$$P(H) = \prod_{i<j} f(h_{ij})$$

$$P(\lambda_1, \cdots, \lambda_N) \Rightarrow \text{本征值联合分布}$$

$$\propto \prod_{i<j} |\lambda_i - \lambda_j|^n e^{-\beta(\sum_j \lambda_j^2)}$$

作业：

2阶随机矩阵的联合概率分布.

1) $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}$ $h_{ij} \in \mathbb{R}$

$$P(\lambda_1, \lambda_2) \sim |\lambda_1 - \lambda_2| e^{-A(\lambda_1^2 + \lambda_2^2)}$$

同样考虑 3×3 Matrix.

2) $H = \begin{pmatrix} h_1 & h_2 + ih_3 \\ h_2 - ih_3 & h_4 \end{pmatrix}$ 2×2.

or 3×3.

~~[illegible]~~ 验证 $P \propto \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-A(\sum_i \lambda_i^2)}$.

3) 1000×1000 的随机矩阵

$\lambda_1 < \lambda_2 < \cdots < \lambda_N$ $N = 1000$

① $\Delta = \lambda_{i+1} - \lambda_i$ 计算分布

$P(\Delta) \sim |\Delta|^\alpha e^{-B\Delta^2}$

② $\Delta = \max\{\lambda_{i+1} - \lambda_i\}$. 求$P(\Delta)$分布.

目标：

① 证明 $P(\lambda_1,\dots,\lambda_N) \propto \prod_{i<j}|\lambda_i-\lambda_j|\, e^{-A\sum_i \lambda_i^2}$

设 θ_i 为H的本征值：

$$P(\theta_1,\dots,\theta_N) \propto \prod_{i<j}|\theta_i-\theta_j|\, e^{-A\sum_i \theta_i^2}$$

② 物理意义：Dyson Brownian motion.

Dyson：目的：给 $P(\theta)$ 找一个合理解释.

补充几个结论：

Jacobian matrix $\left\{ dx_1\cdots dx_N = J\,dy_1\cdots dy_N \right.$

$x_i = x(y)$

$$J=\det\left(\frac{\partial x_i}{\partial y_j}\right).$$

x向量；y向量.

$x = Oy$.

O：正交阵. $O^T=O$.

$dx_i = d(Oy)_i$

$= O_{ij}\,dy_j$

$\Leftrightarrow \frac{\partial x_i}{\partial y_j} = O_{ij}$.

$dX = J\,dY$. $J=\det O = 1$.

推广：X, Y 为矩阵： $X = OY$. ~~dX~~ $dX = f(O)\,dY$ $f(O)=?$

$X_{ij} = (OY)_{ij} = O_{ik}\,y_{kj}$

$$\frac{\partial X_{ij}}{\partial y_{kj}} = O_{ik}.$$

$$X = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix},$$

$$\begin{cases} x_{11} = O_{11} y_{11} + O_{12} y_{21} \\ x_{22} = O_{21} y_{12} + O_{12} y_{22} \\ x_{12} = O_{11} y_{12} + O_{12} y_{22} \\ x_{21} = O_{21} y_{11} + O_{22} y_{21} \end{cases} \qquad \begin{cases} x_1 = O_{11} y_1 + O_{12} y_3 \\ x_2 = O_{11} y_2 + O_{12} y_4 \\ x_3 = O_{21} y_1 + O_{22} y_3 \\ x_4 = O_{21} y_2 + O_{22} y_4 \end{cases}.$$

$x_{11} \Rightarrow x_1$ $\quad y_{21} \Rightarrow x_3$

$x_{12} \Rightarrow x_2$ $\quad x_{22} \Rightarrow x_4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} O_{11} & 0 & O_{12} & 0 \\ 0 & O_{11} & 0 & O_{12} \\ O_{21} & 0 & O_{22} & 0 \\ 0 & O_{21} & 0 & O_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix},$$

$$= O \otimes I \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

$$dX = \det(O \otimes I)\, dY = [\det(O)]^2\, dY = dY.$$

IF $X = OY$. $\quad dX = f(O)\,dY$.

$\Rightarrow Y = O^T X \qquad dY = f(O^T)\,dX$.

$\Leftrightarrow \quad f(O) f(O^T) = 1. \qquad f(O) = (\det(O))^k$ 满足左式.

※ 可用于复数.

另一种可能. $\begin{cases} X = O^T Y O \\ H' = O^T H O \end{cases} \quad \Rightarrow dX = (?)\,dY.$

$Z = OX = YO$.

$dZ = f(O)\,dX$.
$dZ = f(O)\,dY$.
$\Big\}\ dX = f(O) f(O^T)\,dY = dY$.

与之前类似.

解答: $H = O^T \Theta O$ $\quad \Theta = \begin{pmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_N \end{pmatrix}$.
$= (h_{ij})_{N \times N}$.

$H: 1 + 2 + \cdots + N = \frac{(N+1)N}{2}$

求 O 中的 N 个 $\Rightarrow$ O 中有 $1 + 2 + \cdots + N - 1 = \frac{N(N-1)}{2}$

$O = e^h$

$O^T O = 1 \Rightarrow h^T = -h$

$\Big\}\ 1 + 2 + \cdots + N - 1 = N(N-1)/2$ $\Big\}$ 一样.

$\begin{cases} \theta_1, \cdots, \theta_N \\ P_1, P_2, \cdots, P_l \quad l = \frac{N(N-1)}{2}. \end{cases}$

$P_l \in \begin{cases} O \text{ 矩阵} \\ h \text{ 矩阵}. \end{cases}$

$J(\theta, p) = \det\left(\frac{\partial H_{ij}}{\partial \theta}, \frac{\partial H}{\partial p} \right)$

to Calculate $\frac{\partial H_{ij}}{\partial P_\mu}$, $\frac{\partial H_{ij}}{\partial \theta_\nu}$. $\Big\}$ Wigner 1950.

$\Downarrow$ 对 O 中变量 $\quad \Downarrow$ 对 Θ 变量.

$H = O^T \Theta O$. $\quad \frac{\partial H}{\partial P_\mu} = \frac{\partial O^T}{\partial P_\mu} \Theta O + O^T \Theta \frac{\partial O}{\partial P_\mu}$.

$O^T O = 1 \Rightarrow$ $\quad = O^T \left(O \frac{\partial O^T}{\partial P_\mu} \Theta \right) O$

$+ O^T \Theta \frac{\partial O}{\partial P_\mu} O^T O$

$$\left[\begin{array}{ll} O\dfrac{\partial O^T}{\partial P_\mu} = S^\mu & \dfrac{\partial O}{\partial P_\mu}O^T = -S^\mu \\[2ex] O\dfrac{\partial O^T}{\partial P_\mu} + \dfrac{\partial O}{\partial P_\mu}O^T = \dfrac{\partial}{\partial P_\mu}(OO^T) = 0 & \end{array}\right]$$

$$\Rightarrow \frac{\partial H}{\partial P_\mu} = O^T\left[S^\mu \Theta - \Theta S^\mu\right]O$$

$$O\left(\frac{\partial H}{\partial P_\mu}\right)O^T = S^\mu\Theta - \Theta S^\mu$$

$$\frac{\partial H}{\partial \theta_\nu} = O^T\frac{\partial \Theta}{\partial \theta_\nu}O \quad \Rightarrow \quad O\frac{\partial H}{\partial \theta_\nu}O^T = \frac{\partial \Theta}{\partial \theta_\nu}$$

$S^\mu = S^\mu(P)$ 复杂的表达式.

$$O\frac{\partial H}{\partial P_\mu}O^T = S^\mu\Theta - \Theta S^\mu$$

计算 $\alpha\beta$ 元素.

$$\begin{aligned}\left(O\frac{\partial H}{\partial P_\mu}O^T\right)_{\alpha\beta} &= O_{\alpha j}\frac{\partial H_{jk}}{\partial P_\mu}(O^T)_{k\beta} \\ &= O_{\alpha j}O_{\beta k}\frac{\partial H_{jk}}{\partial P_\mu} = S^\mu\Theta - \Theta S^\mu \\ &= S^\mu_{\alpha\beta}(\theta_\beta - \theta_\alpha)\end{aligned}$$

以 $\theta_\alpha - \theta_\beta$ 形式出现.

$$\left(O\frac{\partial H}{\partial \theta_\nu}O^T\right)_{\alpha\beta} = \left(\frac{\partial \Theta}{\partial \theta_\nu}\right)_{\alpha\beta} = \delta_{\alpha\beta}\delta_{\alpha\nu}$$

$$\det\left(\frac{\partial H}{\partial \theta_\nu}, \frac{\partial H}{\partial P_\mu}\right) = \det\left(O^T\frac{\partial H}{\partial \theta_\nu}O,\ O^T\frac{\partial H}{\partial P_\mu}O\right)$$

$$J = \begin{pmatrix} \delta_{\alpha\beta}\delta_{\alpha\nu} \\ S^\mu_{\alpha\beta}(\theta_\beta - \theta_\alpha) \end{pmatrix}$$

展开表示：

$$J = \det\left(\frac{\partial x}{\partial y}\right) = \begin{pmatrix} \frac{\partial H}{\partial \theta} \\ \frac{\partial H}{\partial p} \end{pmatrix}.$$

$$J = \det\begin{pmatrix} d_{\alpha\beta}\, d_{\alpha\nu} \\ \delta^{\mu}_{\alpha\beta}\,(\theta_\alpha - \theta_\beta) \end{pmatrix}. \qquad \det\begin{pmatrix} a & b \\ \lambda c & \lambda d \end{pmatrix} = \lambda \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$\propto \prod_{\alpha<\beta}(\theta_\alpha - \theta_\beta)\ \underbrace{\det\begin{pmatrix} d_{\alpha\beta}\, d_{\alpha\nu} \\ \delta^{\mu}_{\alpha\beta} \end{pmatrix}}_{\text{只与}p\text{有关}}.$$

$$P(\lambda) = \int d(\lambda-\theta)\, e^{-A\sum_j \theta_j^2} \prod_{\alpha<\beta} |\theta_\alpha - \theta_\beta| \cdot f(p)\, d\theta\, dp.$$

$$\propto \int d(\lambda-\theta)\, e^{-A\sum_j \theta_j^2} \prod_{\alpha<\beta} |\theta_\alpha - \theta_\beta|\, d\theta.$$

物理意义.

概率分布 ⟷ Brownian motion.

宏观 ⟷ 微观.

$\theta_\alpha \leftrightarrow x_\alpha$. 本征值 ⟷ 坐标.

IF: $P \propto \prod_{i<j} |x_i - x_j|\, e^{-\frac{\beta}{2}\sum_i x_i^2} \propto e^{-\beta W}$

β：温度. W: potential.

$$E_i = -\left(\frac{\partial W}{\partial x_i}\right) = -x_i + \frac{1}{\beta}\sum_{j\neq i}\left(\frac{1}{x_i - x_j}\right).$$

P是 Fokker-Planck 方程的解：

$$f\frac{\partial P}{\partial t} = \sum_j \beta^{-1}\frac{\partial^2 P}{\partial x_j^2} - \frac{\partial}{\partial x_j}(E_j P)$$

Fokker-Planck eq.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$ 无 Random force.

$$\vec{J} = \vec{v}\rho$$

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2}$$ 无外力，只有扩散.

$$\frac{d^2 x_j}{dt^2} = -f\frac{dx_j}{dt} + E(x_j) + \xi(t)$$

$$E_j = -\frac{\partial W}{\partial x_j}$$

$$\langle \xi(t)\, \xi(t') \rangle = \frac{2k_B T}{f}\delta(t-t')$$

$$k_B T = \beta^{-1}$$

① $f\langle dx_j \rangle = E(x_j)\,dt$

② $f\langle dx_j^2 \rangle = 2k_B T\,dt$. 大数定理.