) Space transitions:

$$x^\mu \to \tilde{x}^\mu = x^\mu + \frac{\eta_{ij}a^ix^j}{R^2}x^\mu - a^i\delta^\mu_i\,, \quad G^i_a = -\frac{\pi_\mu x^\mu}{R^2}x^i + \eta^{ij}\pi_j = m_0\Gamma\dot{x}^i\,. \tag{46}$$

(ii) Time transition charge:

$$x^\mu \to \tilde{x}^\mu = x^\mu + \frac{ctx^\mu}{R^2}a^0 - a^0\delta^\mu_0\,, \quad G_{a^0} = x^0\frac{\pi_\mu x^\mu}{R^2} - \eta^{0\mu}\pi_\mu = m_0 c\Gamma\,. \tag{47}$$

(iii) Lorentz boost charges:

$$x^\mu \to \tilde{x}^\mu = (\gamma(ct-\beta x^1), \gamma(x^1-\beta ct), x^2, x^3)\,, \quad G^i_\beta = -x^i\pi_0 - x^0\pi_i = m_0 c\Gamma(x^i - t\dot{x}^i)\,. \tag{48}$$

(iv) Rotation charges:

$$x^\mu \to \tilde{x}^\mu = (ct, x^1+\theta x^2, x^2-\theta x^1, x^3)\,, \quad G^i_\omega = -m_0\Gamma\epsilon^i{}_{jk}x^j\dot{x}^k\,. \tag{49}$$

Some Remarks

(i) For a free particle that moves with a constant speed $\ddot{x}^i = 0$, we have already proved (see Eq. (31))

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma = 0\,. \tag{50}$$

By using Eq. (50), we can check that those charges derived above are indeed conservative:

$$\frac{\mathrm{d}}{\mathrm{d}t}G_\lambda|_{\lambda=a^0,a^i,\beta^i,\omega^i} = 0\,. \tag{51}$$

(ii) In the limit $R\to\infty$ the Noether charges in $\mathcal{SR}_{cR}$ are the same as those in $\mathcal{SR}_c$, see Eq. (44).

(iii) The mechanical (or physical) momenta and energy in the Lagrangian–Hamiltonian formalism are defined as the Noether charges corresponding to the space transitions, therefore the particle's momenta and energy in $\mathcal{SR}_{cR}$ read

$$p^i_{cR} \equiv G^i_a = m_0\Gamma\dot{x}^i\,, \tag{52}$$

$$E_{cR} \equiv cp^0_{cR} = cG_{a^0} = m_0c^2\Gamma\,, \tag{53}$$

which are conservative quantities. We address that distinguishing from the $\mathcal{SR}_c$, in $\mathcal{SR}_{cR}$ the physical momentum p^i of the particle is different from its canonical momentum π_i. The former is conservative and the latter is space-time-dependent. Combining Eq. (52) with Eq. (53), we have the four-momentum in $\mathcal{SR}_{cR}$ as follows:

$$p^\mu_{cR} \equiv \{p^0_{cR},\ p^i_{cR}\} = m_0\Gamma\dot{x}^\mu = \frac{m_0c}{\sigma(x)}\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = -\frac{1}{\sigma(x)}B^{\mu\nu}\pi_\nu\,, \tag{54}$$

which is consistent with energy-momentum definition in Ref. [2].

(iv) In general the boost Noether charges for $\mathcal{SR}_{cR}$ are

$$K^i_{cR} \equiv G^i_\beta = x^ip^0_{cR} + x^0p^i_{cR} = m_0c\Gamma(x^i - t\dot{x}^i)\,, \tag{55}$$

while the angular momentum is

$$L^i_{cR} \equiv G^i_\omega = -m_0\Gamma\epsilon^i{}_{jk}x^j\dot{x}^k\,. \tag{56}$$

From Eq. (54) we have the dispersion relation

$$\sigma^2(x)B_{\mu\nu}p^\mu_{cR}p^\nu_{cR} = m_0^2c^2\,. \tag{57}$$

And we can rewrite it using the Noether charges

$$E^2_{cR} = m_0^2c^2 + \boldsymbol{p}^2_{cR} + \frac{c^2}{R^2}(\boldsymbol{L}^2_{cR} - \boldsymbol{K}^2_{cR})\,. \tag{58}$$

Here E_{cR}, $\boldsymbol{p}_{cR}$, $\boldsymbol{L}_{cR}$, and $\boldsymbol{K}_{cR}$ are conserved physical energy, momentum, angular-momentum and boost charges respectively.

5 Quantum Mechanics for One Particle in $\mathcal{SR}_{cR}$

Lagrangian–Hamiltonian formulation of mechanics is the foundation of quantization. When the classical Poisson brackets in canonical equations for canonical coordinates and canonical momentum become operator's commutators, i.e., $\{x,\pi\}_{\mathrm{PB}} \Rightarrow (1/\mathrm{i}\hbar)[x,\hat{\pi}]$, the classical mechanics will be quantized. In this way, for instance, the ordinary relativistic (i.e., $\mathcal{SR}_c$) one-particle quantum equations have been derived. To the particle with spin-0, that is just the well-known Klein–Gordon equation. In the canonic quantization formalism for $\mathcal{SR}_{cR}$, the canonic variable operators are x^i, $\hat{\pi}_i$ with $i=1,2,3$, and due to Eq. (40) the basic commutators for the free particle quantization theory of $\mathcal{SR}_{cR}$ are the same as usual, i.e.,

$$[x^i,\hat{\pi}_j] = \mathrm{i}\hbar\delta^i_j\,, \quad [\hat{\pi}_i,\hat{\pi}_j] = 0\,, \quad [x_i,x_j] = 0\,. \tag{59}$$

The Hamiltonian operator $\hat{H}_{cR} \equiv -c\hat{\pi}_0$ represents the generator of time evolution, i.e.,

$$[t,\hat{H}_{cR}] = -\mathrm{i}\hbar\,, \quad \text{or} \quad [x^0,\hat{\pi}_0] = \mathrm{i}\hbar\,. \tag{60}$$

Since the time evolution is independent of the space coordinate displacements whose generators are $\hat{\pi}_i$, we always have

$$[\hat{H}_{cR},\hat{\pi}_i] = 0\,, \quad \text{or} \quad [\hat{\pi}_0,\hat{\pi}_i] = 0\,, \tag{61}$$

which is independent of the dynamics (or the dispersion relation).[9] Combining Eqs. (59), (60), and (61), we have (hereafter the hat notations for operators are removed)

$$[x^\mu,\pi_\nu] = \mathrm{i}\hbar\delta^\mu_\nu\,, \quad [x^\mu,x^\nu] = 0\,, \quad [\pi_\mu,\pi_\nu] = 0\,. \tag{62}$$

The general solution of Eq. (62) is

$$\pi_\mu = -\mathrm{i}\hbar\partial_\mu + (\partial_\mu G(t,x))\,, \tag{63}$$

where $G(t,x)$ is a function of t and x^i. Now, the dynamical Hamiltonian $H_{cR} \equiv -c\pi_0$ is (πx)-product term-dependent (see Eq. (38)), and the ordering of x^i and π^i has to be taken care of. Generally, the most symmetrical ordering (i.e., Weyl ordering) is favored for realistic quantization scheme. To $\mathcal{SR}_{cR}$, we prefer the quantization scheme that protects the de Sitter symmetry SO(1,4).

This requirement will lead to fix the function $G(t,x)$ in Eq. (63). By this consideration, we take[10,11]

$$\pi_\mu = -\mathrm{i}\hbar \acute{D}_\mu = -\mathrm{i}\hbar\Big(\partial_\mu + \frac{\Gamma_\mu}{2}\Big) = -\mathrm{i}\hbar B^{-1/4}\partial_\mu B^{1/4}\,, \quad (64)$$

where $B=\det(B_{\mu\nu})$, $\Gamma_\mu = \Gamma^\nu_{\mu\nu}$. Equation (64) indicates $G(t,x) = -\mathrm{i}\hbar\log(B^{1/4})$. In contrast with the ordinary quantization discussions to the theories in curved space only,[10,11] our treatment presented here is suitable for the theories in generic curved space-time, in which the four-dimensional metric is time- and space-dependent. The classical dispersion relation (37) can be rewritten as symmetric version $B^{-1/4}\pi_\mu B^{1/4}B^{\mu\nu}B^{1/4}\pi_\nu B^{-1/4} = m_0^2c^2$, and then the $\mathcal{SR}_{cR}$-one particle wave equation reads

$$B^{-1/4}\pi_\mu B^{1/4}B^{\mu\nu}B^{1/4}\pi_\nu B^{-1/4}\phi(x,t) = m_0^2c^2\phi(x,t)\,, \quad (65)$$

where $\phi(x,t)$ is the particle's wave function. Substituting Eq. (64) into Eq. (65), we have

$$\frac{1}{\sqrt{B}}\partial_\mu(B^{\mu\nu}\sqrt{B}\partial_\nu)\phi + \frac{m_0^2c^2}{\hbar^2}\phi = 0\,, \quad (66)$$

which is just the Klein–Gordon equation in curved space-time with Beltrami metric $B_{\mu\nu}$, and its explicit form is

$$\Big(\eta^{\mu\nu} - \frac{x^\mu x^\nu}{R^2}\Big)\partial_\mu\partial_\nu\phi - 2\frac{x^\mu}{R^2}\partial_\mu\phi + \frac{m_0^2c^2}{\hbar^2\sigma(x)}\phi = 0\,, \quad (67)$$

which is the desired $\mathcal{SR}_{cR}$-quantum mechanics equation for free particle.

Substituting Eq. (64) into Eq. (54), we obtain the physical momentum and energy operators (noting the subscripts cR for p^μ_{cR}, $L^{\mu\nu}_{cR}$ in Eq. (54) will be moved hereafter),

$$p^\mu = \mathrm{i}\hbar\Big[\Big(\eta^{\mu\nu} - \frac{x^\mu x^\nu}{R^2}\Big)\partial_\nu + \frac{5x^\mu}{2R^2}\Big]\,. \quad (68)$$

p^μ together with operator $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ form an algebra as follows:

$$[p^\mu, p^\nu] = \frac{1}{R^2}L^{\mu\nu}\,,$$
$$[L^{\mu\nu}, p^\rho] = \eta^{\nu\rho}p^\mu - \eta^{\mu\rho}p^\nu\,,$$
$$[L^{\mu\nu}, L^{\rho\sigma}] = \eta^{\nu\rho}L^{\mu\sigma} - \eta^{\nu\sigma}L^{\mu\rho} + \eta^{\mu\sigma}L^{\nu\rho} - \eta^{\mu\rho}L^{\nu\sigma}\,,$$

which is just the de-Sitter algebra SO(1,4). This fact means that the quantization scheme presented in this paper preserves the external space-time symmetry of $\mathcal{SR}_{cR}$.

6 Summery and Discussions

In this paper, we have provided a systemic study to the de Sitter invariant special relativity with Beltrami metric in terms of the Lagrangian–Hamiltonian formalism. In this theory there are two universal parameters c and R, and it was denoted as $\mathcal{SR}_{cR}$. Distinguishing from the Minkowski metric $\eta_{\mu\nu}$, the Beltrami metric is space-time dependent. Therefore the principle of least action for space-time dependent Lagrangian is reexamined in order to make sure the Lagrangian equation is consistent with the geodesic equation in Beltrami space-time $\mathcal{B}$. Following standard procedure in the ordinary special relativity and by means of the Beltrami metric we construct the Lagrangian $L_{cR}(t,\boldsymbol{x},\dot{\boldsymbol{x}})$ for $\mathcal{SR}_{cR}$. The inertial law has been reconfirmed in $\mathcal{SR}_{cR}$ by means of solving its equation of motion in the Lagrangian–Hamiltonian formalism, which leads to well-defined inertial coordinate reference frame in Beltrami space-time $\mathcal{B}$. The canonic momenta and canonic energy (or Hamiltonian) are derived. It is found that both of them are space-time dependent, which is due to that there are no cyclic coordinates in L_{cR} and the L_{cR} is time-dependent. The canonic equations and the corresponding Poisson bracket expressions are obtained. The canonic formulation is useful for quantization of the mechanics in $\mathcal{SR}_{cR}$. The de Sitter transformation in space-time $\mathcal{B}$ (i.e., LZG-transformation) has been used to derive the Noether charges of $\mathcal{SR}_{cR}$. Ten conservative charges are obtained. They are three boost charges, four momentum-energy charges, and three angular momentum charges. In this way and by the symmetry principle, the physical momenta, the physical energy and the physical angular momenta in $\mathcal{SR}_{cR}$ are determined in the Lagrangian–Hamiltonian formalism. It has been found that the Hamiltonian is not equal to the energy, and the canonical momentum is also different from the physical momentum, i.e., $H \neq E$ and $\vec{\pi} \neq \vec{p}$. This is a significant property for $\mathcal{SR}_{cR}$. When $R\to\infty$, all results of the de Sitter invariant special relativity goes back to the ordinary special relativity.

By means of the canonic formulation, the quantum mechanics of $\mathcal{SR}_{cR}$ is achieved. The one particle quantum equation is just the Klein–Gordon equation in curved space-time with Beltrami metric $B_{\mu\nu}$. The quantization scheme with proper $(\pi - x)$-ordering preserves the external space-time symmetry of $\mathcal{SR}_{cR}$. When $R\to\infty$ or $x\to 0$, the theory goes back to ordinary one particle quantum equation of the Einstein's special relativity, i.e., the ordinary Klein–Gordon equation in flat space-time. A further discussion on the solutions of the equation of $\mathcal{SR}_{cR}$-quantum mechanics would be interesting, which, however, is left to be in our coming works.

Physically, since R in the $\mathcal{SR}_{cR}$ could be a very large distance parameter, say the "radius of universe horizon", the existing experiments cannot justify or rule out $\mathcal{SR}_{cR}$. Therefore, how to design experiments to detect the effects of the de Sitter invariant special relativity would be remarkable. We speculate that careful studies on the solutions of $\mathcal{SR}_{cR}$-quantum mechanics may bring us ideas for this aim. For instance, the master equation for the photons emitted from very far away star should be the equation of $\mathcal{SR}_{cR}$-quantum mechanics Eq. (67) with $m_0 = 0$ instead of the ordinary KG-equation of $\mathcal{SR}_c$, because the distance $|x| \sim R$. This difference may lead to reveal some effects to distinguish $\mathcal{SR}_{cR}$ from $\mathcal{SR}_c$.

Finally, we would like to briefly mention the Double Special Relativity (DSR)[12] in comparison with $\mathcal{SR}_{cR}$. DSR is an interesting theory, and is another modified special relativity with also two universal constants: c and Planck length $l_P \equiv \sqrt{\hbar G_N/c^3}$ (or a length $l = \hbar/(\kappa c)$ near l_P, where $\kappa \sim m_P \equiv \sqrt{\hbar c/G_N}$). Obviously, the length parameter of DSR is drastically smaller than one of $\mathcal{SR}_{cR}$: $l_P/R \sim 10^{-120}$. This indicates that the physics discussed in DSR is very different from one in $\mathcal{SR}_{cR}$: DSR is inspired by quantum gravity and by a space-time quantization treatment for over the ultraviolet tragedy in quantum field theory,[13] while $\mathcal{SR}_{cR}$ is motived by naturally extending the space-time and the dynamics theory of Einstein's special relativity $\mathcal{SR}_c$, and the corresponding remarkable physics is related to the cosmology, say the propagation of photons emitted from far away stars with distance $|x| \sim R$. In other words, like $\mathcal{SR}_c$, $\mathcal{SR}_{cR}$ preserves a specific space-time metric (i.e., $B_{\mu\nu}$) and the inertial frames are well-defined. And then like $\mathcal{SR}_c$ further, $\mathcal{SR}_{cR}$ has well-defined Lagrangian–Hamiltonian formulation too. Consequently, a consistent quantum mechanics of $\mathcal{SR}_{cR}$ exists and can be derived by means of the standard quantization procedures relied on the first principle of quantum theory. However, the models of DSR are all different from $\mathcal{SR}_{cR}$ in these aspects. Basically, DSR theories can be understood as particular realizations of deformed κ-Poincaré algebra in momentum spaces,[14] or of a de Sitter geometry in momentum space.[15] Due to this structure, the space coordinates in DSR are non-commutative, i.e., $[\hat{x}^\mu, \hat{x}^\nu] \neq 0$, (which is in conflict with the principle requirement of quantum mechanics (see Eqs. (59) and (62)), and hence there are no Lagrangian–Hamiltonian formulations for DSR yet, which can be constructed consistently. If the length scale for both DSR and $\mathcal{SR}_{cR}$ were denoted as $\mathcal{R}$, then DSR is a theory for $|x| > \mathcal{R}$ $(\equiv l_P)$, and $\mathcal{SR}_{cR}$ is for $|x| < \mathcal{R}$ $(\equiv R)$ (see Eq. (A11)). Therefore, there is no overlapping part for DSR and $\mathcal{SR}_{cR}$, and the theory structures of two theories must be independent each other.

Appendix: Space-Time Transformation to Preserve Beltrami Metric

Now we prove that under the LZG space-time transformation Eq. (41) in the text the Beltrami metric is invariant.

We define the field $\mathfrak{D}_\lambda(m,n)$ to be all $m \times n$ matrix X such that

$$I - \lambda X J X' > 0\,. \tag{A1}$$

Here, I is an $m \times m$ identity matrix, $J = \mathrm{diag}\,[1, -1, \ldots, -1]$ is an $n \times n$ matrix, $\lambda = 1/R^2 \neq 0$ is a real number. A real matrix $A > 0$ means that A is positive-definite. Let A, B, C, and D be $m\times m$, $n\times m$, $m\times n$, $n\times n$ matrices respectively, satisfying

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -\lambda J \end{pmatrix}\begin{pmatrix} A & C \\ B & D \end{pmatrix}' = \begin{pmatrix} I & 0 \\ 0 & -\lambda J \end{pmatrix}. \tag{A2}$$

Writing the entries we get

$$AA' - \lambda CJC' = I\,, \quad AB' = \lambda CJD'\,,$$
$$BB' - \lambda DJD' = -\lambda J\,. \tag{A3}$$

Equation (A2) is also equivalent to

$$\begin{aligned}
&\begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -\lambda J \end{pmatrix}\begin{pmatrix} A & C \\ B & D \end{pmatrix}'\begin{pmatrix} I & 0 \\ 0 & -\lambda^{-1} J \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\
\Leftrightarrow &\begin{pmatrix} I & 0 \\ 0 & -\lambda J \end{pmatrix}\begin{pmatrix} A & C \\ B & D \end{pmatrix}'\begin{pmatrix} I & 0 \\ 0 & -\lambda^{-1} J \end{pmatrix}\begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\
\Leftrightarrow &\begin{pmatrix} A & C \\ B & D \end{pmatrix}'\begin{pmatrix} I & 0 \\ 0 & -\lambda^{-1} J \end{pmatrix}\begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -\lambda^{-1} J \end{pmatrix} \\
\Leftrightarrow &\begin{pmatrix} A & C \\ B & D \end{pmatrix}'\begin{pmatrix} \lambda I & 0 \\ 0 & -J \end{pmatrix}\begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} \lambda I & 0 \\ 0 & -J \end{pmatrix}.
\end{aligned}$$

Writing the entries we get

$$\lambda A'A - B'JB = \lambda I\,, \quad \lambda A'C = B'JD\,, \quad \lambda C'C - D'JD = -J\,. \tag{A4}$$

Therefore, equations (A3) and (A4) are equivalent. Observing that equation (A1) can be written as

$$(\,I \quad X\,)\begin{pmatrix} I & 0 \\ 0 & -\lambda J \end{pmatrix}\begin{pmatrix} I \\ X' \end{pmatrix} > 0\,, \tag{A5}$$

we use Eq. (A2) and get

$$\begin{aligned}
&(\,I \quad X\,)\begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -\lambda J \end{pmatrix}\begin{pmatrix} A & C \\ B & D \end{pmatrix}'\begin{pmatrix} I \\ X' \end{pmatrix} > 0 \\
\Leftrightarrow\ &(A + XB)\,(\,I \quad Y\,)\begin{pmatrix} I & 0 \\ 0 & -\lambda J \end{pmatrix}\begin{pmatrix} I \\ Y' \end{pmatrix}(A + XB)' > 0
\end{aligned}$$

$$\Leftrightarrow \begin{pmatrix} I & Y \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -\lambda J \end{pmatrix} \begin{pmatrix} I \\ Y' \end{pmatrix} > 0\,.$$

Here

$$Y = (A + XB)^{-1}(C + XD)\,. \tag{A6}$$

Therefore the transformation (A6) maps $\mathfrak{D}_\lambda(m,n)$ to itself and is an automorphism. We can define a metric on $\mathfrak{D}_\lambda(m,n)$

$$\mathrm{d}s^2 = \mathrm{tr}\,\{(I - \lambda XJX')^{-1}\mathrm{d}X(J - \lambda X'X)^{-1}\mathrm{d}X'\}\,. \tag{A7}$$

We claim that this metric is invariant under the transformation (A6).

Proof Note that from the above discussion we have

$$(I - \lambda XJX') = (A + XB)(I - \lambda YJY')(A + XB)'\,. \tag{A8}$$

Since $X = (AY - C)(D - BY)^{-1}$, we also have

$$J - \lambda Y'Y = \begin{pmatrix} Y' & -I \end{pmatrix} \begin{pmatrix} -\lambda I & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} Y \\ -I \end{pmatrix} = \begin{pmatrix} Y' & -I \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix}' \begin{pmatrix} -\lambda I & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} Y \\ -I \end{pmatrix}$$

$$= (D - BY)' \begin{pmatrix} X' & -I \end{pmatrix} \begin{pmatrix} -\lambda I & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} X \\ -I \end{pmatrix} (D - BY) = (D - BY)'(J - \lambda X'X)(D - BY)\,,$$

$$\mathrm{d}Y = \mathrm{d}((A + XB)^{-1}(C + XD)) = [-(A + XB)^{-1}\mathrm{d}(A + XB)(A + XB)^{-1}(C + XD) + (A + XB)^{-1}\mathrm{d}(C + XD)]$$

$$= [-(A + XB)^{-1}\mathrm{d}XBY + (A + XB)^{-1}\mathrm{d}XD] = (A + XB)^{-1}\mathrm{d}X(D - BY)\,,$$

$$\mathrm{d}Y' = \mathrm{d}((A + XB)^{-1}(C + XD))' = (D - BY)'\mathrm{d}X'(A + XB)'^{-1}\,.$$

Hence

$$\begin{aligned} &\mathrm{tr}\,\{(I - \lambda YJY')^{-1}\mathrm{d}Y(J - \lambda Y'Y)^{-1}\mathrm{d}Y'\} \\ &= \mathrm{tr}\,\{(A + XB)'(I - \lambda XJX')^{-1}(A + XB)(A + XB)^{-1}\mathrm{d}X(D - BY) \\ &\quad \times (D - BY)^{-1}(J - \lambda X'X)^{-1}(D - BY)'^{-1}(D - BY)'\mathrm{d}X'(A + XB)'^{-1}\} \\ &= \mathrm{tr}\,\{(I - \lambda XJX')^{-1}\mathrm{d}X(J - \lambda X'X)^{-1}\mathrm{d}X'\}\,, \end{aligned}$$

which states that the metric (A7) is invariant under transformation (A6). □

If we let $X_0 = -CD^{-1}$, then

$$Y = (A + XB)^{-1}(C + XD) = A^{-1}(I + XBA^{-1})^{-1}(X + CD^{-1})D\,.$$

The conditions in Eq. (A4) are equivalent to the following:

$$BA^{-1} = (\lambda CD^{-1}J)' = \lambda JX_0'\,, \quad (AA')^{-1} = A'^{-1}(A'A - \lambda^{-1}B'JB)A^{-1} = (I - \lambda X_0JX_0')\,,$$

$$(DJD')^{-1} = D'^{-1}JD^{-1} = D'^{-1}(D'JD - \lambda C'C)D^{-1} = J - \lambda X_0'X_0\,.$$

We get the formula

$$Y = A^{-1}(I - \lambda XJX_0)^{-1}(X - X_0)D\,, \tag{A9}$$

where the matrices A and D satisfy

$$AA' = (I - \lambda X_0JX_0')^{-1}\,, \quad DJD' = (J - \lambda X_0'X_0)^{-1}\,. \tag{A10}$$

For the special case $\mathfrak{D}_\lambda(1,4)$ in our paper, $X = (X^0, X^1, X^2, X^3)$, $\mathfrak{D}_\lambda(1,4)$ is just

$$1 - \lambda\eta_{\mu\nu}x^\mu x^\nu > 0\,. \tag{A11}$$

The metric (A7) now takes the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}X(J - \lambda X'X)^{-1}\mathrm{d}X'}{1 - \lambda XJX'} = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\,, \quad g_{\mu\nu} = \frac{\eta_{\mu\nu}}{1 - \lambda\eta_{\lambda\rho}x^\lambda x^\rho} + \frac{\lambda\eta_{\mu\lambda}\eta_{\nu\rho}x^\lambda x^\rho}{(1 - \lambda\eta_{\alpha\beta}x^\alpha x^\beta)^2}\,. \tag{A12}$$

Comparing Eq. (A7) with Eq. (19) in the text, we see $g_{\mu\nu}$ is just the Beltrami metric, i.e., $g_{\mu\nu} = B_{\mu\nu}(x)$. By our claim, this metric is invariant under the transformation (A9), which now becomes

$$y^\mu = \sqrt{1 - \lambda\eta_{\lambda\rho}a^\lambda a^\rho}\,\frac{(x^\nu - a^\nu)D^\mu_\nu}{1 - \lambda\eta_{\alpha\beta}a^\alpha x^\beta}\,, \tag{A13}$$

where we denote $X_0 = (a^0, a^1, a^2, a^3)$ and $\{D^\mu_\nu\}$ are constants, satisfying

$$\eta_{\lambda\rho}D^\lambda_\mu D^\rho_\nu = \eta_{\mu\nu} + \frac{\lambda\eta_{\mu\lambda}\eta_{\nu\rho}a^\lambda a^\rho}{1 - \lambda\eta_{\alpha\beta}a^\alpha a^\beta}\,. \tag{A14}$$

By using the notations in the text: $\tilde{x}^\mu = y^\mu$, $\sigma(x) = 1 - \lambda\eta_{\alpha\beta}x^\alpha x^\beta$, $\sigma(a,x) = 1 - \lambda\eta_{\alpha\beta}a^\alpha x^\beta$, we rewrite Eqs. (A13) and (A14) as follows:

$$\tilde{x}^\mu = \sqrt{\sigma(a)}\frac{(x^\nu - a^\nu)D^\mu_\nu}{\sigma(a,x)}\,, \tag{A15}$$

$$\eta_{\lambda\rho}D^\lambda_\mu D^\rho_\nu = \eta_{\mu\nu} + \frac{\lambda\eta_{\mu\lambda}\eta_{\nu\rho}a^\lambda a^\rho}{\sigma(a)}\,. \tag{A16}$$

Taking ansatz

$$D^\mu_\nu = \pm(L^\mu_\nu + A\lambda\eta_{\nu\lambda}a^\lambda a^\rho L^\mu_\rho)\,, \tag{A17}$$

where A is a constant which is determined by the normalization constraint Eq. (A16),

$$A = \frac{1}{\sigma(a)+\sqrt{\sigma}}\,. \tag{A18}$$

Substituting Eqs. (A16), (A17), and (A19) into Eq. (A15), we finally obtain

$$\tilde{x}^\mu = \pm\sqrt{\sigma(a)}\sigma(a,x)^{-1}(x^\nu - a^\nu) \times \Big(L^\mu_\nu + R^{-2}\frac{1}{\sigma(a)+\sqrt{\sigma}}\eta_{\nu\rho}a^\rho a^\lambda L^\mu_\lambda\Big)\,, \tag{A19}$$

where $\lambda = R^{-2}$ has been used. Equation (A19) is just Eq. (41) in the text.

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