Week 4

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9.3.2 计算下列映射的 Jacobi 矩阵:
(1)$\boldsymbol{f}(r,\theta)=(r\cos\theta, r\sin\theta)$;
(2)$\boldsymbol{f}(r,\theta,z)=(r\cos\theta, r\sin\theta, z)$;
(3)$\boldsymbol{f}(r,\theta,\phi)=(r\cos\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\varphi)$

解. (1)

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

(2)

$$\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(3)

$$\begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

□

9.3.3 设区域 $D\subset\mathbb{R}^n$, 映射 $\boldsymbol{f},\boldsymbol{g}:D\to\mathbb{R}^n$. 求证:
(3) 当 $m=1$ 时，有 $\mathbf{J}(\boldsymbol{f}\boldsymbol{g})=\boldsymbol{g}\mathbf{J}\boldsymbol{f}+\boldsymbol{f}\mathbf{J}\boldsymbol{g}$;
(4) 当 $m>1$ 时，有

$$\mathbf{J}<\boldsymbol{f},\boldsymbol{g}>=\boldsymbol{g}(\mathbf{J}\boldsymbol{f})+\boldsymbol{f}(\mathbf{J}\boldsymbol{g}),$$

解. (3) 由

$$\frac{\partial\boldsymbol{f}\boldsymbol{g}}{\partial x}=\frac{\partial\boldsymbol{f}}{\partial x}*\boldsymbol{g}+\boldsymbol{f}*\frac{\partial\boldsymbol{g}}{\partial x}$$

这对应于问题中每个分量的值，故成立。
(4)

$$\begin{aligned}\mathbf{J}\langle\boldsymbol{f},\boldsymbol{g}\rangle&=\Big(\frac{\partial\sum_{k=1}^m f_kg_k}{\partial x_1},...,\frac{\partial\sum_{k=1}^m f_kg_k}{\partial x_n}\Big)=\Big(\sum_{k=1}^m\frac{\partial f_kg_k}{\partial x_1},...,\sum_{k=1}^m\frac{\partial f_kg_k}{\partial x_n}\Big)\\&=\Big(\sum_{k=1}^m\frac{\partial f_k}{\partial x_1}\cdot g_k+f_k\cdot\frac{\partial g}{\partial x_1},...,\sum_{k=1}^m\frac{\partial f_k}{\partial x_n}\cdot g_k+f_k\cdot\frac{\partial g_k}{\partial x_n}\Big)=\boldsymbol{g}(\mathbf{J}\boldsymbol{f})+\boldsymbol{f}(\mathbf{J}\boldsymbol{g})\end{aligned}$$

□

9.3.4 设 $\boldsymbol{f}:[a,b]\to\mathbb{R}^n$, 并且对一切 $t\in[a,b]$，有 $||\boldsymbol{f}(t)||=$ 常数。求证：$<\mathbf{J}\boldsymbol{f},\boldsymbol{f}>=0$, 并对此式作出几何解释。

解. 这是因为

$$0=\mathbf{J}||\boldsymbol{f}(t)||=\mathbf{J}<\boldsymbol{f}(t),\boldsymbol{f}(t)>=2<\mathbf{J}\boldsymbol{f},\boldsymbol{f}>$$

几何解释：球面上的切向量总是垂直于径向向量。 □

9.3.6 设映射 $\boldsymbol{f}:\mathbb{R}^n\to\mathbb{R}^m$. 如果

$$\boldsymbol{f}(\lambda\boldsymbol{x}+\mu\boldsymbol{y})=\lambda\boldsymbol{f}(\boldsymbol{x})+\mu\boldsymbol{f}(\boldsymbol{y})$$

对一切 $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$ 和一切 $\lambda,\mu\in\mathbb{R}$ 成立, 则称 $\boldsymbol{f}$ 是线性映射。证明:

(1)$\boldsymbol{f}(0)=0$;

(2)$\boldsymbol{f}(-\boldsymbol{x})=-\boldsymbol{f}(\boldsymbol{x})$ $(\boldsymbol{x}\in\mathbb{R}^n)$;

(3) 映射 $\boldsymbol{f}$ 由 $\boldsymbol{f}(\mathbf{e}_1),...,\boldsymbol{f}(\mathbf{e}_n)$ 完全确定。

解. (1) 取 $\lambda=\mu=0$, 有 $\boldsymbol{f}(0)=0$.

(2) 取 $\lambda=1,\mu=-1,\boldsymbol{y}=\boldsymbol{x}$, 有 $\boldsymbol{f}(-\boldsymbol{x})=-\boldsymbol{f}(\boldsymbol{x})$.

(3)$\forall\boldsymbol{x}\in\mathbb{R}^n$, $\boldsymbol{x}=\boldsymbol{x}_1\mathbf{e}_1+...+\boldsymbol{x}_n\mathbf{e}_n$ 则有 $\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{x}_1\boldsymbol{f}(\mathbf{e}_1)+...+\boldsymbol{x}_n\boldsymbol{f}(\mathbf{e}_n)$ 故映射 $\boldsymbol{f}$ 由 $\boldsymbol{f}(\mathbf{e}_1),...,\boldsymbol{f}(\mathbf{e}_n)$ 完全确定。 □

9.3.7 设 $\boldsymbol{f}:\mathbb{R}^n\to\mathbb{R}^m$ 为线性映射. 试求 $\mathbf{J}\boldsymbol{f}$.

解.

$$\mathbf{J}\boldsymbol{f}=(\boldsymbol{f}(\mathbf{e}_1),\boldsymbol{f}(\mathbf{e}_2),...,\boldsymbol{f}(\mathbf{e}_n))$$

□

9.4.2 设 $u=f(xy)$. 证明:

$$x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=0.$$

解.

$$\frac{\partial u}{\partial x}=f'(xy)y$$
$$\frac{\partial u}{\partial y}=f'(xy)x$$

代入即恒有

$$x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=0.$$

□

9.4.3 设

$$u=f(\log x+\frac{1}{y}),$$

证明:

$$x\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}=0.$$

解.

$$\frac{\partial u}{\partial x} = f'(\log x + \frac{1}{y})\frac{1}{x}$$
$$\frac{\partial u}{\partial y} = -f'(\log x + \frac{1}{y})\frac{1}{y^2}$$

代入即恒有

$$x\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = 0.$$

□

9.4.5 求以下 u 的一切偏导数:

(1)$u = f(x+y, xy)$;

(2)$u = f(x, xy, xyz)$;

(3)$u = f(\frac{x}{y}, \frac{y}{z})$.

解. (1)

$$\frac{\partial u}{\partial x} = f_1'(x+y, xy) + yf_2'(x+y, xy)$$
$$\frac{\partial u}{\partial y} = f_1'(x+y, xy) + xf_2'(x+y, xy)$$

(2)

$$\frac{\partial u}{\partial x} = f_1'(x, xy, xyz) + yf_2'(x, xy, xyz) + yzf_3'(x, xy, xyz),$$
$$\frac{\partial u}{\partial y} = xf_2'(x, xy, xyz) + xzf_3'(x, xy, xyz),$$
$$\frac{\partial u}{\partial z} = xyf_3'(x, xy, xyz).$$

(3)

$$\frac{\partial u}{\partial x} = \frac{1}{y}f_1'(\frac{x}{y}, \frac{y}{z}),$$
$$\frac{\partial u}{\partial y} = -\frac{x}{y^2}f_1'(\frac{x}{y}, \frac{y}{z}) + \frac{1}{z}f_2'(\frac{x}{y}, \frac{y}{z}),$$
$$\frac{\partial u}{\partial z} = -\frac{y}{z^2}f_2'(\frac{x}{y}, \frac{y}{z})$$

□

9.4.7 设 $u = x^2y - xy^2$, 且 $x = r\cos\theta$,$y = r\sin\theta$. 求 $\frac{\partial u}{\partial r}$,$\frac{\partial u}{\partial \theta}$.

解.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

计算有

$$\frac{\partial u}{\partial x} = 2xy - y^2, \quad \frac{\partial u}{\partial y} = x^2 - 2xy.$$
$$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial y}{\partial r} = \sin\theta.$$
$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \qquad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

代入有

$$\frac{\partial u}{\partial r}=3r^2(\cos^2\theta\sin\theta-\sin^2\theta\cos\theta),$$
$$\frac{\partial u}{\partial r}=r^3(\sin^3\theta-2\sin^2\theta\cos\theta-2\sin\theta\cos^2\theta+\cos^3\theta).$$

□

9.4.8 设 $f(x,y,z)=F(u,v,w)$, 其中 $x^2=vw,y^2=wu,z^2=uv$, 求证:

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=u\frac{\partial F}{\partial u}+v\frac{\partial F}{\partial v}+w\frac{\partial F}{\partial w}$$

解. 下面都考虑导数存在，故不考虑 $u,v,w=0$ 的情形由

$$\frac{\partial F}{\partial u}=\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u},\frac{\partial F}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial v},\frac{\partial F}{\partial w}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial w},$$

代入要证的式子，则只用证明

$$x=\frac{\partial x}{\partial v}v+\frac{\partial x}{\partial w}w$$

其余两条同理，由于 $x^2=vw$ 两边分别对 v 和 w 求导

$$2x\frac{\partial x}{\partial v}=w,2x\frac{\partial x}{\partial w}=v.$$

代入即有等式成立. □

9.4.9 求以下的 $\mathbf{J}(\boldsymbol{f}\circ\boldsymbol{g})$.

(2)$\boldsymbol{f}(x,y)=(\varphi(x+y),\varphi(x-y)),\boldsymbol{g}(s,t)=(e^t,e^{-t})$;

(3)$\boldsymbol{f}(x,y,z)=(x^2+y+z,2x+y+z^2,0)$, $\boldsymbol{g}(u,v,w)=(uv^2w^2,w^2\sin v,u^2e^v)$.

解. (2)

$$\mathbf{J}\boldsymbol{f}=\begin{pmatrix}\varphi'(x+y) & \varphi'(x+y)\\ \varphi'(x-y) & -\varphi'(x-y)\end{pmatrix},\mathbf{J}\boldsymbol{g}=\begin{pmatrix}0 & e^t\\ 0 & -e^{-t}\end{pmatrix},$$

由链式法则

$$\mathbf{J}(\boldsymbol{f}\circ\boldsymbol{g})=\mathbf{J}\boldsymbol{f}\mathbf{J}\boldsymbol{g}=\begin{pmatrix}0 & \varphi'(e^t+e^{-t})*e^t-\varphi'(e^t+e^{-t})*e^{-t}\\ 0 & \varphi'(e^t-e^{-t})*e^t+\varphi'(e^t-e^{-t})*e^{-t}\end{pmatrix}.$$

(3)

$$\mathbf{J}\boldsymbol{f}=\begin{pmatrix}2x & 1 & 1\\ x & 1 & 2z\\ 0 & 0 & 0\end{pmatrix},\mathbf{J}\boldsymbol{g}=\begin{pmatrix}v^2w^2 & 2uvw^2 & 2uv^2w\\ 0 & w^2\cos v & 2w\sin v\\ 2ue^v & u^2e^v & 0\end{pmatrix},$$

由链式法则

$$\mathbf{J}(\boldsymbol{f}\circ\boldsymbol{g})=\mathbf{J}\boldsymbol{f}\mathbf{J}\boldsymbol{g}=\begin{pmatrix}2uv^4w^4+2ue^v & 4u^2v^3w^4+w^2\cos v+u^2e^v & 4u^2v^4w^3+2w\sin v\\ 2v^2w^2+4u^3e^{2v} & 4uvw^2+w^2\cos v+2u^4e^{2v} & 4uv^2w+2w\sin v\\ 0 & 0 & 0\end{pmatrix}.$$

□

9.4.10 设函数 $f(x,y,z)$ 在 $\mathbb{R}^3$ 中可微，$\mathbf{u}$ 是一个方向，函数 f 沿方向 $\mathbf{u}$ 的方向导数记作 $\frac{\partial f}{\partial \mathbf{u}}$; 又设 $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ 是 $\mathbb{R}^3$ 中三个互相垂直的方向. 求证:

$$\left(\frac{\partial f}{\partial \mathbf{e}_1}\right)^2 + \left(\frac{\partial f}{\partial \mathbf{e}_2}\right)^2 + \left(\frac{\partial f}{\partial \mathbf{e}_3}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2$$

解. 记

$$A = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$$

则由方向导数计算方法得到

$$\begin{pmatrix} \frac{\partial f}{\partial \mathbf{e}_1} \\ \frac{\partial f}{\partial \mathbf{e}_2} \\ \frac{\partial f}{\partial \mathbf{e}_3} \end{pmatrix} = A^T \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

由 A 为正交阵带入得

$$\left(\frac{\partial f}{\partial \mathbf{e}_1}\right)^2 + \left(\frac{\partial f}{\partial \mathbf{e}_2}\right)^2 + \left(\frac{\partial f}{\partial \mathbf{e}_3}\right)^2 = \begin{pmatrix} \frac{\partial f}{\partial \mathbf{e}_1} & \frac{\partial f}{\partial \mathbf{e}_2} & \frac{\partial f}{\partial \mathbf{e}_3} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial \mathbf{e}_1} \\ \frac{\partial f}{\partial \mathbf{e}_2} \\ \frac{\partial f}{\partial \mathbf{e}_3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} AA^T \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial Z} \end{pmatrix} = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2$$

□

9.5.2 设参数曲线

$$\boldsymbol{r}(t) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, 1\right)$$

求证：在任意 $t \in \mathbb{R}$, 径向量 $\boldsymbol{r}(t)$ 与切向量 $\boldsymbol{r}^{'}(t)$ 互相正交. 问这是一条什么曲线?

解.

$$\boldsymbol{r}'(t) = \left(\frac{2-2t^2}{(1+t^2)^2}, \frac{-4t}{(1+t^2)^2}, 0\right)$$

于是我们有

$$\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = 0$$

故正交. 计算可得 $||\boldsymbol{r}|| = 1$, 且这条曲线落在 xy 平面内, 故为圆. □

9.5.3 讨论平面曲线

$$\boldsymbol{r}(t) = (e^t \cos t, e^t \sin t)$$

求证: 在曲线上的每一点处，切向量和径向量交成定角 $\frac{\pi}{4}$.

解.

$$\boldsymbol{r}'(t) = (e^t(\cos t - \sin t), e^t(\sin t + \cos t))$$

于是我们有

$$\cos\theta = \frac{\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t)}{||\boldsymbol{r}(t)|| \cdot ||\boldsymbol{r}'(t)||} = \frac{1}{\sqrt{2}}$$

故切向量和径向量交成定角 $\frac{\pi}{4}$. □

9.5.5 证明:

(1)$(\lambda(t)\boldsymbol{a}(t))' = \lambda'(t)\boldsymbol{a}(t) + \lambda(t)\boldsymbol{a}'(t)$;

(2)$(\boldsymbol{b}(t)\cdot\boldsymbol{a}(t))' = \boldsymbol{b}'(t)\boldsymbol{a}(t) + \boldsymbol{b}(t)\boldsymbol{a}'(t)$;

(3)$(\boldsymbol{b}(t)\times\boldsymbol{a}(t))' = \boldsymbol{b}'(t)\times\boldsymbol{a}(t) + \boldsymbol{b}(t)\times\boldsymbol{a}'(t)$

解. (1)

$$\begin{aligned}(\lambda(t)\boldsymbol{a}(t))' &= (\lambda(t)\boldsymbol{a}_1(t), \lambda(t)\boldsymbol{a}_2(t), ..., \lambda(t)\boldsymbol{a}_n(t))' \\ &= (\lambda'(t)\boldsymbol{a}_1(t) + \lambda(t)\boldsymbol{a}_1'(t), \lambda'(t)\boldsymbol{a}_2(t) + \lambda(t)\boldsymbol{a}_2'(t), ..., \lambda'(t)\boldsymbol{a}_n(t) + \lambda(t)\boldsymbol{a}_n'(t),) \\ &= \lambda'(t)\boldsymbol{a}(t) + \lambda(t)\boldsymbol{a}'(t)\end{aligned}$$

(2)

$$\begin{aligned}(\boldsymbol{b}(t)\cdot\boldsymbol{a}(t))' &= (\boldsymbol{b}_1(t)\boldsymbol{a}_1(t) + \boldsymbol{b}_2(t)\boldsymbol{a}_2(t) + \cdots + b_n(t)a_n(t))' \\ &= \boldsymbol{b}_1'(t)\boldsymbol{a}_1(t) + \boldsymbol{b}_1(t)\boldsymbol{a}_1'(t) + \boldsymbol{b}_2'(t)\boldsymbol{a}_2(t) + \boldsymbol{b}_2(t)\boldsymbol{a}_2'(t) + \cdots + \boldsymbol{b}_n'(t)\boldsymbol{a}_n(t) + \boldsymbol{b}_n(t)\boldsymbol{a}_n'(t) \\ &= \boldsymbol{b}'(t)\boldsymbol{a}(t) + \boldsymbol{b}(t)\boldsymbol{a}'(t)\end{aligned}$$

(3) 考虑 $n = 3$ 的情形

$$\begin{aligned}(\boldsymbol{b}(t)\times\boldsymbol{a}(t))' &= (\boldsymbol{b}_2(t)\boldsymbol{a}_3(t) - \boldsymbol{b}_3(t)\boldsymbol{a}_2(t), \boldsymbol{b}_3(t)\boldsymbol{a}_1(t) - \boldsymbol{b}_1(t)\boldsymbol{a}_3(t), \boldsymbol{b}_1(t)\boldsymbol{a}_2(t) - \boldsymbol{b}_2(t)\boldsymbol{a}_1(t))' \\ &= (\boldsymbol{b}_2'(t)\boldsymbol{a}_3(t) - \boldsymbol{b}_3'(t)\boldsymbol{a}_2(t) + \boldsymbol{b}_2(t)\boldsymbol{a}_3'(t) - \boldsymbol{b}_3(t)\boldsymbol{a}_2'(t), \\ &\boldsymbol{b}_3'(t)\boldsymbol{a}_1(t) - \boldsymbol{b}_1'(t)\boldsymbol{a}_3(t) + \boldsymbol{b}_3(t)\boldsymbol{a}_1'(t) - \boldsymbol{b}_1(t)\boldsymbol{a}_3'(t), \boldsymbol{b}_1'(t)\boldsymbol{a}_2(t) - \boldsymbol{b}_2'(t)\boldsymbol{a}_1(t) + \boldsymbol{b}_1(t)\boldsymbol{a}_2'(t) - \boldsymbol{b}_2(t)\boldsymbol{a}_1'(t)) \\ &= \boldsymbol{b}'(t)\times\boldsymbol{a}(t) + \boldsymbol{b}(t)\times\boldsymbol{a}'(t).\end{aligned}$$

□

9.5.7 讨论椭圆

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, (0 \le t \le 2\pi).$$

(1) 求椭圆在每一点处的切向量;

(2) 证明椭圆的光学性质.

解. (1)

$$\boldsymbol{r}'(t) = (-a\sin t, b\cos t)$$

(2) 椭圆的焦点为 $F_1(-c,0)$, $F_2(c,0)$. 椭圆在 $P = \boldsymbol{x}(t)$ 点处的切向量为

$$\boldsymbol{n} = (b\cos t, a\sin t)$$

证明夹角相等只需证明

$$\frac{\overrightarrow{F_1P}\cdot\boldsymbol{n}}{||\overrightarrow{F_1P}||} = \frac{\overrightarrow{F_2P}\cdot\boldsymbol{n}}{||\overrightarrow{F_2P}||},$$

代入 $\overrightarrow{F_1P} = (a\cos t + c, b\sin t)$,$\overrightarrow{F_2P} = (a\cos t - c, b\sin t)$ 即可得到等式成立. □

9.5.8 求下列曲线的曲率:

(2)$\boldsymbol{r}(t) = (a(3t - t^3), 3at^2, a(3t + t^3))$, 其中常数 $a > 0$.

解. 分别计算 $\boldsymbol{r}'(t)$ 和 $\boldsymbol{r}''(t)$ 有

$$\boldsymbol{r}'(t) = (3a(1-t^2), 6at, 3a(1+t^2)), \boldsymbol{r}''(t) = (-6at, 6a, 6at),$$

则有

$$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = (18a^2(t^2-1), -36a^2t, 18a^2(t^2+1)),$$

代入曲率的计算公式有

$$\begin{aligned} k(t) &= \frac{||\boldsymbol{r}^{'}(t) \times \boldsymbol{r}^{''}(t)||}{||\boldsymbol{r}^{'}(t)||^3} \\ &= \frac{1}{3a(t^2+1)^2}. \end{aligned}$$

□

9.5.10 求法向量和切平面方程。

(3) $3x^2 + 2y^2 - 2z - 1 = 0$ 在 $(1,1,2)$ 处；

(4) $z = y + \log(x/z)$ 在 $(1,1,1)$ 处.

解. (3)$F(x,y,z) = 3x^2 + 2y^2 - 2z - 1$, 故其偏导

$$\frac{\partial F}{\partial x} = 6x, \frac{\partial F}{\partial y} = 4y, \frac{\partial F}{\partial z} = -2.$$

代入点的坐标得

$$\mathbf{J}F(\boldsymbol{p}_0) = (6, 4, -2)$$

故单位法向量为

$$\boldsymbol{n} = \frac{1}{\sqrt{15}}(3, 2, -1),$$

切平面为 $3(x-1) + 2(y-1) - (z-2) = 0$, 化简得

$$3x + 2y - z - 3 = 0.$$

(4)$F(x,y,z) = y + \log(x/z)$, 故其偏导

$$\frac{\partial F}{\partial x} = \frac{1}{x}, \frac{\partial F}{\partial y} = 1, \frac{\partial F}{\partial z} = -\frac{1}{z} - 1.$$

代入点的坐标得

$$\mathbf{J}F(\boldsymbol{p}_0) = (1, 1, -2)$$

故单位法向量为

$$\boldsymbol{n} = \frac{1}{\sqrt{6}}(1, 1, -2),$$

切平面为 $(x-1) + (y-1) - 2(z-1) = 0$, 化简得

$$x + y - 2z = 0.$$

□

9.5.11 求 $x^2 + 2y^2 + 3z^2 = 21$ 上所有平行于 $x + 4y + 6z = 0$ 的切平面。

解. $\mathbf{J}F(x,y,z)=(2x,4y,6z)$, 在 (x_0,y_0,z_0) 处的切平面方程为

$$x_0(x-x_0)+2y_0(y-y_0)+3z_0(z-z_0)=0.$$

故得到方程组

$$\begin{cases}x_0:2y_0:3z_0=1:4:6\\x_0^2+2y_0^2+3z_0^2=21\end{cases}$$

解得

$$(x_0,y_0,z_0)=\pm(1,2,2)$$

得到2 个满足条件得切平面 $(x-1)+4(y-2)+6(z-2)=0$ 和 $-(x+1)-4(y+2)-6(z+2)=0$, 化简得

$$x+4y+6z\pm 21=0.$$

□

注. 本次作业难度不高，但是计算量庞大，并且 *A2* 的重点也就是在于计算，所以大家一定要保证自己计算每一步都是正确的！