11:00

1.16 晚上7:00

exam ~~[illegible]~~ ~~周五上午~~

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开卷.

第七周四

9选4.

Bogoliubov 变换.

1947 ~~on the~~ on the theory of superfluidity

Boson.
Fermion } 相似的.

Fermion 一半一半问题 → 激发 $\begin{cases} F \to F \\ B \to B \end{cases}$

⇓

保证：Bogoliubov 变换.

$$H = \frac{1}{2}\begin{pmatrix} H_0 & \Delta \\ \Delta^+ & -H_0^* \end{pmatrix}$$

$$H = t_{ij}\, c_i^+ c_j + \Delta_{ij}\, c_i^+ c_j^+ + \Delta^*_{ij}\, c_j c_i$$

$$= \sum_i \varepsilon_i \gamma_i^+ \gamma_i$$

$$\begin{cases} c_i = u_{ij}\gamma_j + v_{ij}\gamma_j^+ \\ c_i^+ = u_{ij}^+ \gamma_j^+ + v_{ij}^+ \gamma_j \end{cases} \Rightarrow \text{Bogoliubov 变换}$$

$$\psi = \begin{pmatrix} c \\ c^+ \end{pmatrix} = \left(\begin{array}{c|c} u & v \\ \hline v^* & u^* \end{array}\right)\begin{pmatrix} \gamma \\ \gamma^+ \end{pmatrix}$$

B. F 都成立.

要求：$\{c_i, c_j\} = 0$
$\{c_i, c_j^+\} = \delta_{ij}$ 　同时 γ_i 也成立.

$$\{c_i, c_j\} = \{u_{ik}\gamma_k + v_{ik}\gamma_k^+ , u_{jl}\gamma_l + v_{jl}\gamma_l^+\}$$

$$= u_{ik}v_{jk} + v_{ik}u_{jk} = 0.$$
$$= (uv^T + vu^T)_{ij} = 0.$$

$$\therefore uv^T + vu^T = 0$$

$$\{c_i, c_j^+\} = \{u_{ik}\gamma_k + v_{ik}\gamma_k^+ , u_{jl}^*\gamma_l^+ + v_{jl}^*\gamma_l\}_\pm$$
$$= u_{ik}u_{jk}^* \pm v_{ik}v_{jk}^*$$
$$= (uu^+ \pm vv^+)_{ij} = \delta_{ij}$$
$$\Leftrightarrow uu^+ \pm vv^+ = 1.$$

① $t_{ij} c_i^+ c_j = t_{ij}(u_{ik}^*\gamma_k^+ + v_{ik}^*\gamma_k)(u_{jl}\gamma_l + v_{jl}\gamma_l^+)$

② $\Delta_{ij} c_i^+ c_j$ +h.c. $= \Delta_{ij}(u_{ik}^*\gamma_k^+ + v_{ik}^*\gamma_k)(u_{jl}^*\gamma_l^+ + v_{jl}^*\gamma_l)$ +h.c.

$H =$ ①+② $= \text{const} + \sum_j \varepsilon_j \gamma_j^+\gamma_j.$

$$\left.\begin{array}{l} t_{ij}u_{ik}^* v_{jl}\,\gamma_k^+\gamma_l^+ \\ \Delta_{ij}u_{ik}^* u_{jk}^*\,\gamma_k^+\gamma_l^+ + \cdots \end{array}\right\} = 0.$$

eg: $H = \varepsilon(a^+a + b^+b) + \Delta a^+b^+ + \Delta^* ba.$ for boson

Δ, ε real.

$$H = (a^+\ b^+\ a\ b)\begin{pmatrix} \varepsilon & 0 & & \Delta \\ 0 & \varepsilon & \Delta & \\ & \Delta & \varepsilon & 0 \\ \Delta & & 0 & \varepsilon \end{pmatrix}\begin{pmatrix} a \\ b \\ a^+ \\ b^+ \end{pmatrix} + \text{const}.$$

Spectra $\frac{1}{2}(\varepsilon \pm \Delta) \Rightarrow$ 结果有问题

$$i\dot{a} = [a, H] = \frac{\partial H}{\partial a^+} = \varepsilon a + \Delta b^+$$

$$i\dot{b} = \varepsilon b + \Delta a^+$$

$$\begin{cases} i\dot{a}^+ = -\varepsilon a^+ - \Delta b \\ i\dot{b}^+ = -\varepsilon b^+ - \Delta a. \end{cases}$$

$$i\frac{d}{dt}\begin{pmatrix} a \\ b \\ a^+ \\ b^+ \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & 0 & \Delta \\ 0 & \varepsilon & \Delta & 0 \\ 0 & -\Delta & \varepsilon & 0 \\ -\Delta & 0 & 0 & -\varepsilon \end{pmatrix}\begin{pmatrix} a \\ b \\ a^+ \\ b^+ \end{pmatrix}.$$

$\pm\sqrt{\varepsilon^2 - \Delta^2}$ 本征频率.

$$\cos wt = (e^{iwt} + e^{-iwt})/2.$$

$$H = \varepsilon(a^+a + b^+b) + \Delta a^+b^+ + \Delta ba.$$

令: $\begin{cases} a = u\alpha + v\beta^+ \\ b = x\beta^+ + y\alpha. \end{cases}$ 设 $u.\ v.\ x,\ y$. real $\in R$.

$$\begin{cases} a^+ = u\alpha^+ + v\beta \\ b^+ = x\beta + y\alpha^+ \end{cases}$$

$$[a, a^+] = 1 = [u\alpha + v\beta^+,\ u\alpha^+ + v\beta]$$
$$= u^2 - v^2 = 1$$

$$[a, b^+] = 0 \Rightarrow [u\alpha + v\beta^+,\ x\beta + y\alpha^+] = 0$$
$$uy - vx = 0$$

$$\begin{cases} u^2 - v^2 = 1 \\ x^2 - y^2 = 1 \quad \Rightarrow x = u.\ y = v \\ uy - vx = 0 \quad \Rightarrow y \propto v \quad x \propto u \end{cases}$$

$$\Rightarrow \begin{cases} a = u\alpha + v\beta^+ \\ b = u\beta^+ + v\alpha \end{cases}$$

$$\begin{cases} a^+ = u\alpha^+ + v\beta \\ b^+ = u\beta + v\alpha^+ \end{cases}$$

$$u^2 - v^2 = 1.$$

$$\xi(u\alpha^+ + v\beta)(u\alpha + v\beta^+) + \Delta(u\alpha^+ + v\beta)(u\beta^+ + v\alpha^+)$$

$$\xi(u\beta + v\alpha^+)(u\beta^+ + v\alpha) + \cancel{\Delta(u\alpha + v\beta^+)(u\beta^+ + v\alpha)}$$

$$\Delta(u\beta + v\alpha^+)(u\alpha + v\beta^+)$$

$\alpha^+\alpha$ 4项 $\xi u^2 + \xi v^2 + \Delta uv + \Delta vu = \xi(u^2+v^2) + 2\Delta uv$

$\beta^+\beta$ 4项 $= \sqrt{\xi^2 - \Delta^2}$

$\alpha^+\beta^+$ 4

$\alpha\beta$ 4 消失. $[2\xi uv + \Delta(u^2+v^2)]\alpha\beta = 0$.

$u^2 - v^2 = 1$. $u = \cosh x$. $v = \sinh x$.

$$\left(\begin{cases} \cos ix = \cosh x \\ \sin ix = i\sinh x \end{cases} \right).$$

知乎上: bogolinbov 变换.

$$H = t_{ij} c_i^{+} c_j + \Delta_{ij} c_i^{+} c_j^{+} + h.c.$$

$$= \frac{1}{2} \psi^{+} M \psi$$

$$\psi = \begin{pmatrix} c \\ c^{+} \end{pmatrix} \quad M = \left(\begin{array}{c|c} t & \Delta \\ \hline \Delta^{+} & t \end{array} \right).$$

$$M \to (M\Lambda).$$

$$\Lambda = 1 \ (F), \quad \Lambda = \sigma_z \ (B)$$

$$\psi = \begin{pmatrix} c \\ c^{+} \end{pmatrix} = T(\phi) = T\begin{pmatrix} b \\ b^{+} \end{pmatrix} = \begin{pmatrix} u & v \\ v^{*} & u^{*} \end{pmatrix} \begin{pmatrix} b \\ b^{+} \end{pmatrix}$$

$$\begin{cases} uv^{T} - vu^{T} = 0 \\ uu^{+} - vv^{+} = 1. \end{cases}$$

可以证明： $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

我们， $T \Lambda T^{+} = \Lambda$.

$$\begin{pmatrix} u & v \\ v^{*} & u^{*} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u^{+} & v^{T} \\ v^{+} & u^{T} \end{pmatrix}$$

$$= \begin{pmatrix} u & v \\ v^{*} & u^{*} \end{pmatrix} \begin{pmatrix} u^{+} & v^{T} \\ -v^{+} & -u^{T} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Lambda$$

Fermion $T \Lambda T^{+} = \Lambda$,

$\Lambda = 1 \ (F) \Rightarrow TT^{+} = 1$.

$T\Lambda T^+ = \Lambda$ $\qquad$ $T\Lambda = \Lambda (T^+)^{-1}$

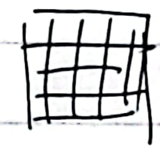
$$\frac{1}{2}\psi^+ M \psi = \frac{1}{2}\phi^+ T^+ M T \phi = \frac{1}{2}\phi^+ T^+ M T \Lambda \Lambda \phi$$

$$= \frac{1}{2}\phi^+ T^+ (M\Lambda)(T^+)^{-1} \Lambda \phi$$

$$= \frac{1}{2}\phi^+ \underbrace{P(M\Lambda)P^{-1}}_{\text{相似变换对角化}} \Lambda \phi$$

作业2

50×50的格子. $t=1$. Fermion & Boson.



Δ_{ij} 随机 本征值，波函数给出.

F → M. B → $M\Lambda$.

$\sqrt{\xi^2-\Delta^2}$ 为复数，不稳定.（Δ足够大）

$e^{i\varepsilon t}$. 随着时间演化会到无穷大.

为什么会不稳定？

$$H = \alpha(a^+a + b^+b) + \Delta a^+b^+ + \Delta ba.$$

$$= \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 x_2^2$$

$$+ \lambda x_1 x_2.$$

$$V = x^2 + y^2 + \lambda xy$$

$$= \left(x + \frac{\lambda}{2}y\right)^2 + \left(1 - \frac{\lambda^2}{4}\right)y^2$$

正定矩阵，稳定性.

$(p_1 x_1, p_2 x_2) \to (y_1\ y_2\ y_3\ y_4)$