Week6

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9.9.1 求下列函数的二阶偏导数：
(2) $z = \tan \frac{x^2}{y}$;
(4) $z = \arctan \frac{y}{x}$;
(6) $u = xy + yz + zx$;
(8) $u = x^{yz}$;
(10) $u = \arcsin(x_1^2 + x_2^2 + ... + x_n^2)$.

解. (2)

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{2}{y^2 \cos^2 \frac{x^2}{y}}(y + 4x^2 \tan \frac{x^2}{y}), \\
\frac{\partial^2 z}{\partial x \partial y} &= -\frac{2x}{y^2 \cos^2 \frac{x^2}{y}}(1 + \frac{2x^2}{y} \tan \frac{x^2}{y}), \\
\frac{\partial^2 z}{\partial y^2} &= \frac{2x^2}{y^4 \cos^2 \frac{x^2}{y}}(y + x^2 \tan \frac{x^2}{y}).
\end{aligned}$$

(4)

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{2xy}{(x^2+y^2)^2}, \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{y^2 - x^2}{(x^2+y^2)^2}, \\
\frac{\partial^2 z}{\partial y^2} &= -\frac{2xy}{(x^2+y^2)^2}.
\end{aligned}$$

(6)

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= 0, \\
\frac{\partial^2 u}{\partial y^2} &= 0, \\
\frac{\partial^2 u}{\partial z^2} &= 0, \\
\frac{\partial^2 u}{\partial x \partial y} &= 1, \\
\frac{\partial^2 u}{\partial x \partial z} &= 1, \\
\frac{\partial^2 u}{\partial y \partial z} &= 1,
\end{aligned}$$

(8)

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= yz(yz-1)x^{yz-2},\\
\frac{\partial^2 u}{\partial y^2} &= z^2 x^{yz}(\ln x)^2,\\
\frac{\partial^2 u}{\partial z^2} &= y^2 x^{yz}(\ln x)^2,\\
\frac{\partial^2 u}{\partial x\partial y} &= z x^{yz-1}(1+yz\ln x),\\
\frac{\partial^2 u}{\partial x\partial z} &= y x^{yz-1}(1+yz\ln x),\\
\frac{\partial^2 u}{\partial y\partial z} &= (1+yz\ln x)x^{yz}\ln x.
\end{aligned}$$

(10)

$$\begin{aligned}
\frac{\partial^2 u}{\partial x_i^2} &= \frac{2(1+(x_1^2+x_2^2+...+x_n^2)(2x_i^2-(x_1^2+x_2^2+...+x_n^2)))}{(1-(x_1^2+x_2^2+...+x_n^2)^2)^{\frac{3}{2}}},\\
\frac{\partial^2 z}{\partial x_i\partial x_j} &= \frac{4x_i x_j(x_1^2+x_2^2+...+x_n^2)}{(1-(x_1^2+x_2^2+...+x_n^2)^2)^{\frac{3}{2}}}.
\end{aligned}$$

□

9.9.3 设 $u = e^{a\theta}\cos(a\ln r)$(a 为常数). 求证:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0.$$

解.

$$\begin{aligned}
\frac{\partial u}{\partial r} &= -e^{a\theta}\sin(a\ln r)\frac{a}{r},\\
\frac{\partial^2 u}{\partial r^2} &= e^{a\theta}(\frac{a}{r^2}\sin(a\ln r) - \frac{a^2}{r^2}\cos(a\ln r)),\\
\frac{\partial^2 u}{\partial \theta^2} &= a^2 e^{a\theta}\cos(a\ln r),
\end{aligned}$$

带入即有

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0.$$

□

9.9.4 设 u 是 x,y,z 的函数，令

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},$$

我们称

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

为 Laplace 算子.

(1) 设 $p = \sqrt{x^2+y^2+z^2}$. 证明:

$$\Delta p = \frac{2}{p}, \qquad \Delta \ln p = \frac{1}{p^2}, \qquad \Delta(\frac{1}{p}) = 0,$$

其中 $p > 0$.

(2) 设 $u = f(p)$. 求 Δu.

解. (1)

$$\frac{\partial^2 p}{\partial x^2}=\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$
$$\frac{\partial^2 p}{\partial y^2}=\frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$
$$\frac{\partial^2 p}{\partial z^2}=\frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

故得

$$\Delta p=\frac{2}{p},$$

同理有

$$\frac{\partial^2 \ln p}{\partial x^2}=\frac{1}{p^2}\frac{y^2+z^2-x^2}{x^2+y^2+z^2},$$
$$\frac{\partial^2 \ln p}{\partial y^2}=\frac{1}{p^2}\frac{x^2+z^2-y^2}{x^2+y^2+z^2},$$
$$\frac{\partial^2 \ln p}{\partial z^2}=\frac{1}{p^2}\frac{x^2+y^2-z^2}{x^2+y^2+z^2},$$

故得

$$\Delta \ln p=\frac{1}{p^2},$$

同理有

$$\frac{\partial^2 \frac{1}{p}}{\partial x^2}=\frac{1}{p^3}\frac{-y^2-z^2+2x^2}{x^2+y^2+z^2},$$
$$\frac{\partial^2 \frac{1}{p}}{\partial y^2}=\frac{1}{p^3}\frac{-x^2-z^2+2y^2}{x^2+y^2+z^2},$$
$$\frac{\partial^2 \frac{1}{p}}{\partial z^2}=\frac{1}{p^3}\frac{-x^2-y^2+2z^2}{x^2+y^2+z^2},$$

故得

$$\Delta\frac{1}{p}=0,$$

(2)

$$\frac{\partial u}{\partial x}=f'(p)\frac{\partial p}{\partial x},$$
$$\frac{\partial^2 u}{\partial x^2}=f''(p)(\frac{\partial p}{\partial x})^2+f'(p)\frac{\partial^2 p}{\partial x^2},$$

代入得

$$\begin{aligned}\Delta u&=f''(p)||\nabla p||^2+f'(p)\Delta p\\&=f''(p)+\frac{2}{p}f'(p).\end{aligned}$$

□

注. 有些同学这里写了 $\frac{\partial f}{\partial p}$ 的记号，这个是不好的，因为 f 是只关于 p 的单变量函数，从而也就没有求偏导这一说。

9.9.6 解下列方程，其 u 是 x,y,z 的函数:

(1) $\frac{\partial^2 u}{\partial x^2}=0;$

(2) $\frac{\partial^2 u}{\partial x\partial y}=0;$

(3) $\frac{\partial^3 u}{\partial x\partial y\partial z}=0.$

解. (1) 由 $\frac{\partial^2 u}{\partial x^2}=0$, 两边对 x 积分，得到

$$\frac{\partial u}{\partial x}=f(y,z),$$

两边在关于 x 积分, 得到

$$u(x,y,z)=f(y,z)x+g(y,z),$$

注：上面对于 x 计算过程中把 y,z 都试为常数。

(2) 由 $\frac{\partial^2 u}{\partial x\partial y}=0$, 两边对 y 积分，得到

$$\frac{\partial u}{\partial x}=f(x,z),$$

两边在关于 x 积分, 得到

$$u(x,y,z)=\int f(x,z)dy=g(x,z)+h(y,z),$$

其中 g(x,z) 是关于 x 的可导函数.

(3) 由 $\frac{\partial^3 u}{\partial x\partial y\partial z}=0$, 两边对 z 积分，得到

$$\frac{\partial^2 u}{\partial x\partial y}=f(x,y),$$

两边在关于 y 积分, 得到

$$\frac{\partial u}{\partial x}=\int f(x,y)dy=g(x,y)+h(x,z),$$

两边在关于 z 积分, 得到

$$u=\int g(y,z)+h(x,z)dz=p(y,z)+q(x,z)+r(x,y),$$

其中 p(y,z) 是关于 y,z 的可导函数,q(x,z) 是关于 x,z 的可导函数. □

注. *这道题需要注意的是，每一步积分 (如对 x 积分) 都只能得到一个多变量函数 $h(y,z)$，而不能写成类似 $f(y)+g(z)$ 这种变量分离的形式。*

9.9.7 求解偏微分方程

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z.$$

解. 做换元

$$\begin{cases} x=p \\ y=\frac{p}{q}, \end{cases}$$

记换元之后的函数

$$\tilde{z}(p,q)=z(x(p,q),y(p,q)),$$

则由链式法则得到

$$\frac{\partial \tilde{z}}{\partial p}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial p}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial p},$$
$$\frac{\partial \tilde{z}}{\partial q}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial q}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial q},$$

带入即有

$$\frac{\partial \tilde{z}}{\partial p} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{1}{q},$$
$$\frac{\partial \tilde{z}}{\partial q} = -\frac{\partial z}{\partial y}\frac{p}{q^2},$$

反解得到

$$\frac{\partial z}{\partial x} = \frac{\partial \tilde{z}}{\partial p} + \frac{\partial \tilde{z}}{\partial q}\frac{q}{p},$$
$$\frac{\partial z}{\partial y} = -\frac{\partial \tilde{z}}{\partial q}\frac{q^2}{p},$$

带入原方程得到

$$p\frac{\partial \tilde{z}}{\partial p} = \tilde{z},$$

解得

$$\tilde{z}(p,q) = pf(q),$$

带入即有

$$z(x,y) = xf(\frac{x}{y}).$$

□

9.9.8. 设 a,b,c 满足 $b^2 - ac > 0$,λ_1,λ_2 是二次方程 $cx^2 + 2bx + a = 0$ 的两个根. 试通过引进新变量

$$\xi = x + \lambda_1 y, \qquad \eta = x + \lambda_2 y,$$

解二阶偏微分方程

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = 0.$$

解. 做换元

$$\begin{cases} \xi = x + \lambda_1 y, \\ \eta = x + \lambda_2 y, \end{cases}$$

记换元之后的函数

$$\tilde{u}(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta)),$$

则有

$$u(x,y) = \tilde{u}(\xi(x,y), \eta(x,y)),$$

则由链式法则得到

$$\frac{\partial u}{\partial x} = \frac{\partial \tilde{u}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \tilde{u}}{\partial \eta}\frac{\partial \eta}{\partial x},$$
$$\frac{\partial u}{\partial y} = \frac{\partial \tilde{u}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \tilde{u}}{\partial \eta}\frac{\partial \eta}{\partial y},$$

带入即有

$$\frac{\partial u}{\partial x} = \frac{\partial \tilde{u}}{\partial \xi}(\xi(x,y), \eta(x,y)) + \frac{\partial \tilde{u}}{\partial \eta}(\xi(x,y), \eta(x,y)),$$
$$\frac{\partial u}{\partial y} = \lambda_1\frac{\partial \tilde{u}}{\partial \xi}(\xi(x,y), \eta(x,y)) + \lambda_2\frac{\partial \tilde{u}}{\partial \eta}(\xi(x,y), \eta(x,y)),$$

再由链式法则求导得到

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \tilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{u}}{\partial \eta^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = \lambda_1 \frac{\partial^2 \tilde{u}}{\partial \xi^2} + (\lambda_1 + \lambda_2)\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} + \lambda_2 \frac{\partial^2 \tilde{u}}{\partial \eta^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \lambda_1^2 \frac{\partial^2 \tilde{u}}{\partial \xi^2} + 2\lambda_1\lambda_2 \frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} + \lambda_2^2 \frac{\partial^2 \tilde{u}}{\partial \eta^2},$$

带入原方程得到

$$\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} = 0,$$

解得

$$\tilde{u}(\xi, \eta) = f(\xi) + g(\eta),$$

带入即有

$$u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y).$$

□

9.10.1 将下列多项式在指定点处展开为 Taylor 多项式 (写出前三项);

(1)$2x^2 - xy - y^2 - 6x - 3y + 5$, 在点 (1,-2) 处;

(2)$x^3 + y^3 + z^3 - 3xyz$, 在点 (1,1,1) 处.

解. (1)$f(x, y) = 2(x-1)^2 - (x-1)(y+2) - (y+2)^2 + 5$

(2)$g(x, y, z) = 3(x-1)^2 + 3(y-1)^2 + 3(z-1)^2 - 3(x-1)(y-1) - 3(x-1)(z-1) - 3(y-1)(z-1)$ □

9.10.2 考察二次多项式

$$f(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} A & D & F \\ D & B & E \\ F & E & C \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

试将 $f(x + \Delta x, y + \delta y, z + \delta z)$ 按 Δx,Δy,Δz 的正整数幂展开.

解. 带入有 f 为一个二次型

$$f(x, y, z) = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fxz$$

, 则可以计算其各阶偏导数在 (x, y, z) 处的值，带入 Taylor 公式即有

$$\begin{aligned}
&f(x + \Delta x, y + \Delta y, z + \Delta z) \\
&= f(x, y, z) + Jf(x, y, z)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{1}{2}(\Delta x, \Delta y, \Delta z)Hf(x, y, z)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= f(x, y, z) + 2\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} A & D & F \\ D & B & E \\ F & E & C \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \\
&+ \begin{pmatrix} \Delta x & \Delta y & \Delta z \end{pmatrix}\begin{pmatrix} A & D & F \\ D & B & E \\ F & E & C \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}
\end{aligned}$$

□

9.10.3 将 x^y 在点 (1,1) 处作 Taylor 展开, 写到二次项.

解.

$$f(x,y) = 1 + (x-1) + (x-1)(y-1) + o(||h||^2)$$

□

9.10.4 证明：当 $|x|$ 和 $|y|$ 充分小时，有近似式

$$\frac{\cos x}{\cos y} = 1 - \frac{1}{2}(x^2 - y^2) + o(x^2 + y^2).$$

解. 我们记 $f(x,y) = \frac{\cos x}{\cos y}$, 计算各界偏导数有:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -\frac{\sin x}{\cos y}, \\
\frac{\partial f}{\partial y} &= -\frac{\cos x \sin y}{\cos^2 y}, \\
\frac{\partial^2 f}{\partial x^2} &= -\frac{\cos x}{\cos y}, \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\sin x \sin y}{\cos^2 y}, \\
\frac{\partial^2 f}{\partial y^2} &= \frac{cosx(1 + \sin^2 y)}{\cos^3 y},
\end{aligned}$$

在 (0,0) 处取值带入 Taylor 公式得到

$$\frac{\cos x}{\cos y} = 1 - \frac{1}{2}(x^2 - y^2) + o(x^2 + y^2).$$

□