Week 13&14

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11.3.4 $f \in C^1, \Gamma$ 是任意一条分段光滑的封闭曲线。证明:
(1) $\displaystyle\int_\Gamma f(xy)(y\mathrm{d}x + x\mathrm{d}y) = 0$;
(2) $\displaystyle\int_\Gamma f(x^2+y^2)(x\mathrm{d}x + y\mathrm{d}y) = 0$;
(3) $\displaystyle\int_\Gamma f(x^n+y^n)(x^{n-1}\mathrm{d}x + y^{n-1}\mathrm{d}y) = 0$.

解. (1) $P(x,y) = f(xy)y, Q(x,y) = f(xy)x$,

$$\frac{\partial Q}{\partial x} = f(xy) + yf'(xy)x, \frac{\partial P}{\partial y} = f(xy) + xf'(xy)y$$

故

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

由 Green 公式知 $\displaystyle\int_\Gamma f(xy)(y\mathrm{d}x + x\mathrm{d}y) = 0$.
(3) $P(x,y) = f(x^n+y^n)x^{n-1}, Q(x,y) = f(x^n+y^n)y^{n-1}$,

$$\frac{\partial Q}{\partial x} = y^{n-1}f'(x^n+y^n)nx^{n-1}, \frac{\partial P}{\partial y} = x^{n-1}f'(x^n+y^n)ny^{n-1},$$

故

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

由 Green 公式知 $\displaystyle\int_\Gamma f(x^n+y^n)(x^{n-1}\mathrm{d}x + y^{n-1}\mathrm{d}y) = 0$. □

11.3.5 Γ 是 $\mathbb{R}^2$ 中的光滑封闭曲线，$\boldsymbol{n}$ 是单位外法向量，设 $\boldsymbol{a}$ 是一个固定的单位向量，求证:

$$\int_\Gamma \cos(\boldsymbol{a}, \boldsymbol{n})\mathrm{d}s = 0.$$

解. 设 $\boldsymbol{t}$ 是弧长参数下的切向量，则 $\boldsymbol{t}\mathrm{d}s = (\mathrm{d}x, \mathrm{d}y)$, 又有 $||\boldsymbol{t}|| = ||\boldsymbol{n}||$ 和 $\boldsymbol{n} \perp \boldsymbol{t}$, 故 $\boldsymbol{n}\mathrm{d}s = \pm(\mathrm{d}y, -\mathrm{d}x)$, 由 Green 公式

$$\int_\Gamma \cos(\boldsymbol{a}, \boldsymbol{n})\mathrm{d}s = \pm\int_\Gamma a_1\mathrm{d}y - a_2\mathrm{d}x = 0.$$

□

11.3.6 Γ 是光滑封闭曲线，$\boldsymbol{n}$ 是单位外法向量，计算

$$\int_\Gamma x\cos(\boldsymbol{n}, \boldsymbol{i}) + y\cos(\boldsymbol{n}, \boldsymbol{j})\mathrm{d}s.$$

解.

$$\int_{\Gamma} x\cos(\boldsymbol{n},\boldsymbol{i})+y\cos(\boldsymbol{n},\boldsymbol{j})\mathrm{d}s=\int_{\Gamma} x\boldsymbol{n}\cdot\boldsymbol{i}+y\boldsymbol{n}\cdot\boldsymbol{j}\mathrm{d}s=\int_{\Gamma} x\mathrm{d}y-y\mathrm{d}x=2\sigma(\Omega).$$

□

11.3.7 (1)$\int_L (x^2+2xy-y^2)\mathrm{d}x+(x^2-2xy-y^2)\mathrm{d}y, L$ 是连接 $A=(0,0), B=(2,1)$ 的任意光滑线段；

(2) $\int_L \frac{(x+y)\mathrm{d}x-(x-y)\mathrm{d}y}{x^2+y^2}$, 其中 L 是位于上半平面从点 $(-1,0)$ 到 $(1,0)$ 的任意光滑线段。

解. (1)$P(x,y)=x^2+2xy-y^2, Q(x,y)=x^2-2xy-y^2$.

$$\frac{\partial Q}{\partial x}=2(x-y), \frac{\partial P}{\partial y}=2(x-y)$$

故该积分与积分路径无关。选择道路为 $(0,0)\to(2,0)\to(2,1)$ 的折线，则

$$\int_L (x^2+2xy-y^2)\mathrm{d}x+(x^2-2xy-y^2)\mathrm{d}y=\int_0^2 x^2\mathrm{d}x+\int_0^1 (4-4y-y^2)\mathrm{d}y=\frac{13}{3}$$

(2)$P(x,y)=\frac{x+y}{x^2+y^2}, Q(x,y)=\frac{y-x}{x^2+y^2}$.

$$\frac{\partial Q}{\partial x}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2}, \frac{\partial P}{\partial y}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2}$$

故该积分与积分路径无关。选择道路为 $(-1,0)\to(1,0)$ 的单位圆弧（上半平面内），则

$$\int_L \frac{(x+y)\mathrm{d}x-(x-y)\mathrm{d}y}{x^2+y^2}=\int_{-\pi/2}^{\pi/2} -(\cos\theta+\sin\theta)\sin\theta\mathrm{d}\theta-(\cos\theta-\sin\theta)\cos\theta\mathrm{d}\theta=\pi$$

□

11.3.8 计算

$$\int_{\Gamma} \frac{e^x}{x^2+y^2}\big((x\sin y-y\cos y)\mathrm{d}x+(x\cos y+y\sin y)\mathrm{d}y\big),$$

其中 Γ 是原点在其内部的分段光滑的闭曲线。

解. 设 Γ_ε 为 Γ 围成的区域去掉小圆盘 $B_\varepsilon(0)$ 所得区域的边界。则

$$\begin{aligned}&\int_{\Gamma} \frac{e^x}{x^2+y^2}\big((x\sin y-y\cos y)\mathrm{d}x+(x\cos y+y\sin y)\mathrm{d}y\big)\\ =&\int_{\Gamma_1} \frac{e^x}{x^2+y^2}\big((x\sin y-y\cos y)\mathrm{d}x+(x\cos y+y\sin y)\mathrm{d}y\big)\\ &+\int_{\partial B_\varepsilon(0)} \frac{e^x}{x^2+y^2}\big((x\sin y-y\cos y)\mathrm{d}x+(x\cos y+y\sin y)\mathrm{d}y\big),\end{aligned}$$

$P(x,y)=\frac{e^x}{x^2+y^2}(x\sin y-y\cos y), Q(x,y)=\frac{e^x}{x^2+y^2}(x\cos y+y\sin y)$, 计算易得 $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$, 故

$$\int_{\Gamma_1} \frac{e^x}{x^2+y^2}\big((x\sin y-y\cos y)\mathrm{d}x+(x\cos y+y\sin y)\mathrm{d}y\big)=0,$$

$$
\begin{aligned}
&\int_{\partial B_\varepsilon(0)} \frac{e^x}{x^2+y^2}\big((x\sin y - y\cos y)\mathrm{d}x + (x\cos y + y\sin y)\mathrm{d}y\big)\\
=&\frac{1}{\varepsilon^2}\int_{\partial B_\varepsilon(0)} e^x\big((x\sin y - y\cos y)\mathrm{d}x + (x\cos y + y\sin y)\mathrm{d}y\big)\\
\overset{\text{Green}}{=}&\frac{1}{\varepsilon^2}\iint_{B_\varepsilon(0)} e^x 2\cos y\mathrm{d}x\mathrm{d}y \xrightarrow{\varepsilon\to 0+} \frac{1}{\varepsilon^2}\cdot 2\pi\varepsilon^2 = 2\pi
\end{aligned}
$$

故

$$
\begin{aligned}
&\int_\Gamma \frac{e^x}{x^2+y^2}\big((x\sin y - y\cos y)\mathrm{d}x + (x\cos y + y\sin y)\mathrm{d}y\big)\\
=&\int_{\Gamma_1} \frac{e^x}{x^2+y^2}\big((x\sin y - y\cos y)\mathrm{d}x + (x\cos y + y\sin y)\mathrm{d}y\big)\\
+&\int_{\partial B_\varepsilon(0)} \frac{e^x}{x^2+y^2}\big((x\sin y - y\cos y)\mathrm{d}x + (x\cos y + y\sin y)\mathrm{d}y\big)\\
=&2\pi
\end{aligned}
$$

□

12.1.1 锥面 $z=\sqrt{x^2+y^2}$ 被圆柱面 $x^2+y^2=2x$ 截下的部分的面积。

解.

$$
\begin{aligned}
&\frac{\partial z}{\partial x} = \frac{x}{z}, \frac{\partial z}{\partial y} = \frac{y}{z}\\
&\mathrm{d}\sigma = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\mathrm{d}x\mathrm{d}y = \sqrt{2}\mathrm{d}x\mathrm{d}y\\
&\sigma(D) = \iint\limits_{\substack{x^2+y^2\le 2x\\ z=\sqrt{x^2+y^2}}} \mathrm{d}\sigma = \iint\limits_{x^2+y^2\le 2x} \sqrt{2}\mathrm{d}x\mathrm{d}y = \sqrt{2}\pi
\end{aligned}
$$

□

12.1.3 圆柱面 $x^2+y^2=a^2$ 介于平面 $x\pm z=0$ 之间的部分的面积。

解.

$$
\begin{aligned}
&\boldsymbol{r} = (a\cos\theta, a\sin\theta, z)\\
&\text{if } x\ge 0, -x\le z\le x, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}.\\
&\boldsymbol{r}_\theta = (-a\sin\theta, a\cos\theta, 0), \boldsymbol{r}_z = (0,0,1)\\
&\boldsymbol{r}_\theta\times\boldsymbol{r}_z = (a\cos\theta, a\sin\theta, 0), ||\boldsymbol{r}_\theta\times\boldsymbol{r}_z|| = a,\\
&\sigma(D) = \iint\limits_{\substack{x^2+y^2=a^2\\ -|x|\le z\le|x|}} \mathrm{d}\sigma = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-a\cos\theta}^{a\cos\theta} a\mathrm{d}z\mathrm{d}\theta = 8a^2
\end{aligned}
$$

□

12.1.5 马鞍面 $az=xy$ 被圆柱面 $x^2+y^2=a^2$ 截下的部分的面积。

解.

$$
\begin{aligned}
&\boldsymbol{r} = \Big(x, y, \frac{xy}{a}\Big), x^2 + y^2 \leq a \\
&\boldsymbol{r}_x = \Big(1, 0, \frac{y}{a}\Big), \boldsymbol{r}_y = \Big(0, 1, \frac{x}{a}\Big) \\
&\boldsymbol{r}_x \times \boldsymbol{r}_y = \Big(-\frac{y}{a}, \frac{x}{a}, 1\Big), ||\boldsymbol{r}_x \times \boldsymbol{r}_y|| = \sqrt{1 + \frac{x^2 + y^2}{a^2}} \\
&\sigma(D) = \iint\limits_{x^2+y^2 \leq a} \sqrt{1 + \frac{x^2 + y^2}{a^2}} \mathrm{d}x\mathrm{d}y = \frac{2\pi a^2(2\sqrt{2} - 1)}{3}
\end{aligned}
$$

□

12.1.7 螺旋面 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = h\theta \end{cases}, (0 < r < a, 0 \leq \theta \leq 2\pi)$ 的面积。

解.

$$
\begin{aligned}
&\boldsymbol{R}(r, \theta) = (r\cos\theta, r\sin\theta, h) \\
&\boldsymbol{R}_r = (\cos\theta, \sin\theta, 0), \boldsymbol{R}_\theta = (-r\sin\theta, r\cos\theta, h) \\
&\boldsymbol{R}_r \times \boldsymbol{R}_\theta = (h\sin\theta, -h\cos\theta, r), ||\boldsymbol{R}_r \times \boldsymbol{R}_\theta|| = \sqrt{h^2 + r^2} \\
&\sigma(D) = \iint\limits_{\substack{0<r<a \\ 0\leq\theta\leq 2\pi}} \sqrt{h^2 + r^2}\mathrm{d}r\mathrm{d}\theta = 2\pi \int_0^{\frac{a}{h}} h^2\sqrt{1 + t^2}\mathrm{d}t
\end{aligned}
$$

计算 $\int_0^x \sqrt{1 + t^2}\mathrm{d}t$: 作换元 $t = \sinh u$, 则 $u = \log(t + \sqrt{1 + t^2})$.

$$
\int_0^x \sqrt{1 + t^2}\mathrm{d}t = \int_0^{u_0} \cosh u \mathrm{d}\sinh u = \int_0^{u_0} \cosh^2 u\mathrm{d}u = \int_0^{u_0} \cosh^2 u\mathrm{d}u = \frac{1}{4}\sinh 2u_0 + \frac{u_0}{2} = \frac{1}{2}x\sqrt{1 + x^2} + \frac{1}{2}\log(x + \sqrt{1 -
$$

故带入上式得

$$
\sigma(D) = 2\pi h^2\left[\frac{1}{2}\frac{a}{h}\sqrt{1 + \frac{a^2}{h^2}} + \frac{1}{2}\log\left(\frac{a}{h} + \sqrt{1 + \frac{a^2}{h^2}}\right)\right] = \pi a\sqrt{h^2 + a^2} + \pi h^2\log\left(\frac{a}{h} + \sqrt{1 + \frac{a^2}{h^2}}\right)
$$

□

12.2.1 $\int_\Sigma \frac{\mathrm{d}\sigma}{(1 + x + y)^2}, \Sigma$ 是四面体 $x + y + z \leq 1(x, y, z \geq 0)$ 的边界。

解. 令 $\Sigma_1 = \Sigma \cap \{x + y + z = 1\}, \Sigma_2 = \Sigma \cap \{x = 0\}, \Sigma_3 = \Sigma \cap \{y = 0\}, \Sigma_4 = \Sigma \cap \{z = 0\}$. 分别计算。
Σ_1:

$$
\begin{aligned}
&\boldsymbol{r} = (x, y, 1 - x - y), \boldsymbol{r}_x = (1, 0, -1), \boldsymbol{r}_y = (0, 1, -1) \\
&\boldsymbol{r}_x \times \boldsymbol{r}_y = (1, 1, 1), ||\boldsymbol{r}_x \times \boldsymbol{r}_y|| = \sqrt{3} \\
&\int_{\Sigma_1} \frac{\mathrm{d}\sigma}{(1 + x + y)^2} = \iint\limits_{\substack{0\leq x+y\leq 1 \\ 0\leq x\leq 1 \\ 0\leq y\leq 1}} \frac{\sqrt{3}}{(1 + x + y)^2}\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-x} \frac{\sqrt{3}}{(1 + x + y)^2}\mathrm{d}y\mathrm{d}x = \sqrt{3}\big(\log 2 - \frac{1}{2}\big).
\end{aligned}
$$

Σ_2:

$$\boldsymbol{r}=(0,y,z),||\boldsymbol{r}_y\times\boldsymbol{r}_z||=1$$

$$\int_{\Sigma_2}\frac{\mathrm{d}\sigma}{(1+x+y)^2}=\iint\limits_{\substack{0\le y+z\le 1\\0\le y\le 1\\0\le z\le 1}}\frac{1}{(1+y)^2}\mathrm{d}y\mathrm{d}z=\int_0^1\int_0^{1-z}\frac{1}{(1+y)^2}\mathrm{d}y\mathrm{d}z=1-\log 2.$$

Σ_3: 由于 x 和 y 对称，故 $\displaystyle\int_{\Sigma_3}\frac{\mathrm{d}\sigma}{(1+x+y)^2}=\int_{\Sigma_2}\frac{\mathrm{d}\sigma}{(1+x+y)^2}$.

Σ_4:

$$\boldsymbol{r}=(x,y,0),||\boldsymbol{r}_x\times\boldsymbol{r}_y||=1$$

$$\int_{\Sigma_4}\frac{\mathrm{d}\sigma}{(1+x+y)^2}=\iint\limits_{\substack{0\le x+y\le 1\\0\le x\le 1\\0\le y\le 1}}\frac{1}{(1+x+y)^2}\mathrm{d}x\mathrm{d}y=\int_0^1\int_0^{1-x}\frac{1}{(1+x+y)^2}\mathrm{d}y\mathrm{d}x=\log 2-\frac{1}{2}$$

故

$$\begin{aligned}\int_{\Sigma}\frac{\mathrm{d}\sigma}{(1+x+y)^2}&=\int_{\Sigma_1}\frac{\mathrm{d}\sigma}{(1+x+y)^2}+\int_{\Sigma_2}\frac{\mathrm{d}\sigma}{(1+x+y)^2}+\int_{\Sigma_3}\frac{\mathrm{d}\sigma}{(1+x+y)^2}+\int_{\Sigma_4}\frac{\mathrm{d}\sigma}{(1+x+y)^2}\\&=(\sqrt{3}-1)\log 2+\frac{3-\sqrt{3}}{2}.\end{aligned}$$

□

12.2.2 $\displaystyle\int_{\Sigma}|xyz|\mathrm{d}\sigma,\Sigma:z=x^2+y^2,z\le 1.$

解.

$$\begin{aligned}&\boldsymbol{r}=(x,y,x^2+y^2),\boldsymbol{r}_x=(1,0,2x),\boldsymbol{r}_y=(0,1,2y)\\&\boldsymbol{r}_x\times\boldsymbol{r}_y=(-2x,-2y,1),||\boldsymbol{r}_x\times\boldsymbol{r}_y||=\sqrt{1+4(x^2+y^2)}\\&\int_{\Sigma}|xyz|\mathrm{d}\sigma=\iint\limits_{x^2+y^2\le 1}|xy|(x^2+y^2)\sqrt{1+4(x^2+y^2)}\mathrm{d}x\mathrm{d}y\\&=\int_0^{2\pi}\int_0^1 r^5\sqrt{4r^2+1}|\cos\theta\sin\theta|\mathrm{d}r\mathrm{d}\theta\\&=2\int_0^1 r^5\sqrt{4r^2+1}\mathrm{d}r\\&\overset{u=\sqrt{4r^2+1}}{=}\int_1^{\sqrt{5}}u^2\Big(\frac{u^2-1}{4}\Big)^2\mathrm{d}u\\&=\frac{125\sqrt{5}-1}{420}.\end{aligned}$$

□

12.2.3 $\displaystyle\int_{\Sigma}(xy+yz+zx)\mathrm{d}\sigma,\Sigma:z=\sqrt{x^2+y^2}$ 被圆柱面 $x^2+y^2=2x$ 截下的部分。

解.

$$\begin{aligned}&\boldsymbol{r}=(x,y,\sqrt{x^2+y^2}),\boldsymbol{r}_x=(1,0,\frac{x}{z}),\boldsymbol{r}_y=(0,1,\frac{y}{z})\\&\boldsymbol{r}_x\times\boldsymbol{r}_y=(-\frac{x}{z},-\frac{y}{z},1),||\boldsymbol{r}_x\times\boldsymbol{r}_y||=\sqrt{1+\frac{x^2+y^2}{z^2}}=\sqrt{2}\end{aligned}$$

$$\int_\Sigma (xy+yz+zx)\mathrm{d}\sigma = \iint\limits_{x^2+y^2\le 2x} \sqrt{2}\big(xy+(x+y)\sqrt{x^2+y^2}\big)\mathrm{d}x\mathrm{d}y$$
$$=\sqrt{2}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\cos\theta\sin\theta + r^3(\cos\theta+\sin\theta)\mathrm{d}r\mathrm{d}\theta$$
$$=\frac{64\sqrt{2}}{15}$$

□

12.3.1 计算第二型曲面积分。

(1) $\iint_\Sigma x^4\mathrm{d}y\mathrm{d}z + y^4\mathrm{d}z\mathrm{d}x + z^4\mathrm{d}x\mathrm{d}y, \Sigma: x^2+y^2+z^2=a^2$, 内侧；

(2) $\iint_\Sigma xz\mathrm{d}y\mathrm{d}z + yz\mathrm{d}z\mathrm{d}x + x^2\mathrm{d}x\mathrm{d}y, \Sigma: x^2+y^2+z^2=a^2$, 外侧；

(3) $\iint_\Sigma f(x)\mathrm{d}y\mathrm{d}z + g(y)\mathrm{d}z\mathrm{d}x + h(z)\mathrm{d}x\mathrm{d}y, \Sigma: [0,a]\times[0,b]\times[0,c]$ 的边界，外侧；

(4) $\iint_\Sigma z\mathrm{d}x\mathrm{d}y, \Sigma: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, 外侧；

(5) $\iint_\Sigma (y-z)\mathrm{d}y\mathrm{d}z + (z-x)\mathrm{d}z\mathrm{d}x + (x-y)\mathrm{d}x\mathrm{d}y, \Sigma: z=\sqrt{x^2+y^2}, z\le h$, 外侧。

解. (1) $\iint_\Sigma x^4\mathrm{d}y\mathrm{d}z = \iint_{\Sigma\cap\{x>0\}} x^4\mathrm{d}y\mathrm{d}z + \iint_{\Sigma\cap\{x<0\}} x^4\mathrm{d}y\mathrm{d}z, \boldsymbol{n} = -\frac{(x,y,z)}{a}$, 故在 $\Sigma\cap\{x>0\}$ 和 $\Sigma\cap\{x<0\}$ 上对应位置的法向量是反向的，由对称性知 $\iint_\Sigma x^4\mathrm{d}y\mathrm{d}z = 0$, 又因为 x,y,z 三者对称，故

$$\iint_\Sigma x^4\mathrm{d}y\mathrm{d}z + y^4\mathrm{d}z\mathrm{d}x + z^4\mathrm{d}x\mathrm{d}y = 0.$$

(2) $\boldsymbol{F} = (xz, yz, x^2), \boldsymbol{n} = \frac{(x,y,z)}{a}$.

$$\iint_\Sigma xz\mathrm{d}y\mathrm{d}z + yz\mathrm{d}z\mathrm{d}x + x^2\mathrm{d}x\mathrm{d}y = \iint_\Sigma \frac{1}{a}(x^2z+y^2z+x^2z)\mathrm{d}\sigma = \frac{1}{a}\iint_\Sigma (2x^2+y^2)z\mathrm{d}\sigma \overset{\text{关于}z\text{奇}}{=\!=\!=\!=} 0.$$

(3) 根据各面的法向量得到

$$\iint_\Sigma f(x)\mathrm{d}y\mathrm{d}z + g(y)\mathrm{d}z\mathrm{d}x + h(z)\mathrm{d}x\mathrm{d}y$$
$$= \iint\limits_{\Sigma\cap\{x=a\}} f(x)\mathrm{d}\sigma + \iint\limits_{\Sigma\cap\{x=0\}} -f(x)\mathrm{d}\sigma + \iint\limits_{\Sigma\cap\{y=b\}} g(y)\mathrm{d}\sigma + \iint\limits_{\Sigma\cap\{y=0\}} -g(y)\mathrm{d}\sigma + \iint\limits_{\Sigma\cap\{z=c\}} h(z)\mathrm{d}\sigma + \iint\limits_{\Sigma\cap\{z=0\}} -h(z)\mathrm{d}\sigma$$
$$=\big(f(a)-f(0)\big)bc + \big(g(b)-g(0)\big)ac + \big(h(c)-h(0)\big)ab.$$

(4)

$$\iint_\Sigma z\mathrm{d}x\mathrm{d}y = \iint\limits_{\Sigma\cap\{z>0\}} z\mathrm{d}x\mathrm{d}y + \iint\limits_{\Sigma\cap\{z<0\}} z\mathrm{d}x\mathrm{d}y$$
$$= 2c\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\mathrm{d}x\mathrm{d}y$$
$$= \frac{4\pi abc}{3}$$

(5)

$$\iint_{\Sigma}(y-z)\mathrm{d}y\mathrm{d}z=\iint_{\Sigma\cap\{x>0\}}(y-z)\mathrm{d}y\mathrm{d}z+\iint_{\Sigma\cap\{x<0\}}(y-z)\mathrm{d}y\mathrm{d}z$$

由图形知 $x>0$ 和 $x<0$ 时法向相反，故 $\iint_{\Sigma\cap\{x>0\}}(y-z)\mathrm{d}y\mathrm{d}z=-\iint_{\Sigma\cap\{x<0\}}(y-z)\mathrm{d}y\mathrm{d}z$, 从而

$$\iint_{\Sigma}(y-z)\mathrm{d}y\mathrm{d}z=\iint_{\Sigma\cap\{x>0\}}(y-z)\mathrm{d}y\mathrm{d}z+\iint_{\Sigma\cap\{x<0\}}(y-z)\mathrm{d}y\mathrm{d}z=0.$$

再由 x 和 y 的对称性知 $\iint_{\Sigma}(z-x)\mathrm{d}z\mathrm{d}x=0.$

$$\iint_{\sigma}(x-y)\mathrm{d}x\mathrm{d}y=-\iint_{\sigma}(y-x)\mathrm{d}\sigma=-\iint\limits_{x^2+y^2\le h^2}(y-x)\mathrm{d}x\mathrm{d}y\overset{*}{=}0,$$

其中 * 处是因为积分关于 x,y 是对称的。 □

12.3.2 给定流速场 $\boldsymbol{F}=(y,z,x)$, 封闭曲面 $x^2+y^2=R^2, z=0, z=h$. 计算 $\boldsymbol{F}$ 流向曲面之外的流量。

解. 设 $\boldsymbol{n}$ 是曲面的外法向，则流量为 $\iint_{\Sigma}\boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}\sigma.$

记 $\Sigma_1: x^2+y^2=R^2, 0\le z\le h, \Sigma_2: x^2+y^2\le R^2, z=0, \Sigma_3: x^2+y^2\le R^2, z=h.$

在这三部分上的法向量分别是 $\boldsymbol{n}_1=\frac{(x,y,0)}{R}, \boldsymbol{n}_2=(0,0,-1), \boldsymbol{n}_3=(0,0,1).$

分别计算

$$\begin{aligned}
\iint_{\Sigma_1}\boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}\sigma&=\iint_{\Sigma_1}\frac{xy+yz}{R}\mathrm{d}\sigma=0 &&(\text{关于 } y \text{ 奇});\\
\iint_{\Sigma_2}\boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}\sigma&=\iint_{\Sigma_2}-x\mathrm{d}\sigma=0 &&(\text{关于 } x \text{ 奇});\\
\iint_{\Sigma_3}\boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}\sigma&=\iint_{\Sigma_3}x\mathrm{d}\sigma=0 &&(\text{关于 } x \text{ 奇}).
\end{aligned}$$

故总流量为 0. □

12.3.3 设 $\Sigma: z=f(x,y), (x,y)\in D$, 法向量向上，求证：

(1) $\iint\limits_{\Sigma}P\mathrm{d}y\mathrm{d}z=-\iint\limits_{D}P\big(x,y,f(x,y)\big)\dfrac{\partial f}{\partial x}\mathrm{d}x\mathrm{d}y;$

(2) $\iint\limits_{\Sigma}Q\mathrm{d}z\mathrm{d}x=-\iint\limits_{D}Q\big(x,y,f(x,y)\big)\dfrac{\partial f}{\partial y}\mathrm{d}x\mathrm{d}y.$

解.

$$\boldsymbol{r}=\big(x,y,f(x,y)\big), \boldsymbol{r}_x=(1,0,f_x), \boldsymbol{r}_y=(0,1,f_y)$$

$$\boldsymbol{r}_x\times\boldsymbol{r}_y=(-f_x,-f_y,1), \boldsymbol{n}=\frac{\boldsymbol{r}_x\times\boldsymbol{r}_y}{||\boldsymbol{r}_x\times\boldsymbol{r}_y||}$$

(1)

$$\iint\limits_{\Sigma}P\mathrm{d}y\mathrm{d}z=\iint\limits_{\Sigma}P\cdot\frac{-f_x}{||\boldsymbol{r}_x\times\boldsymbol{r}_y||}\mathrm{d}\sigma=\iint\limits_{\Sigma}P\cdot\frac{-f_x}{||\boldsymbol{r}_x\times\boldsymbol{r}_y||}||\boldsymbol{r}_x\times\boldsymbol{r}_y||\mathrm{d}x\mathrm{d}y=-\iint\limits_{D}P\big(x,y,f(x,y)\big)\frac{\partial f}{\partial x}\mathrm{d}x\mathrm{d}y.$$

(2) 同理。 □