上一节课

① Runge-kutta.

ode
rk23
rk45

$$\begin{cases} y' = f(t, y) \\ y_i = f_i(t, \vec{y}) \end{cases}$$

Mathematica NDSolve

② Crank-Nicolson 方法 $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = a^2 \frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{\Delta x^2}.$$

核心技术.

$$\begin{cases} \frac{\partial}{\partial t} f = \frac{f(t+dt) - f(t)}{dt} \\ \frac{\partial^2 f}{\partial x^2} = \frac{f(x+dx) + f(x-dx) - 2f(x)}{dx^2}. \end{cases}$$

概念差分.

$$\frac{\partial f}{\partial t} = f(t, y).$$

$$f_{n+1} - f_n = dt\, f(t_n, y_n)$$

又. $f_{n+1} = f_n + dt\, f(t_n, y_n)$

$\begin{cases} \text{差分方程} \\ \text{级数.} \end{cases}$ $\frac{dx}{dt} = x - x^2$

$$x_{n+1} - x_n = dt(x_n - x_n^2).$$

出发点 Motivation.

1) 找方程的解.

2) 轨迹.

3) 高中级数/数列 ⇒ limit.

Chaos lorenz 63.

$\Big|$ lorentz

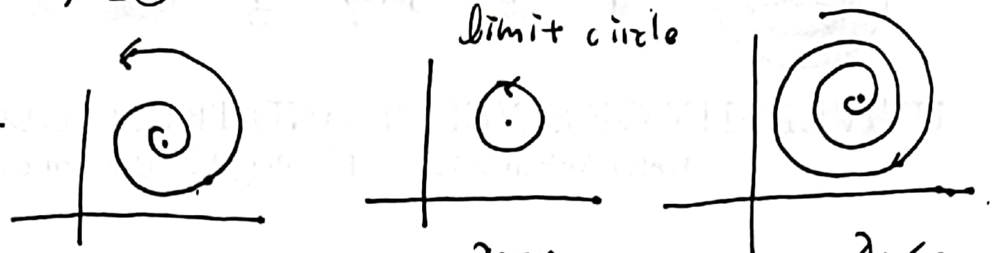
↓

相对论.

Lyapunov exponent.

$\frac{d\vec{x}}{dt} = F(\vec{x})$. $\Rightarrow F(\vec{x}^*) = 0$

$\vec{x}_{n+1} = \vec{x}_n + \vec{F}(\vec{x}_n)dt$



$\lambda_i > 0$. (其中一个本征值)

$\lambda_i = 0$

$\lambda_i < 0$. (所有本征值).

$\vec{x} = \vec{x}^* + \vec{y}$

$$\frac{d\vec{y}}{dt} = \dot{F}(\vec{x}^* + \vec{y}) = F(\vec{x}^*) + A\vec{y} . \Rightarrow \frac{d\vec{y}}{dt} = A\vec{y}.$$

$$A = P^{-1}\lambda P.$$

$$\frac{d\vec{y}}{dt} = P^{-1}\lambda P\vec{y} \qquad z = P\vec{y}$$

$$\frac{dz}{dt} = \lambda z.$$

$$\frac{dz}{dt} = \lambda z \Rightarrow \frac{dz_i}{dt} = \lambda_i z \Rightarrow z_i = e^{\lambda_i t} z_i(0)$$

两个例子:

1) $\frac{dx}{dt} = x - x^2$. (目的: 计算 exponent).

2) Lorenz model. (目的: 理解 chaos).

$$\begin{cases} \frac{dx}{dt} = \sigma(y-x) \\ \frac{dy}{dt} = rx - y - xz \\ \frac{dz}{dt} = -bz + xy. \end{cases}$$

1): $\frac{dx}{dt} = x - x^2 \Rightarrow x^* = 0, 1$.

令 $x = x^* + y = y$ ($x^* = 0$)

$$\frac{dy}{dt} = y - y^2 = y \quad (|y| \ll 1, y \to 0)$$

$\Rightarrow y(t) = e^t y_0$.

令 $x = 1+y$

$$\frac{dy}{dt} = (1+y) - (1+y)^2 = -y \Rightarrow y(t) = e^{-t} y_0 .$$

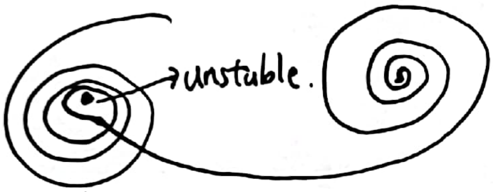
⇒ 回答：为什么迭代中有时很难找到方程的解.

找根 ① Newton

② 迭代法

~~2) Lorenz model.~~ 2) Lorenz model.

butterfly effect.



↳ 被高阶效应吸引. 靠近后因为不稳定点, 又跑出去了.

极限环: 跑不出某一个范围.

$$\frac{dx}{dt} = 0 \Rightarrow y = x$$

$$\gamma x - x - xz = 0$$

$$-bz + x^2 = 0 .$$

平衡解: $(x, y, z) = (0, 0, 0)$

② $x \neq 0 \quad z = \gamma - 1$.

$$x^2 = b(\gamma - 1) \qquad b(\gamma-1) > 0 \Rightarrow x = \pm\sqrt{(\gamma-1)b}$$

对于平衡解: $x, y, z \to 0$ $\left\{ \begin{array}{l} xy = 0 \\ xz = 0 \end{array} \right.$

$$\left\{ \begin{array}{l} \frac{dx}{dt} = \sigma(y-x) \\ \frac{dy}{dt} = \gamma x - y \\ \frac{dz}{dt} = -bz . \end{array} \right. \quad \Big\} \; \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ \gamma & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

例如取 $\sigma = 10$, $b = \frac{8}{3}$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} -10 & 10 & 0 \\ \gamma & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$\cancel{\det(A) = 0}$

$\det(\lambda - A) = 0$

$$\lambda_1 = -\frac{8}{3}$$

$$\lambda_2 = \frac{-11 - \sqrt{81 + 40\gamma}}{2}$$

$$\lambda_3 = \frac{-11 + \sqrt{81 + 40\gamma}}{2}.$$

$\gamma < 1$ 所有 eigenstates < 0

☆ $e^{\lambda t} = e^{\lambda_r t + i\lambda_i t}$

$\therefore$ $\gamma = 1$ $\quad \lambda_3 = 0$ 临界

$\gamma > 1$ $\quad \lambda_3 > 0 \Rightarrow$ unstable.

另一个点 $\vec{x}^* = \left(\sqrt{\frac{8}{3}(\gamma - 1)}, \sqrt{\frac{8}{3}(\gamma - 1)}, \gamma - 1\right)$.

$\vec{x} = \vec{x}^* + (u, v, w)$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{8(\gamma-1)/3} \\ \sqrt{8(\gamma-1)/3} & \sqrt{8(\gamma-1)/3} & -\frac{8}{3} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$\det(\lambda - A) = 0 \Rightarrow \mathrm{Det}(\lambda I - A) = 0.$

$3\lambda^3 + 41\lambda^2 + 8(\gamma + 10)\lambda + 160(\gamma - 1) = 0.$

IF $\lambda > 1$ $\begin{cases} 1 < \sigma < 1.3456 & \text{3个实负根.} \\ 1.3456 < \sigma < 24.737 & \text{三个根 } Re(\lambda_i) < 0. \\ \sigma > 24.737 & \text{有一个 } Re(\lambda_i) > 0 \end{cases}$

Unstable.

作业 画图 Lorenz eq(63)

mathematica 有现成代码.(自己调参数,观察)..

含时间问题 简单→复杂.

简单 ①  单摆.

②  弹簧单摆.

③ o~~~~o → v 以一定的速度抛出.

↓g.

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2)$$

$$U = mg z_1 + mg z_2 + \frac{1}{2}k(d_2 - x_0)^2.$$

$$d_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

$$\mathcal{L} = T - U. \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}.$$

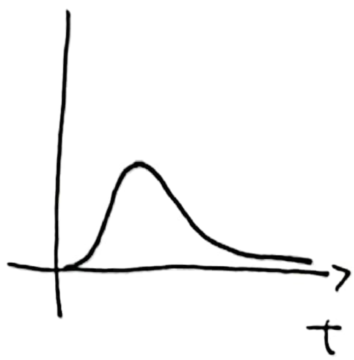
SIR 病毒传播 model.

S: 易感染人群.

I: 感染人.

R: ~~感染人~~ 康复人.

$$\begin{cases} \frac{dS}{dt} = -cSI. \\ \frac{dI}{dt} = \cancel{SS}cSI - gI. \\ \frac{dR}{dt} = gI. \end{cases}$$



N→多.

① 天体 $T=\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}$

$U = G m_i m_j / |\vec{r}_i - \vec{r}_j|$

$\Rightarrow m_i \ddot{\vec{r}}_i = -\sum_j \vec{F}_{ij}$

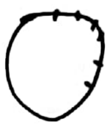
$$\vec{F}_{ij} = \frac{G m_i m_j}{|\vec{r}_i - \vec{r}_j|^3}(\vec{r}_i - \vec{r}_j).$$

$\vec{r}_i^{\,n+1}$ ← 时间

↳ 空间.

$$\frac{\vec{r}_i^{\,n+1} + \vec{r}_i^{\,n-1} - 2\vec{r}_i^{\,n}}{(dt)^2} = \frac{1}{2m_i}\left(\sum_j \vec{F}_{ij}^{\,n} + \sum_j \vec{F}_{ij}^{\,n+1}\right).$$

② 同步 kuramoto model.



$$\frac{d\theta_i}{dt} = w_i - \frac{K}{n}\sum_j \sin(\theta_i - \theta_j)$$

$$\Leftrightarrow \theta_i^{n+1} = \theta_i^n + dt\left[w_i - \frac{K}{n}\sum_j \sin(\theta_i^n - \theta_j^n)\right].$$

下一次课:

① 分子动力学.

② Flocking. 鱼群. 鸟群 集聚运动.

③ 随机数. 回顾概率论.

↳ 大约讲一个月.

④ 二次量子化.