Anderson Localization.

1) 单粒子问题: $|K\rangle = \gamma\, c_i^{\dagger}|0\rangle = |0\,0\,0\cdots0\,1_i\,0\,0\cdots0\rangle$

2) 求解运动方程.

$$E\alpha_i = V_i\alpha_i - t(\alpha_{i+1}+\alpha_{i-1}).$$

$\left\{\begin{array}{l}\text{迭代方程.}\\ V_i \text{ 随机数.}\end{array}\right.$

3) 转移矩阵:
$$\begin{pmatrix}\alpha_{i+1}\\ \alpha_i\end{pmatrix} = \begin{pmatrix}E-V_i & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha_i\\ \alpha_{i-1}\end{pmatrix}$$
$$= T_i\begin{pmatrix}\alpha_i\\ \alpha_{i-1}\end{pmatrix} = T_i T_{i-1}\cdots T_1\begin{pmatrix}\alpha_1\\ \alpha_1\end{pmatrix}.$$

求解 $i\sim10^6$

技巧: QR分解

$$\tilde{T} = T_N T_{N-1}\cdots T_1 = Q_N R_N R_{N-1}\cdots R_1 = QR$$
$$\tilde{T}^{+}\tilde{T} = R^{+}R.$$
$$\tilde{T}\tilde{T}^{+} = QRR^{+}Q^{+}$$

$\cancel{\lambda_i = \sum_j R_j}$

$$\lambda_i = \prod_j (R_j^{+}R_j)_{ii}.$$

$$\begin{pmatrix}\alpha_{N+1}\\ \alpha_N\end{pmatrix} = T_N\cdots T_1\begin{pmatrix}\alpha_1\\ \alpha_1\end{pmatrix} \sim e^{-N/\xi}(\ )$$

$$\frac{1}{\xi_i} = \frac{1}{N}\sum_j \ln(R_j^{+}R_j)_{ii}.$$

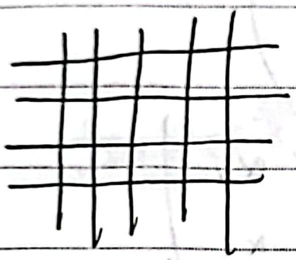
推广 d维.

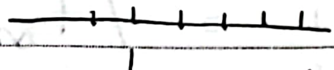
$-t c_i^+ c_{i+1} + h.c + V_i c_i^+ c_i$

$$H = -t_i^{\sigma\sigma'} c_{i\sigma}^+ c_{i+1\sigma'} + V_i^{\sigma\sigma'} c_{i\sigma}^+ c_{i\sigma'}$$

$$= -c_i^+ t_i c_{i-1} + c_i^+ V_i c_i$$

$$H = -t_i^{jj'} c_{ij}^+ c_{i+1j'} + V_i^{jj'} c_{ij}^+ c_{ij'}$$





$$c_i = \begin{pmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{iN} \end{pmatrix}.$$

$$\begin{pmatrix} \alpha_{i+1} \\ \alpha_i \end{pmatrix} = \begin{pmatrix} E - V_i & -t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_i \\ \alpha_{i-1} \end{pmatrix}$$

$\alpha_i$ 不再是一个数. 而是一个面上所有的点.

$$\begin{pmatrix} \alpha_{i+1} \\ \alpha_i \end{pmatrix} = \begin{pmatrix} E - [V_i] & -[t_i] \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_i \\ \alpha_{i-1} \end{pmatrix}$$

理想情况 $V_i = V$

$\lambda \to \infty$ 可解

$$H = -t(c_i^+ c_{i+1} + h.c) + V c_i^+ c_i$$

$\downarrow$ FT

$$\sum_k [-2t\cos k + V] c_k^+ c_k.$$

$$\epsilon_k = -2t\cos k + V$$

简化为: $T_1 = T_2 = \cdots = T_N = T$

$$T = \begin{pmatrix} E - V & -1 \\ 1 & 0 \end{pmatrix}^N$$

$$T^N = P\lambda^N P^{-1} \quad T = P\lambda P^{-1}$$

$$\lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

取 $V=0$ $\quad T = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix} = U \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} U^+$

$$\lambda_1 = \left( \frac{E}{2} \pm \sqrt{\frac{E^2}{4} - 1} \right)$$

$$\frac{E}{2} \pm \sqrt{\frac{E^2}{4} - 1} = x.$$

$$E = \frac{x^2+1}{x}.$$

$$x = e^{ik} \Rightarrow E = 2\cos k.$$

$$e^{-\frac{N}{\lambda}}. \quad \lambda = \left( \frac{E}{2} \pm \sqrt{\frac{E^2}{4} - 1} \right)$$ $\lambda$ 复数. $\Rightarrow$ extended.

$\lambda$ 实数 $\Rightarrow$ localized.

局域长度与态密度的关系.（刘川普适）.

Ref: Thouless 1972.

$$H = \sum_i \left( -V_{i,i+1} (c_i^+ c_{i+1} + h.c) + \varepsilon_i c_i^+ c_i \right)$$

$$\downarrow$$

$$E^\alpha \alpha_i^\alpha = \varepsilon_i \alpha_i^\alpha - V_{i,i+1} \alpha_{i+1}^\alpha - V_{i+1,i} \alpha_i^\alpha$$

DATE

$G(E-H)=1.$

$\hat{L}G=\delta(x-x') \Rightarrow \hat{L}G=1 \Rightarrow G=1/L.$

$((E-H)G)_{ij}=(1)_{ij}=\delta_{ij}$

$(E-\varepsilon_i)G_{ij}\cancel{-(E-H)}-H_{i\,i-1}G_{i-1\,j}-H_{i\,i+1}G_{i+1\,j}=\delta_{ij}$

$(E-\varepsilon_i)G_{ij}+V_{i\,i+1}G_{i+1\,j}+V_{i+1\,i}G_{i+1\,j}=\delta_{ij}.$

求 $G_{1N}$ ⇒ 从1开始传播到N的可能性.

$G=\frac{1}{E-H}. \quad \sim e^{-N/\xi}$

$\langle 1|G|N\rangle=\langle 1|\frac{1}{E-H}|N\rangle=\sum_j\frac{\langle 1|\psi_j\rangle\langle\psi_j|N\rangle}{E-E_j}.$

$H|\psi_j\rangle=E_j|\psi_j\rangle.$

求 $\langle 1|G|N\rangle=\langle 1|\frac{1}{E-H}|N\rangle.$

矩阵 cofactor 余子式

$A^{-1}=\frac{1}{\det(A)}(C)^t$ ← cofactor matrix.

$$E-H=\begin{pmatrix} E-\varepsilon_1 & V_{12} & & 0 \\ V_{21} & E-\varepsilon_2 & V_{23} & \\ & V_{32} & \ddots & \ddots \\ 0 & & \ddots & \end{pmatrix}$$

$$\langle 1 | \frac{1}{E-H} | 4 \rangle$$

$$= \frac{1}{\det(E-H)} \det \begin{pmatrix} V_{21} & \cdots & \cdots \\ & V_{32} & \vdots \\ 0 & & V_{43} \end{pmatrix}$$

$$= \frac{1}{\det(E-H)} (V_{21})(V_{32})(V_{43})$$

$$= \frac{1}{\det(E-H)} \left( \prod_{j=1}^{3} V_{j+1,j} \right).$$

$$\langle 1 | G | N \rangle = \frac{1}{\det(E-H)} \left( \prod_{j=1}^{N-1} V_{j+1,j} \right)$$

$$= \sum_k \frac{\langle 1 | k \rangle \langle k | N \rangle}{E - E_k}$$

$$= \sum_k \frac{e^{-N/\xi} f_k}{E - E_k}.$$

$$= \left[ \frac{1}{\prod_j (E-E_j)} \right] \left( \prod_j V_{j+1,j} \right).$$

$$E \to E_\beta \quad \frac{1}{(E-E_\beta) \prod_{j \neq \beta} (E_\beta - E_j)} \left( \prod_j V_{j+1,j} \right)$$

$$= \frac{1}{E-E_\beta} e^{-N/\xi} f_\beta + \sum_{j \neq \beta} \frac{e^{-N/\xi} f_j}{E_\beta - E_j}.$$

$$\frac{1}{\prod_{j \neq \beta} (E_\beta - E_j)} \prod_j V_{j+1,j} = e^{-N/\xi} f_\beta.$$

$$\frac{1}{N}\sum_{j=1}^{N-1} \ln V_{j+1,j} - \frac{1}{N}\sum_{j} \ln|\epsilon_\beta - \epsilon_j| = -\frac{1}{\xi} + \frac{\ln f}{N}$$

$$V_{i,i+1} = V$$

~~$V_{i,i+1}$~~ $\Leftrightarrow \ln V - \int \rho(x) \ln|\epsilon - x|\,dx = -\frac{1}{\xi}$.

$$\frac{1}{\xi} = \int \rho(x) \ln|x - E|\,dx - \ln V$$

$$\int \rho(x)\,dx = 1.$$

$$\Rightarrow \int \rho(x) \ln\frac{|x-E|}{|V|}\,dx.$$

是一个严格的结果.

作业用Mathematica求:

1) $V_i = 0$. $\begin{cases} \epsilon_k = -2t\cos k \\ V = t. \end{cases} \Rightarrow \frac{1}{\xi} = 0.$

$\Rightarrow$ 严格用数值证明.

2) $\rho(x)$ 为Lorentz分布. 求 $\frac{1}{\xi(E)}$ $\Rightarrow$ Thouless paper eq. 9.

$$\frac{\gamma/\pi}{x^2+\gamma^2}.$$

画 $\xi(E)$ 与 $E$ 的图.

AL标度理论.

$$R = \frac{U}{I} \quad e^2/\hbar$$

$$g = \frac{1}{R}\left(\frac{\hbar}{e^2}\right)$$ 电导 dimensionless.

$$R \propto \frac{\text{长度}}{\text{截面积}} \propto \frac{L}{L^{d-1}} \propto L^{2-d}.$$

$$\left.\begin{array}{ll} \text{metal} & g = \sigma L^{d-2} = (L/\xi)^{d-2} \\ \text{insulator} & g \propto G_N \sim e^{-(L/\xi)} \end{array}\right\} f\left(\frac{L}{\xi}\right)$$

β fuction $\beta = \frac{d\ln g}{d\ln L} = d-2.$ $\quad g = g_0 L^{\beta}.$

$$g = \sigma L^{d-2}$$

$$\ln g = \ln\sigma + (d-2)\ln L.$$

for $g = \sigma e^{-L/\xi}$

$$\ln g = \ln\sigma - \frac{L}{\xi}.$$

$$\beta = \frac{d\ln g}{d\ln L} = \frac{d\ln g}{dL} \Big/ \frac{d\ln L}{dL} = -\frac{L}{\xi} < 0.$$