Week8/9/10

潘晨翔、王曹励文

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10.2.2 设有界集$B \subset \mathbb{R}^2$且B'是零面积集，求证$\overline{B}$也是零面积集。

解. 因为B'是零面积集，所以对任意的$\epsilon > 0$,存在有限个开矩形$I_1, I_2, \cdots, I_m$使得$B' \subset \bigcup_{i=1}^{m} I_i$,并且$\sum_{i=1}^{m} \sigma(I_i) < \epsilon/2$.于是$\overline{B} \setminus \bigcup_{i=1}^{m} I_i$是一个有限集，否则由聚点定理知它与$B'$有交集，矛盾！于是可作有限个开矩形$I_{m+1}, I_{m+2}, \cdots, I_{m+k}$使得$\overline{B} \setminus \bigcup_{i=1}^{m} I_i \subset \bigcup_{i=m+1}^{m+k} I_i$,且$\sum_{i=m+1}^{m+k} \sigma(I_i) < \epsilon/2$.因此$\overline{B} \subset \bigcup_{i=1}^{m+k} I_i$,并且$\sum_{i=1}^{m+k} \sigma(I_i) < \epsilon$,所以$\overline{B}$是零面积集。 □

10.2.4 闭矩形$J \subset I$,f在I上可积，求证f在J上也可积。

解. $J \subset I$,故若x是f在J上的间断点，则x也是f在I上的间断点，即$D_J(f) \subset D_I(f)$.
f在I上可积，则f有界且$D_I(f)$零测，故$D_J(f)$零测，因而f在J上可积。 □

10.2.5 I上可积函数$f > 0$,求证$\int_I f \mathrm{d}\sigma > 0$.

解. f可积，故f有界且$D(f)$零测，取$x_0 \in I \setminus D(f)$,则$f(x_0) > 0$且x_0是连续点，故存在x_0的邻域$U_0 \subset I$,当$x \in U_0$时，有$f(x) > f(x_0)/2$.则

$$\int_I f \mathrm{d}\sigma \geq \int_{U_0} f \mathrm{d}\sigma \geq \frac{1}{2} f(x_0) \sigma(U_0) > 0.$$

□

10.2.6 $I = [0,1]^2, f(x,y) = \begin{cases} \sin \dfrac{1}{xy}, & x \neq 0 \text{ 且 } y \neq 0 \\ 0, & x = 0 \text{ or } y = 0 \end{cases}$ 的可积性。

解. $\lim\limits_{x \to 0} \sin \dfrac{1}{xy}$和$\lim\limits_{y \to 0} \sin \dfrac{1}{xy}$不存在，在$x \neq 0$ 且 $y \neq 0$时均为连续函数，故$D(f) = \{0\} \times [0,1] \cup [0,1] \times \{0\}$为$\mathbb{R}^2$中的零测集，又$f$有界，故$f$在$I$上可积。 □

10.2.7 I有界矩形，$B = \{\boldsymbol{p}_1, \cdots, \boldsymbol{p}_n, \cdots\} \subset I$,函数$f(\boldsymbol{p}) = \begin{cases} 0, & \boldsymbol{p} \notin B \\ \dfrac{1}{n}, & \boldsymbol{p} = \boldsymbol{p}_n \end{cases}$ 的可积性。

解. f有界，且$D(f) = B$零测，故f在I上可积。 □

10.2.8 f,g在I上可积，则fg可积，$\dfrac{f}{g}$在$g\neq 0$且有界时可积。

解. f,g在I上可积,则f,g在I上有界，则fg在I上有界。
若f,g连续，则fg连续，当$g\neq 0$时$\dfrac{f}{g}$连续，因此

$$\begin{aligned} &D(fg)\subset D(f)\cup D(g)\\ &D\Big(\frac{f}{g}\Big)\subset D(f)\cup D(g)\ \text{if}\ g\neq 0\end{aligned}$$

因此fg可积，$\dfrac{f}{g}$在$g\neq 0$且有界时可积。 □

10.3.1 计算积分。
(1)$\displaystyle\iint_I \frac{x^2}{1+y^2}\mathrm{d}x\mathrm{d}y, I=[0,1]^2$;
(2)$\displaystyle\iint_I x\cos xy\mathrm{d}x\mathrm{d}y, I=[0,\frac{\pi}{2}]\times[0,1]$;
(3)$\displaystyle\iint_I \sin(x+y)\mathrm{d}x\mathrm{d}y, I=[0,\pi]^2$.

解.

$$\begin{aligned}
(1)\qquad \iint_I \frac{x^2}{1+y^2}\mathrm{d}x\mathrm{d}y &= \int_0^1 x^2\mathrm{d}x\int_0^1\frac{1}{y^2+1}\mathrm{d}y\\
&=\frac{1}{3}\arctan y\Big|_0^1=\frac{\pi}{12}.\\
(2)\qquad \iint_I x\cos xy\mathrm{d}x\mathrm{d}y &= \int_0^{\frac{\pi}{2}}\int_0^1 x\cos xy\mathrm{d}y\mathrm{d}x\\
&=\int_0^{\frac{\pi}{2}}\sin x\mathrm{d}x=1.\\
(3)\quad \iint_I \sin(x+y)\mathrm{d}x\mathrm{d}y &= \int_0^{\pi}\int_0^{\pi}\sin(x+y)\mathrm{d}x\mathrm{d}y\\
&=\int_0^{\pi}\cos y-\cos(y+\pi)\mathrm{d}y\\
&=2\int_0^{\pi}\cos y\mathrm{d}y=0.
\end{aligned}$$

□

10.3.2 f在$I=[a,b]\times[c,d]$上有连续的二阶导，计算$\displaystyle\iint_I\frac{\partial^2}{\partial x\partial y}f(x,y)\mathrm{d}x\mathrm{d}y$.

解.

$$\begin{aligned}
\iint_I\frac{\partial^2}{\partial x\partial y}f(x,y)\mathrm{d}x\mathrm{d}y &= \int_c^d\int_a^b\frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x,y)\mathrm{d}x\mathrm{d}y\\
&=\int_c^d\frac{\partial}{\partial y}f(b,y)-\frac{\partial}{\partial y}f(a,y)\mathrm{d}y\\
&=f(b,d)-f(b,c)-f(a,d)+f(a,c).
\end{aligned}$$

□

10.3.3 计算$\displaystyle\int_I f\mathrm{d}\sigma, I=[0,1]^2$.

(1) $f(x,y)=\begin{cases}1, & y\le x^2\\ 0, & y>x^2\end{cases}$; (2) $f(x,y)=\begin{cases}x+y, & x^2\le y\le 2x^2\\ 0, & \text{else}\end{cases}$.

解.

$$
\begin{aligned}
(1)\quad \int_I f\mathrm{d}\sigma &= \int_0^1\int_0^{x^2} 1\mathrm{d}y\mathrm{d}x=\int_0^1 x^2\mathrm{d}x=\frac{1}{3}.\\
(2)\quad \int_I f\mathrm{d}\sigma &= \int_0^{\frac{\sqrt{2}}{2}}\int_{x^2}^{2x^2}(x+y)\mathrm{d}y\mathrm{d}x+\int_{\frac{\sqrt{2}}{2}}^1\int_{x^2}^1(x+y)\mathrm{d}y\mathrm{d}x\\
&= \int_0^{\frac{\sqrt{2}}{2}}(x^3+\frac{3}{2}x^4)\mathrm{d}x+\int_{\frac{\sqrt{2}}{2}}^1(\frac{1}{2}+x-x^3-\frac{x^4}{2})\mathrm{d}x\\
&= \frac{21}{40}-\frac{\sqrt{2}}{5}.
\end{aligned}
$$

□

10.3.4 利用定理10.3.4的Minkowski不等式，证明：$a_k\ge 0, b_k\ge 0, k=1,2,\cdots,n, p\ge 1$,有

$$\Big(\sum_{k=1}^n(a_k+b_k)^p\Big)^{\frac{1}{p}}\le\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}+\Big(\sum_{k=1}^n b_k^p\Big)^{\frac{1}{p}}.$$

解. 令$f(x,y)=\begin{cases}a_k, & 0\ge y\ge 1, k-1\ge x\ge k\\ b_k, & 1\ge y\ge 2, k-1\ge x\ge k\end{cases}$,代入定理10.3.4中左右两式得

$$
\begin{aligned}
\Big(\int_0^n\Big(\int_0^2 f(x,y)\mathrm{d}y\Big)^p\Big)^{\frac{1}{p}} &= \Big(\sum_{k=1}^n\int_{k-1}^k(a_k+b_k)^p\mathrm{d}x\Big)^{\frac{1}{p}}=\Big(\sum_{k=1}^n(a_k+b_k)^p\mathrm{d}x\Big)^{\frac{1}{p}}\\
\int_0^2\Big(\int_0^n f^p(x,y)\mathrm{d}x\Big)^{\frac{1}{p}}\mathrm{d}y &= \int_0^1\Big(\sum_{k=1}^n\int_{k-1}^k a_k^p\Big)^{\frac{1}{p}}\mathrm{d}y+\int_1^2\Big(\sum_{k=1}^n\int_{k-1}^k b_k^p\Big)^{\frac{1}{p}}\mathrm{d}y\\
&= \Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}+\Big(\sum_{k=1}^n b_k^p\Big)^{\frac{1}{p}}.
\end{aligned}
$$

故

$$\Big(\sum_{k=1}^n(a_k+b_k)^p\Big)^{\frac{1}{p}}\le\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}+\Big(\sum_{k=1}^n b_k^p\Big)^{\frac{1}{p}}.$$

□

10.3.5 f,g都在$[a,b]$上可积，用$\displaystyle\iint_{[a,b]^2}\big(f(x)g(y)-f(y)g(x)\big)^2\mathrm{d}x\mathrm{d}y\ge 0$证明：

$$\Big(\int_a^b f(x)g(x)\mathrm{d}x\Big)^2\le\int_a^b f(x)^2\mathrm{d}x\int_a^b g(x)^2\mathrm{d}x.$$

解.

$$
\begin{aligned}
0 &\le \iint_{[a,b]^2} \big(f(x)g(y)-f(y)g(x)\big)^2 \mathrm{d}x\mathrm{d}y\\
&= \iint_{[a,b]^2} \big(f(x)g(y)\big)^2 - 2f(x)f(y)g(x)g(y) + \big(f(y)g(x)\big)^2 \mathrm{d}x\mathrm{d}y\\
&= \int_a^b f(x)^2\mathrm{d}x \int_a^b g(y)^2\mathrm{d}y - 2\int_a^b f(x)g(x)\mathrm{d}x \int_a^b f(y)g(y)\mathrm{d}y + \int_a^b f(y)^2\mathrm{d}y \int_a^b g(x)^2\mathrm{d}x\\
&= 2\int_a^b f(x)^2\mathrm{d}x \int_a^b g(x)^2\mathrm{d}x - 2\Big(\int_a^b f(x)g(x)\mathrm{d}x\Big)^2
\end{aligned}
$$

□

10.4.2 证明：$1.96 < \iint\limits_{|x|+|y|\le 10} \dfrac{\mathrm{d}x\mathrm{d}y}{100+\cos^2 x+\cos^2 y} < 2.$

解. $E := \{(x,y) : |x|+|y| \le 10\}, \sigma(E) = 200,$又有$\dfrac{1}{102} \le \dfrac{1}{100+\cos^2 x+\cos^2 y} \le \dfrac{1}{100}$, 故

$$1.96 < \frac{200}{102} \le \iint\limits_{|x|+|y|\le 10} \frac{\mathrm{d}x\mathrm{d}y}{100+\cos^2 x+\cos^2 y} \le \frac{200}{100} = 2,$$

由于$\dfrac{1}{100+\cos^2 x+\cos^2 y}$的最大值和最小值不是处处取到，因而

$$1.96 < \iint\limits_{|x|+|y|\le 10} \frac{\mathrm{d}x\mathrm{d}y}{100+\cos^2 x+\cos^2 y} < 2.$$

□

10.4.3 $B \subset \mathbb{R}^2$是有界集，证明B有面积集当且仅当对$I^\circ \supset B$的矩形I的任何矩形分割π,均有$\lim\limits_{||\pi||\to 0} \sum\limits_{I_j\cap B\neq\emptyset} \sigma(I_j) = \lim\limits_{||\pi||\to 0} \sum\limits_{I_j\subset B} \sigma(I_j)$.

解. B有面积当且仅当∂B是零面积集。对任何矩形分割π,

$$\lim_{||\pi||\to 0} \sum_{I_j\cap B\neq\emptyset} \sigma(I_j) = \lim_{||\pi||\to 0} \sum_{I_j\subset B} \sigma(I_j) + \lim_{||\pi||\to 0} \sum_{\substack{I_j\cap B\neq\emptyset\\ I_j\cap B^c\neq\emptyset}} \sigma(I_j),$$

故$\lim\limits_{||\pi||\to 0} \sum\limits_{I_j\cap B\neq\emptyset} \sigma(I_j) = \lim\limits_{||\pi||\to 0} \sum\limits_{I_j\subset B} \sigma(I_j)$当且仅当$\lim\limits_{||\pi||\to 0} \sum\limits_{\substack{I_j\cap B\neq\emptyset\\ I_j\cap B^c\neq\emptyset}} \sigma(I_j) = 0.$由定义可知该式等价于$\partial B$是零测集，又$B$是有界集，故$\partial B$是有界闭集，因而$\partial B$是零测集等价于$\partial B$是零面积集。 □

10.5.1计算下列积分：

(2) $\iint_D xy^2 dxdy$, D由$y^2 = 4x$和$x = 1$围成；

(4) $\iint_D |xy| dxdy$, $D = \{(x,y) : x^2+y^2 \le a^2\}$;

(6) $\iint_D |\cos(x+y)|\, dxdy$, $D = [0,1]^2$;

(7) $\iint_D y^2dxdy$, D由滚轮线

$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \sin t) \end{cases} \qquad (0 \le t \le 2\pi)$$

与$y=0$围成;

(8)$\iint_D [x+y]dxdy$, $D = [0,2]^2$.

解. (2)

$$\begin{aligned}\iint_D xy^2dxdy &= \int_{-2}^{2} dy \int_{\frac{y^2}{4}}^{1} xy^2dx \\ &= \frac{1}{2}\int_{-2}^{2} y^2(1 - \frac{1}{16}y^4)dy \\ &= \frac{32}{21}.\end{aligned}$$

(4)

$$\begin{aligned}\iint_D |xy|dxdy &= 4 \iint\limits_{\substack{x^2+y^2\le a^2 \\ x,y\ge 0}} xydxdy \\ &= 4\int_0^a ydy \int_0^{\sqrt{a^2-y^2}} xdx \\ &= 2\int_0^a y(a^2 - y^2)dy \\ &= \frac{1}{2}a^4.\end{aligned}$$

(6) 记$E = \{(x,y) \in D | x + y \le \frac{\pi}{2}\}$ $\qquad F = \{(x,y) \in D | x + y > \frac{\pi}{2}\}$,

$$\begin{aligned}\iint\limits_D \cos(x+y)dxdy &= \iint\limits_E \cos(x+y)dxdy - \iint\limits_F \cos(x+y)dxdy \\ &= \iint\limits_D |\cos(x+y)|dxdy - 2\iint\limits_F |\cos(x+y)|dxdy \\ &= \int_0^1 dy \int_0^1 \cos(x+y)dx - 2\int_{\frac{\pi}{2}-1}^{1} dx \int_{\frac{\pi}{x}-x}^{1} cos(x+y)dy \\ &= \cos 2 + 2\cos 1 + 3 - \pi.\end{aligned}$$

(7)

$$\begin{aligned}\iint\limits_D y^2dxdy &= \int_0^{2\pi a} dx \int_0^{y(x)} y^2dxdy \\ &= \int_0^{2\pi a} \frac{1}{3}(y(x))^3dx \\ &= \int_0^{2\pi} \frac{1}{3}(y(x(t)))^3\frac{dx}{dt}dt \\ &= \int_0^{2\pi} \frac{a^4}{3}(1 - \cos t)^4dt \\ &= \frac{35\pi a^4}{12}.\end{aligned}$$

(8)
$$\iint_D [x+y]dxdy = 6.$$
$\square$

10.5.2改变下列累次积分的次序:

(1)$\int_0^1 dx \int_0^{x^2} f(x,y)dy$;

(3)$\int_0^1 dx \int_{x^2}^{x} f(x,y)dy$;

(5)$\int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)dy$;

(7)$\int_0^{\pi} dx \int_0^{\cos x} f(x,y)dy$.

解. (1)$\int_0^1 dy \int_{\sqrt{y}}^1 f(x,y)dx$;

(3)$\int_0^1 dy \int_y^{\sqrt{y}} f(x,y)dx$;

(5)$\int_{-1}^0 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx + \int_0^1 dy \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)dx$;

(7)$\int_0^1 dy \int_0^{\arccos y} f(x,y)dx - \int_{-1}^0 dy \int_{\arccos y}^{\pi} f(x,y)dx$.

$\square$

10.5.3设f为一元连续函数.求证:
$$\int_0^a dx \int_0^x f(x)f(y)dy = \frac{1}{2}\Big(\int_0^a f(t)dt\Big)^2.$$
解. 记 $F(x) = \int_0^x f(t)dt$,
$$\begin{aligned}\int_0^a dx \int_0^x f(x)f(y)dxdy &= \int_0^a f(x)dx \int_0^x f(y)dy\\ &= \int_0^a f(x)F(x)dx\\ &= \frac{1}{2}F(a)^2\\ &= \frac{1}{2}\Big(\int_0^a f(t)dt\Big)^2.\end{aligned}$$
$\square$

10.5.4设f为连续函数.证明:
$$\int_0^a dx \int_0^x f(y)dy = \int_0^a (a-t)f(t)dt.$$
解.
$$\begin{aligned}\int_0^a dx \int_0^x f(y)dy &= \int_0^a f(y)dy \int_y^a dx\\ &= \int_0^a (a-y)f(y)dy.\end{aligned}$$

□

10.5.5设f为连续函数.证明：

$$\int_a^b dx \int_a^x dy \int_a^y f(x,y,z)dz = \int_a^b dz \int_z^b dy \int_y^b f(x,y,z)dx.$$

解. 交换积分顺序即得

$$\int_a^b dx \int_a^x dy \int_a^y f(x,y,z)dz = \int_a^b dz \int_z^b dy \int_y^b f(x,y,z)dx.$$

□

10.5.6设f为连续函数.证明：

$$\int_0^a dx \int_0^x dy \int_0^y f(x)f(y)f(z)dz = \frac{1}{3!}\Big(\int_0^a f(t)dt\Big)^3.$$

解. 记 $F(x) = \int_0^a f(t)dt$,

$$\begin{aligned}\int_0^a dx \int_0^x dy \int_0^y f(x)f(y)f(z)dz &= \int_0^a f(x)dx \int_0^x F(y)f(y)dy \\ &= \int_0^a \frac{1}{2}f(x)\big(F(x)^2 - F(0)^2\big)dx \\ &= \frac{1}{2}\int_0^a F(x)^2 f(x)dx \\ &= \frac{1}{3!}\big(F(a)^3 - F(0)^3\big) \\ &= \frac{1}{3!}\Big(\int_0^a f(t)dt\Big)^3.\end{aligned}$$

□

10.5.7设f为连续函数.证明：

$$\int_0^a dx \int_0^x dy \int_0^y f(z)dz = \frac{1}{2}\int_0^a (a-t)^2 f(t)dt.$$

解.

$$\begin{aligned}\int_0^a dx \int_0^x dy \int_0^y f(z)dz &= \int_0^a f(z)dz \int_z^a dy \int_y^a dx \\ &= \int_0^a f(z)dz \int_z^a (a-y)dy \\ &= \frac{1}{2}\int_0^a (a-z)^2 f(z)dz.\end{aligned}$$

□

10.5.8设f为连续函数.求极限

$$\lim_{r\to 0}\frac{1}{\pi r^2}\iint\limits_{x^2+y^2\le r^2} f(x,y)dxdy.$$

解. 由积分中值定理得

$$\begin{aligned}\lim_{r\to 0}\frac{1}{\pi r^2}\iint\limits_{x^2+y^2\le r^2} f(x,y)dxdy &= \lim_{r\to 0}\frac{1}{\pi r^2}f(\xi,\eta)\pi r^2 \\ &= \lim_{r\to 0} f(\xi,\eta) \\ &= f(0,0).\end{aligned}$$

□