Week 11

潘晨翔、王曹励文

2024 年 5 月 20 日

10.6.1 计算下列二重积分：

(1) $\displaystyle\iint_D (x-y)^2\sin(x+y)dxdy$, 其中 D 是由四点 $(\pi,0),(2\pi,\pi),(\pi,2\pi),(0,\pi)$ 顺次连成的正方形.

(2) $\displaystyle\iint_D (x^2+y^2)dxdy$, 其中 D 是由曲线 $x^2-y^2=1$,$x^2-y^2=2$,$xy=1$ 和 $xy=2$ 围成的图像在第一象限的那一部分.

解. (1) 令

$$\begin{cases} u = x - y \\ v = x + y, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2},$$

对应的积分区域化为

$$F = [-\pi,\pi]\times[\pi,3\pi],$$

带入由换元公式得到

$$\begin{aligned}\iint\limits_D (x-y)^2\sin(x+y)dxdy &= \iint\limits_F u^2\sin v|\frac{1}{2}|dudv \\ &= \frac{1}{2}\int_{-\pi}^{\pi} u^2du\int_{\pi}^{3\pi}\sin vdv \\ &= 0.\end{aligned}$$

(2) 令

$$\begin{cases} u = x^2 - y^2 \\ v = xy, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2) \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(x^2+y^2)},$$

对应的积分区域化为

$$F = [1,2]\times[1,2],$$

带入由换元公式得到

$$\begin{aligned}\iint\limits_D x^2+y^2dxdy &= \iint\limits_F |\frac{1}{2}|dudv \\ &= \frac{1}{2}\iint\limits_F dudv \\ &= \frac{1}{2}.\end{aligned}$$

□

10.6.2 计算下列围成的图像的面积:
(1)$(a_1x+b_1y+c_1)^2+(a_2x+b_2y+c_2)^2=1$, 其中 $a_1b_2\neq a_2b_1$;
(2)$\sqrt{x}+\sqrt{y}=\sqrt{a}$,$x=0$ 和 $y=0$.

解. (1) 令

$$\begin{cases} u=a_1x+b_1y+c_1 \\ v=a_2x+b_2y+c_2, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1 \qquad \frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{a_1b_2-a_2b_1},$$

对应的积分区域化为

$$F=\{(u,v)\in\mathbb{R}^2|u^2+v^2\leq 1\},$$

带入由换元公式得到

$$\begin{aligned}\iint\limits_D dxdy &= \iint\limits_F |\frac{1}{a_1b_2-a_2b_1}|dudv \\ &= \frac{\pi}{|a_1b_2-a_2b_1|}.\end{aligned}$$

(2) 令

$$\begin{cases} x=r^2\cos^4\theta \\ y=r^2\sin^4\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix} 2r\cos^4\theta & -4r^2\cos^3\theta\sin\theta \\ 2r\sin^4\theta & 4r^2\sin^3\theta\cos\theta \end{vmatrix}=8r^3\sin^3\theta\cos^3\theta,$$

对应的积分区域化为

$$F=[0,\sqrt{a}]\times[0,\frac{\pi}{2}],$$

带入由换元公式得到

$$\begin{aligned}\iint\limits_D dxdy &= \int_0^{\sqrt{a}}dr\int_0^{\frac{\pi}{2}}8r^3\sin^3\theta\cos^3\theta d\theta \\ &= \frac{a^2}{6}.\end{aligned}$$

□

10.6.3 求证:

$$\iint\limits_{|x|+|y|\le 1} f(x+y)dxdy = \int_{-1}^{1} f(t)dt.$$

解. (1) 令

$$\begin{cases} u = x+y \\ v = x-y, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2 \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2},$$

对应的积分区域化为

$$F = [-1,1]\times[-1,1],$$

带入由换元公式得到

$$\begin{aligned} \iint\limits_{|x|+|y|\le 1} f(x+y)dxdy &= \iint\limits_{[-1,1]^2} f(u)|\frac{1}{2}|dudv \\ &= \int_{-1}^{1} f(t)dt. \end{aligned}$$

□

10.6.5 计算下列二重积分:

(2) $\iint\limits_{x^2+y^2\le Rx} \sqrt{R^2-x^2-y^2}dxdy$;

(4) $\iint\limits_{x^2+y^2\le x+y} \sqrt{x^2+y^2}dxdy$.

解. (2) 令

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

对应的积分区域化为

$$\{(r,\theta)\in\mathbb{R}^2 | 0\le r\le R\cos\theta, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\},$$

带入由换元公式得到

$$\begin{aligned} \iint\limits_{x^2+y^2\le Rx} \sqrt{R^2-x^2-y^2}dxdy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{R\cos\theta} \sqrt{R^2-r^2}rdr \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}(R^3 - R^3|\sin^3\theta|)d\theta \\ &= (\frac{\pi}{3}-\frac{4}{9})R^3. \end{aligned}$$

(4) 令

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

对应的积分区域化为

$$\{(r,\theta)\in\mathbb{R}^2 | 0\le r\le(\cos\theta+\sin\theta), -\frac{\pi}{4}\le\theta\le\frac{3\pi}{4}\},$$

带入由换元公式得到

$$\begin{aligned}\iint\limits_{x^2+y^2\le x+y} \sqrt{x^2+y^2}dxdy &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^{\cos\theta+\sin\theta} r^2 dr \\ &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{(\cos\theta+\sin\theta)^3}{3} d\theta \\ &= \frac{8\sqrt{2}}{9}.\end{aligned}$$

□

10.6.7 设常数 $a, b > 0$,

$$D = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \ x \ge y \ge 0\}.$$

计算二重积分

$$\iint_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}dxdy.$$

解. 令

$$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr,$$

对应的积分区域化为

$$\{(r,\theta)\in\mathbb{R}^2 | 0\le r\le 1, 0\le\theta\le\arctan\frac{a}{b}\},$$

带入由换元公式得到

$$\begin{aligned}\iint\limits_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}dxdy &= \int_0^{\arctan\frac{a}{b}} d\theta \int_0^1 abr^2 dr \\ &= \frac{1}{3}ab\arctan\frac{a}{b}.\end{aligned}$$

□

10.7.1 计算以下积分:

(1)$\int_V xyzd\mu$,V 为球体 $x^2+y^2+z^2\le 1$ 在第一卦限中的部分;

(2)$\int_V (x+y+z)d\mu$,V 为平面 $x+y+z=1$ 和上和坐标平面所围成的立体;

(3)$\int_V xy^2z^3dxdydz$,V 由 $z=xy$ 和 $z=0$ 以及两张平面 $x=1$ 和 $x=y$ 围成.

解. (1) 令

$$\begin{cases} x=r\sin\theta\cos\phi \\ y=r\sin\theta\sin\phi \\ z=r\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}=r^2\sin\theta,$$

对应的积分区域化为

$$D=[0,1]\times[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}],$$

带入由换元公式得到

$$\begin{aligned}\int_V xyzd\mu &= \iiint\limits_D r^5\sin^3\theta\cos\theta\sin\phi\cos\phi drd\theta d\phi \\ &= \frac{1}{48}.\end{aligned}$$

(2)

$$\begin{aligned}\int_V (x+y+z)d\mu &= \int_V xd\mu+\int_V yd\mu+\int_V zd\mu \\ &= 3\int_V xd\mu \\ &= 3\int_0^1 xdx\iint\limits_{\substack{y+z\le 1-x\\ y,z\ge 0}} dydz \\ &= 3\int_0^1 \frac{1}{2}x(1-x)^2dx \\ &= \frac{1}{8}.\end{aligned}$$

(3)

$$\begin{aligned}\int_V xy^2z^3d\mu &= \int_0^1 xdx\int_0^x y^2dy\int_0^{xy} z^3dz \\ &= \frac{1}{364}.\end{aligned}$$

□

10.7.2 计算下列曲面围成的立体的体积:

(1)$z^2=x^2+\frac{y^2}{4}, 2z=x^2+\frac{y^2}{4}$;

(2)$x^2+y^2=a^2, |x|+|y|=a$,

解. (1)

$$V=\iint\limits_{x^2+\frac{y^2}{4}\leq 4}(\sqrt{x^2+\frac{y^2}{4}}-\frac{x^2}{2}-\frac{y^2}{8})dxdy$$

令

$$\begin{cases}x=r\cos\theta\\ y=2r\sin\theta,\end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ 2\sin\theta & 2r\cos\theta\end{vmatrix}=2r,$$

对应的积分区域化为

$$D=[0,2]\times[0,2\pi],$$

带入由换元公式得到

$$\iint\limits_{x^2+\frac{y^2}{4}\leq 4}(\sqrt{x^2+\frac{y^2}{4}}-\frac{x^2}{2}-\frac{y^2}{8})dxdy=\int_0^{2\pi}d\theta\int_0^2(r-\frac{r^2}{2})2rdr\quad=\frac{8\pi}{3}.$$

(2)

$$\begin{aligned}V&=8\iint\limits_{\substack{x^2+y^2\leq a^2\\ x,z\geq 0}}dxdy\int_0^{a-x}dy\\&=8\iint\limits_{\substack{x^2+y^2\leq a^2\\ x,z\geq 0}}(a-x)dxdz\\&=8\iint\limits_{[0,a]\times[0,2\pi]}(a-r\cos\theta)rdrd\theta\\&=(2\pi-\frac{8}{3})a^3.\end{aligned}$$

□

10.7.3 设 f 为连续函数, 求极限:

$$\lim_{r\to 0}\frac{3}{4\pi r^3}\iiint\limits_{x^2+y^2+z^2\leq r^2}f(x,y,z)dxdydz.$$

解. 由积分中值定理得

$$\begin{aligned}\lim_{r\to 0}\frac{3}{4\pi r^3}\iiint\limits_{x^2+y^2+z^2\leq r^2}f(x,y,z)dxdydz&=\lim_{r\to 0}\frac{3}{4\pi r^3}f(\xi,\eta,\phi)\frac{4\pi r^3}{3}\\&=\lim_{r\to 0}f(\xi,\eta,\phi)\\&=f(0,0,0).\end{aligned}$$

□

10.7.4 计算下列积分:

(1) $\displaystyle\iiint\limits_{x^2+y^2+z^2\leq 2}\sqrt{x^2+y^2+z^2}dxdydz$;

(2) $\displaystyle\iiint\limits_{D}(x^2+y^2)dxdydz$, 其中 $D=\{(x,y,z): z\geq 0, a^2\leq x^2+y^2+z^2\leq b^2\}$;

(3) $\displaystyle\iiint\limits_{D}\left(1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\right)^{1/2}dxdydz$, 其中 D 为椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 包围的立体.

解. (1) 令

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta,$$

对应的积分区域化为

$$D = [0,2]\times[0,\pi]\times[0,2\pi],$$

带入由换元公式得到

$$\begin{aligned}\iiint\limits_{x^2+y^2+z^2\leq 2}\sqrt{x^2+y^2+z^2}dxdydz &= \iiint\limits_{D} r^3\sin\theta drd\theta d\phi \\ &= 4\pi.\end{aligned}$$

(2) 令

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta,$$

对应的积分区域化为

$$E = [a,b]\times[0,\frac{\pi}{2}]\times[0,2\pi],$$

带入由换元公式得到

$$\begin{aligned}\iiint\limits_{D}(x^2+y^2)dxdydz &= \iiint\limits_{E} r^4\sin^3\theta drd\theta d\phi \\ &= \frac{4(b^5-a^5)\pi}{15}.\end{aligned}$$

(3) 令

$$\begin{cases} x = ar\sin\theta\cos\phi \\ y = br\sin\theta\sin\phi \\ z = cr\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} a\sin\theta\cos\phi & ar\cos\theta\cos\phi & -ar\sin\theta\cos\phi \\ b\sin\theta\sin\phi & br\cos\theta\sin\phi & br\sin\theta\cos\phi \\ c\cos\theta & -cr\sin\theta & 0 \end{vmatrix} = abcr^2\sin\theta,$$

对应的积分区域化为

$$E = [0,1]\times[0,\frac{\pi}{2}]\times[0,2\pi],$$

带入由换元公式得到

$$\begin{aligned} \iiint\limits_D \Big(1-\Big(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\Big)\Big)^{1/2} dxdydz &= abc\iiint\limits_E r^2\sqrt{1-r^2}\sin\theta drd\theta d\phi \\ &= \frac{abc\pi^2}{4}. \end{aligned}$$

□

10.7.5 计算由下列曲面围成的立体的体积:
(1)$a_ix+b_iy+c_iz=\pm h_i(i=1,2,3)$, 设行列式

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0;$$

(4)$(x^2+y^2+z^2)^n = z^{2n-1}(n\in\mathbf{N}^*)$;
(5)$\Big(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\Big)^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}$.

解. (1) 令

$$\begin{cases} u = a_1x+b_1y+c_1z \\ v = a_2x+b_2y+c_2z \\ w = a_3x+b_3y+c_3z, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

对应的积分区域化为

$$E = [-h_1,h_1]\times[-h_2,h_2]\times[-h_3,h_3],$$

带入由换元公式得到

$$V = \iiint\limits_E \left| \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^{-1} \right| dudvdw$$
$$= \frac{8h_1h_2h_3}{\left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} \right\|}.$$

(4) 令

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta,$$

对应的积分区域化为

$$E = \{(r,\theta,\phi)|0 \le r \le \cos^{2n-1}\theta, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le 2\pi\},$$

带入由换元公式得到

$$V = \iiint\limits_E r^2\sin\theta drd\theta d\phi$$
$$= \frac{\pi}{3(3n-1)}.$$

(5) 令

$$\begin{cases} x = ar\sin\theta\cos\phi \\ y = br\sin\theta\sin\phi \\ z = cr\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} a\sin\theta\cos\phi & ar\cos\theta\cos\phi & -ar\sin\theta\cos\phi \\ b\sin\theta\sin\phi & br\cos\theta\sin\phi & br\sin\theta\cos\phi \\ c\cos\theta & -cr\sin\theta & 0 \end{vmatrix} = abcr^2\sin\theta,$$

对应的积分区域化为

$$E = \{(r,\theta,\phi)|0 \le r^2 \le \sin^2\theta, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi\},$$

带入由换元公式得到

$$V = \iiint\limits_E abcr^2\sin\theta drd\theta d\phi$$
$$= 8abc\int_0^{\frac{\pi}{2}}\sin\theta d\theta\int_0^{\sin\theta}r^2dr\int_0^{\frac{\pi}{2}}d\phi \quad = \frac{\pi^2}{4}abc.$$

$\square$

10.8.1 计算下列 n 重积分:

(1) $\underset{[0,1]^n}{\int\cdots\int}(x_1^2+...+x_n^2)dx_1...dx_n$;

(2) $\underset{[0,1]^n}{\int\cdots\int}(x_1+...+x_n)^2dx_1...dx_n$.

解. (1)

$$\begin{aligned}\underset{[0,1]^n}{\int\cdots\int}(x_1^2+...+x_n^2)dx_1...dx_n &= n\underset{[0,1]^n}{\int\cdots\int}x_1^2dx_1...dx_n\\ &= n\int_0^1 x_1^2dx_1\underset{[0,1]^{n-1}}{\int\cdots\int}dx_2...dx_n\\ &= n\int_0^1 x_1^2dx_1\\ &= \frac{n}{3}.\end{aligned}$$

(2)

$$\begin{aligned}\underset{[0,1]^n}{\int\cdots\int}(x_1+...+x_n)^2dx_1...dx_n &= \underset{[0,1]^n}{\int\cdots\int}\sum_{k=1}^n x_k^2+\sum_{i\neq j}x_ix_jdx_1...dx_n\\ &= n\underset{[0,1]^n}{\int\cdots\int}x_1^2dx_1...dx_n+\frac{n(n-1)}{2}\underset{[0,1]^n}{\int\cdots\int}x_1x_2dx_1...dx_n\\ &= \frac{n}{3}+\frac{n(n-1)}{4}\\ &= \frac{n^2}{4}+\frac{n}{12}.\end{aligned}$$

$\square$

10.8.2 计算累次积分:

$$\int_0^1 dx_1\int_0^{x_1}...\int_0^{x_{n-1}}x_1...x_{n-1}x_ndx_n.$$

解.

$$\begin{aligned}f_n=\int_0^1 dx_1\int_0^{x_1}...\int_0^{x_{n-1}}x_1...x_{n-1}x_ndx_n &= \int_0^1 x_1dx_1\int_0^{x_1}...\int_0^{x_{n-1}}x_2...x_ndx_n\\ &= \int_0^1 x_1x_1^{2n-2}dx_1\int_0^1 dx_2\int_0^{x_1}...\int_0^{x_{n-1}}x_2...x_{n-1}dx_{n-1}\\ &= \frac{1}{2n}f_{n-1}\\ &= \frac{1}{(2n)!!}\end{aligned}$$

$\square$

10.8.3 计算下列 $\mathbb{R}^n$ 中区域的体积 $(a_1, a_2, ..., a_n > 0)$:

(1)$V_n = (x_1, x_2, ..., x_n) : \dfrac{x_1}{a_1} + \dfrac{x_2}{a_2} + ... + \dfrac{x_n}{a_n} \le 1, x_1, x_2, ..., x_n \ge 0;$

(2)$V_n(a) = \left\{(x_1, x_2, ..., x_n) : |x_1| + |x_2| + ... + |x_n| \le a\right\}.$

解. (1)

$$\begin{aligned}
f_n(a_1, ..., a_n) &= \int\cdots\int\limits_{\substack{\frac{x_1}{a_1}+...+\frac{x_n}{a_n}\le 1\\ xi\ge 0}} dx_1...dx_n \\
&= \int_0^{a_1} dx_1 \int\cdots\int\limits_{\substack{\frac{x_2}{a_2}+...+\frac{x_n}{a_n}\le 1-\frac{x_1}{a_1}\\ xi\ge 0}} dx_2...dx_n \\
&= \int_0^{a_1} \big(1 - \frac{x_1}{a_1}\big)^{n-1} dx_1 f_{n-1}(a_2, ..., a_n) \\
&= \frac{a_1}{n} f_{n-1}(a_2, ..., a_n) \\
&= \frac{a_1...a_n}{n!}.
\end{aligned}$$

(2)

$$\begin{aligned}
f_n(a) &= \int\limits_{V_n(a)} d\mu \\
&= 2\int_0^a f_{n-1}(a - x_n) dx_n \\
&= 2\int_0^a (a - x_n)^{n-1} f_{n-1}(1) dx_n \\
&= \frac{2a}{n} f_{n-1}(a) \\
&= \frac{(2a)^n}{n!}.
\end{aligned}$$

□

10.8.4 设 K 为二元连续函数, 对 $n \in \mathbf{N}^*$, 令

$$K_n(x, y) = \int\cdots\int\limits_{[a,b]^n} K(x, t_1)K(t_1, t_2)...K(t_n, y)dt_1...dt_n.$$

求证: 对任意 $m, n \in \mathbf{N}^*$, 有

$$K_{m+n+1}(x, y) = \int_a^b K_m(x, t)K_n(t, y)dt.$$

解.

$$\begin{aligned}
&\int_a^b K_n(x, t)K_m(t, y)dt \\
&= \int_a^b \int\cdots\int\limits_{[a,b]^n} K(x, t_1)K(t_1, t_2)...K(t_n, t)dt_1...dt_n \int\cdots\int\limits_{[a,b]^m} K(t, t_{n+1})K(t_{n+1}, t_{n+2})...K(t_{n+m}, t)dt_{n+1}...dt_{n+m}dt \\
&= \int\cdots\int\limits_{[a,b]^{n+m}} K(x, t_1)K(t_1, t_2)...K(t_{n+m}, y)dt_1...dt_{n+m}.
\end{aligned}$$

$\square$

10.8.5 设 $a_1, a_2, ..., a_n > 0$,

$$V_n = \left\{(x_1, x_2, ..., x_n) : \frac{|x_i|}{a_2} + |\frac{|x_n|}{a_n} \le 1(i-1, 2, ..., n-1)\right\}.$$

求 V_n 的体积.

解.

$$\begin{aligned}
\int_{V_n} d\mu &= 2^n \int_0^{a_n} dx_n \int_0^{a_1(1-\frac{x_n}{a_n})} dx_1 ... \int_0^{a_{n-1}(1-\frac{x_n}{a_n})} dx_{n-1} \\
&= 2^n a_1 ... a_{n-1} \int_0^{a_n} (1 - \frac{x_n}{a_n}) dx_n \\
&= \frac{2^n}{n} a_1 ... a_n
\end{aligned}$$

$\square$