second quantization
二次量子化 / Canonical quantization 正则~

| 量子力学
| 多体物理..

目标：解释二次量子化.

方法，类比，核心：找共轭量 x_i, p_i.

正则坐标 (x, p) $(z, \bar{z})$, (a, a^*)
动量

热力学 (P, V), (S, T).

公式

①. Hamiltonian eq:

$$\dot{x} = \frac{\partial H}{\partial p} ; \dot{p} = -\frac{\partial H}{\partial x}.$$

$$= \{x, H\} \qquad = \{p, H\}.$$

②复数中. 柯西-黎曼定理.

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} , \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}.$$

目的：①对经典model量子化.
②对角化. 求Bloch能带.
③理解无序. Anderson Localization
④常见多体模型 对角化.
⑤Bethe - ansatz 数值.
⑥平均场理论与相变.

量子化 quantization.

标志性史 ① 黑体辐射 Planck

② 光电实验、量子化粒子性. 1904-1905. Einstein 1924

Bohr model. de Broglie → Born, Schrödinger, Heisenberg, Jordan, wigner

1911年. 1924.

1927 ~~Dirac~~ Dirac ⇒ QED开端. 场论开端.

The quantum theory of the emission and absorption of ~~Radio~~ Radiation.

Hamiltonian eq:

$$H = \frac{p^2}{2m} + V(x)$$

$$\mathcal{L} = \frac{p^2}{2m} - V(x) = \frac{m}{2}\dot{x}^2 - V(x)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \qquad \therefore m\ddot{x} = -V'(x)$$

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} = -\frac{\partial H}{\partial x} = -V'(x) \end{cases} \qquad H(p_i, q_i) \Rightarrow \begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\} \end{cases}$$

Poisson Braket. $\{A,B\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$

③ 基本性质. $\{q_i, q_j\} = 0$ $\{p_i, p_j\} = 0$.

$\{q_i, p_j\} = \delta_{ij}$

$\Rightarrow [a_i^+, a_j] = 0$. $[a_i^+, a_j^+] = 0$. $[a_i, a_j^+] = \delta_{ij}$

谐振子：

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \hbar\omega(a^+a + \frac{1}{2})$$

$$[x, p] = i\hbar \quad \Downarrow \quad ~~[a^+, a]~~ \; [a, a^+] = 1.$$

构造：之前讲到过. $a^+ = \sqrt{\frac{m\omega}{2\hbar}} x - \frac{ip}{\sqrt{2\hbar m\omega}}$

$$a = \sqrt{\frac{m\omega}{2\hbar}} x + \frac{ip}{\sqrt{2\hbar m\omega}}$$

$\Rightarrow$ ~~$[a, a^+]$~~

A , $-B$.

$$[A, B] = \sqrt{\frac{m\omega}{2\hbar}} \sqrt{\frac{1}{2\hbar m\omega}} [x, -ip]$$

$$= \frac{1}{2\hbar} (-i) i\hbar = \frac{1}{2}.$$

① 应用到微扰系统：

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \lambda x^4$$

$$= \hbar\omega(a^+a + \frac{1}{2}) + \lambda(\frac{\hbar}{2m\omega})^2 (a + a^+)^4 \qquad x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+)$$

$$= \hbar\omega(a^+a + \frac{1}{2}) + \lambda(\frac{\hbar}{2m\omega})^2 [aa + a^+a^+ + aa^+ + a^+a]^2$$

$$= \hbar\omega(a^+a + \frac{1}{2}) + \lambda(\frac{\hbar}{2m\omega})^2 \underbrace{(a^2 + a^{+2} + 2a^+a + 1)^2}_{V'}$$

λ 小量 $E_0 = \hbar\omega(n + \frac{1}{2}) + \langle n | V' | n \rangle + \cdots$

$x, p \Rightarrow a, a^+$

基矢 $\Rightarrow |n\rangle$ 表示 代数

$$a^+a|n\rangle = n|n\rangle$$

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^+|n\rangle = \sqrt{n+1}|n+1\rangle$$

为什么不用 $\phi_n(x)$ 而用 $|n\rangle$?

1) 简单.

$$\phi_n(x) \sim H_n(x) e^{-Ax^2}$$

$$\langle \phi_n | x^4 | \phi_n \rangle \sim \int H_n^2(x) e^{-2Ax^2} dx.$$

$$\lambda \left(\frac{\hbar}{2m\omega}\right)^4 \langle n | (a+a^+)^4 | n \rangle = \lambda\left(\frac{\hbar}{2m\omega}\right)^4 \langle n | (a+a^+)^2 (a+a^+)^2 | n\rangle$$

算子: $\lambda\left(\frac{\hbar}{2m\omega}\right)^4$

$(a+a^+)|n\rangle = \sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle$

$(a+a^+)|n\rangle = \sqrt{n}\,(a+a^+)|n-1\rangle + \sqrt{n+1}\,(a+a^+)|n+1\rangle$

$= \sqrt{n(n-1)}\,|n-2\rangle + (2n+1)|n\rangle + \sqrt{(n+1)(n+2)}\,|n+2\rangle$

$$\Rightarrow \propto \lambda\left(\frac{\hbar}{2m\omega}\right)^2 \left[n(n-1) + (2n+1)^2 + (n+1)(n+2) \right]$$

2) 多体系统.

$\phi_n(x)\phi_m(y) + \phi_n(y)\phi_m(x) \Rightarrow |n\rangle \otimes |m\rangle$ or $|nm\rangle$.

N个粒子.

$$\frac{\sum\limits_{\text{轮换}} \phi_{n_1}(x_{p_1})\phi_{n_2}(x_{p_2}) \cdots \phi_{n_L}(x_{p_L})}{l}$$

复杂 $\Rightarrow |n_1, n_2, \cdots, n_L\rangle$

全同性的体现: $a_i |n_1, \cdots, n_L\rangle = \sqrt{n_i}\,|n_1, \cdots, n_i - 1, \cdots n_L\rangle$

二次量子化 $\Big|$ $x, p \Rightarrow a, a^+$
多体物理, 全同性.

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m w_1^2 x_1^2 + \frac{1}{2}m w_2^2 x_2^2 + \lambda x_1^2 x_2^2.$$

$$= \hbar w_1 (a_1^+ a_1 + \tfrac{1}{2}) + \hbar w_2 (a_2^+ a_2 + \tfrac{1}{2})$$
$$+ \lambda \frac{\hbar}{2m w_1} \frac{\hbar}{2m w_2} (a_1 + a_1^+)^2 (a_2 + a_2^+)^2.$$

[illegible] ↙

$$E = \hbar W_1 (n_1 + \tfrac{1}{2}) + \hbar w_2 (n_2 + \tfrac{1}{2}) + \lambda \frac{\hbar^2}{(2m)^2 w_1 w_2} \langle n_1 n_2 | (a_1 + a_1^+)^2 (a_2 + a_2^+)^2 | n_1 n_2 \rangle$$

$$a_1 |n_1 n_2\rangle = \sqrt{n_1}\, |n_1 - 1, n_2\rangle$$
$$a_2 |n_1 n_2\rangle = \sqrt{n_2}\, |n_1, n_2 - 1\rangle$$
$$a_1^+ |n_1 n_2\rangle = \sqrt{n_1 + 1}\, |n_1 + 1, n_2\rangle$$
$$a_2^+ |n_1 n_2\rangle = \sqrt{n_2 + 1}\, |n_1, n_2 + 1\rangle.$$

$$H = \frac{p^2}{2m} + v \cos x \; ?$$