2021. 10.12.

Review.

① Runge-Kutta

$$\left\{ y' = f(x,y) \right.$$

Mathematica NDSolve.

② CN方法.

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u$$

PDE

$$\begin{cases} \frac{\partial}{\partial t}f(x,t) = \frac{f(t+\delta t) - f(t)}{\delta t} \\ \frac{\partial^2}{\partial x^2}f(x,t) = \frac{f(x+dx) + f(x-dx) - 2f(x)}{(dx)^2} \end{cases}$$

概念

差分 $\frac{\partial f}{\partial t} = f(y,t)$

$$\Rightarrow f_{n+1} = f_n + f(y_n, t_n)\Delta t$$

e.g. $\frac{dx}{dt} = x - x^2$

$$x_{n+1} = x_n + (x_n - x_n^2)\cdot\Delta t$$

不动点 x^* 是否存在.

$$x^* = x^* + (x^* - x^{*2})\Delta t$$

$$x^* = 0, 1$$

是一个求根问题.

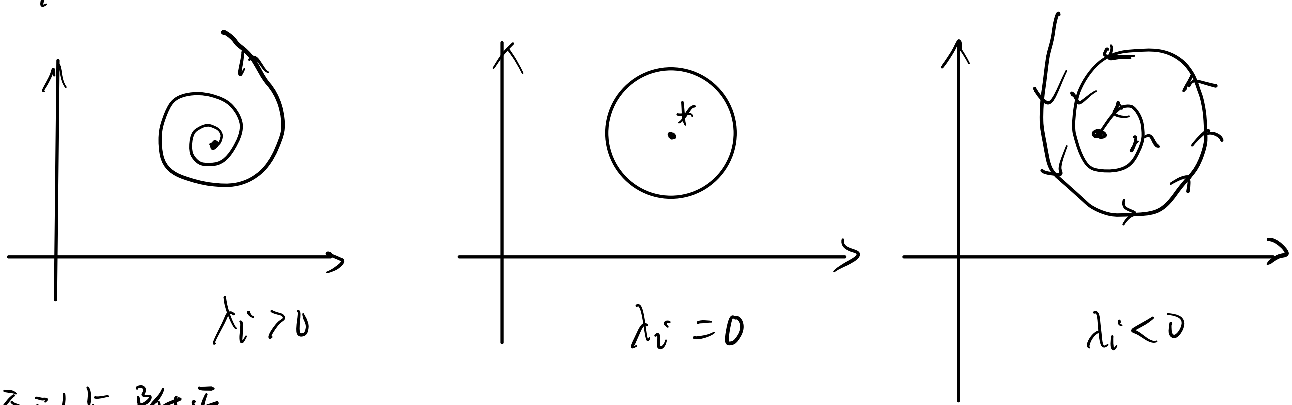
① 求根

② 求轨迹.

Lyapunov 指数.

$\frac{d\vec{x}}{dt} = F(\vec{x}) \qquad F(\vec{x}^*) = 0$

$\vec{X}_{n+1} = \vec{X}_n + F(\vec{x}_n)dt$



$\lambda_i > 0 \qquad \lambda_i = 0 \qquad \lambda_i < 0$

在不动点附近.

$\vec{x} = \vec{x}^* + \vec{y}$

$\frac{d\vec{y}}{dt} = F(\vec{x}^* + \vec{y}) = F(\vec{x}^*) + A\vec{y}$

$\frac{d\vec{y}}{dt} = A\vec{y} \xrightarrow{\text{对角化}} \frac{d\vec{z}}{dt} = \lambda \vec{z}$.

$\max(\lambda_i) = \gamma$. 称为 Laypunov 指数.

两个例子.

1) $\frac{dx}{dt} = x - x^2$

2) Lorenz Model.

$$\begin{cases} \frac{dx}{dt} = \sigma(y-x) \\ \frac{dy}{dt} = \gamma x - y - xz \\ \end{cases}$$

目的:

① 计算 exponent

② 了解 chaos.

$$\frac{dz}{dt} = -bz + xy$$

1). $\frac{dx}{dt} = x - x^2$. 令 $\frac{dx^*}{dt} = 0 \Rightarrow x^* = 0, 1$.

令 $x = x^* + y$, y 是小量.

$$\frac{dy}{dt} = \frac{d}{dx}(x - x^2)\Big|_{x=x^*} y \Rightarrow y = e^{\lambda t} y_0.$$

$$= (1 - 2x^*) y \qquad \lambda = \begin{cases} 1, & x^* = 0, \text{ 非稳} \\ -1, & x^* = 1, \text{ 稳定} \end{cases}$$

2). Lorenz Model.

$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \\ \frac{dz}{dt} = 0 \end{cases} \Rightarrow \begin{cases} y - x = 0 \\ rx - y - xz = 0 \\ -bz + xy = 0 \end{cases} \Rightarrow (x^*, y^*, z^*) = \begin{cases} \langle 1 \rangle (0, 0, 0) \\ \langle 2 \rangle x^2 = y^2 = b(r-1) \\ \qquad z = r - 1 \end{cases}$$

对于平衡的解, $(0, 0, 0)$

$$\begin{cases} \frac{dx}{dt} = \sigma(y - x) \\ \frac{dy}{dt} = rx - y \\ \frac{dz}{dt} = -bz \end{cases} \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

取 $\sigma = 10$, $b = \frac{8}{3}$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} \lambda_1 = -\frac{8}{3} \\ \lambda_2 = \frac{-11-\sqrt{81+40r}}{2} \\ \lambda_3 = \frac{-11+\sqrt{81+40r}}{2}. \end{cases}$$

$r<1$. 所有 eigenvalue <0

$r>1$. $\lambda_3>0 \Rightarrow$ unstable

另一个点、

$$\vec{x}^* = \left(\sqrt{\tfrac{8}{3}(r-1)},\ \sqrt{\tfrac{8}{3}(r-1)},\ r-1\right)$$

$$\vec{x} = \vec{x}^* + (u, v, w)$$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{8(r-1)/3} \\ \sqrt{8(r-1)/3} & \sqrt{8(r-1)/3} & -8/3 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

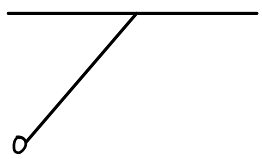
$$\Rightarrow\ 3\lambda^3 + 41\lambda^2 + 8(r+10)\lambda + 160(r-1) = 0$$

If $\lambda > 1$ $\begin{cases} 1<r<1.3456 & \text{3个实根} \\ 1.3456<r<24.737. & \text{3个根 } Re(\lambda_i)<0 \\ r>24.737. & \text{有一个 } Re(\lambda_i)>0.\ \text{unstable} \end{cases}$

作业：画图 Lorenz 方程的轨迹 MMa.

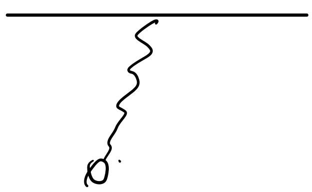
含时演化问题 简单 ⇒ 复杂

简单问题 ①



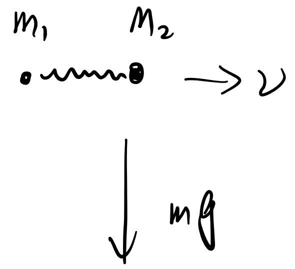
单摆

②



弹簧单摆

③



以一定的初速度抛出.

$$\mathcal{L} = \frac{1}{2}m_1(\dot{\vec{x}}_1)^2 + \frac{1}{2}m_2(\dot{\vec{x}}_2)^2 + mgz_1 + m_2 g z_2 + \frac{1}{2}k|\vec{x}_1 - \vec{x}_2|^2$$

运动方程.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

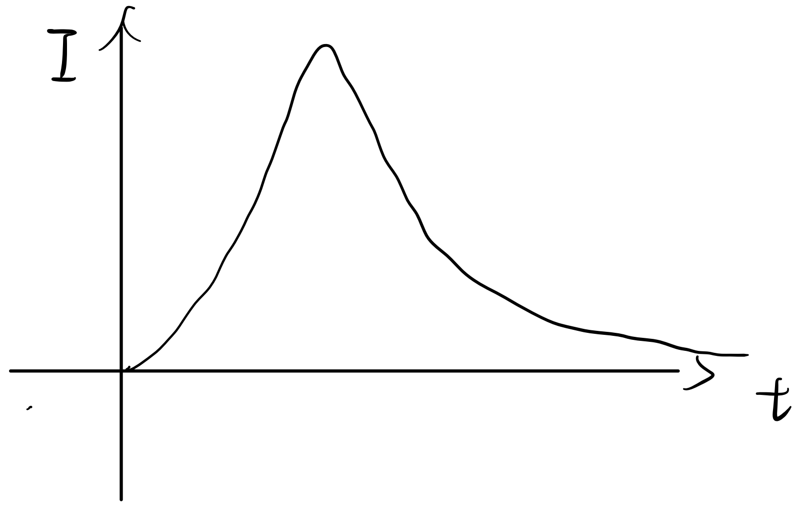
4. SIR 病毒传播 Model.

S：易感染人群

I：感染人

R：康复.

$$\begin{cases} \frac{dS}{dt} = -cSI \\ \frac{dI}{dt} = cSI - gI \\ \frac{dR}{dt} = gI \end{cases}$$



5. N → 多.

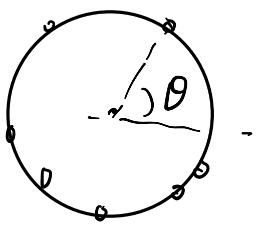
① 天体 $T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}$

$U = G\frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|}$ $\Rightarrow$ $m_i \ddot{\vec{r}}_i = -\sum_j \vec{F}_{ij}$

$$\vec{F}_{ij} = \frac{G m_i m_j}{|\vec{r}_i - \vec{r}_j|^3}(\vec{r}_i - \vec{r}_j)$$

$$\Rightarrow \frac{r_i^{n+1} + r_i^{n-1} - 2r_i^n}{(dt)^2} = -\frac{1}{m_i}\left(\sum_j \vec{F}_{ij}^{\,n} + \sum_j \vec{F}_{ij}^{\,n+1}\right)$$

② 同步模型 Kuramoto model



$$\frac{d\theta_i}{dt} = w_i - \frac{K}{N}\sin(\theta_i - \theta_j)$$

$$\Leftrightarrow \theta_i^{n+1} = \theta_i^n + dt\left[w_i - \frac{K}{N}\sum_j \sin(\theta_i - \theta_j)\right]$$

分子动力学，次量子化
Flocking
随机数，