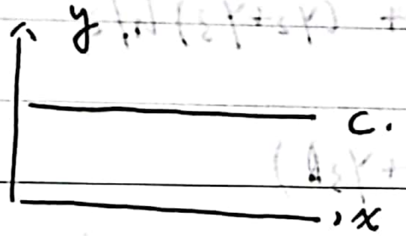
离散化 1积分.
偏微分方程. 矩阵.
⋮

多项式插值: ~~f(x)~~ $f(x) = poly + \underbrace{O(x^N)}_{error}$

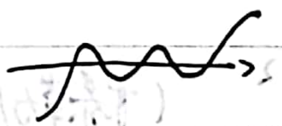
Runge.

常数函数:



多项式插值的结果: C. 没有问题.

f(x) = 多项式



Integrate[f, x].

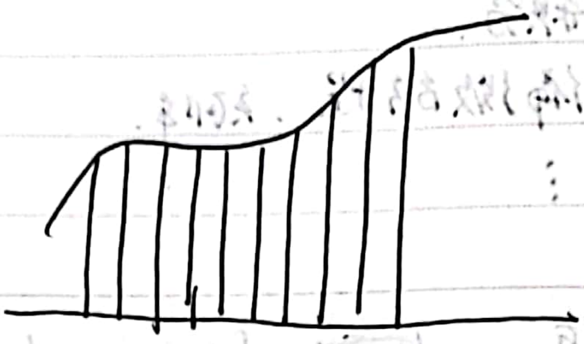
Integrate[f, {x, a, b}].

InterpolatingPolynomial.

☆☆ 积分的 ppt; 给例子.

1.1) simpson.

1.2) 高阶.



simpson. $\frac{h}{3}(y_1+4y_2+y_3)$

梯形: $(y_1+y_2)h/2 + (y_2+y_3)h/2$

$= \frac{h}{2}(y_1+2y_2+y_3)$

相减:

$= \frac{h}{6}(2y_2-y_1-y_3) \sim \frac{h^3}{6}y_2''$

$y_2 = y_1 - y_2'h + \frac{y_2''}{2}h^2$ (泰勒展开)

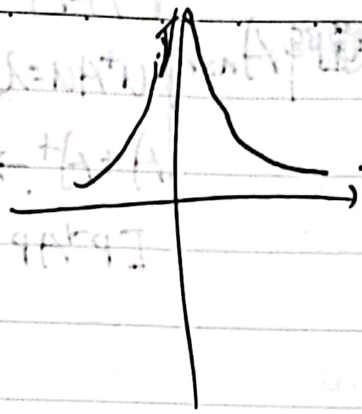
$y_2 = y_3 + y_2'h + \frac{y_2''}{2}h^2$.

推广 1d: $S = h\sum_i c_i f_i$. $c_i = (1, 2, \dots, 2, 1)$

$c_i = (1, 4, 2, 4, 2, \dots, 1)$

2d: $S = h^2 \sum c_{ij} f_{ij}$.

例: $\int_0^1 \frac{1}{\sqrt{x}}dx$



积分.

$\int_{-1}^{1} \frac{1}{\sqrt{|x|}}dx$

$= \int_{\varepsilon}^{1} \frac{1}{\sqrt{x}}dx$

$+ \int_{-1}^{-\varepsilon} \frac{1}{\sqrt{-x}}dx.$

奇点问题.

变步长.

矩阵:

核心: 1) 整体图像.

2). 程序.

矩阵:

1. $Ax=b$. $x=A^{-1}b$. $O(N^3)$

2. 特征值问题: $Ax=\lambda x$

对应两个函数库 (BLAS, LAPACK)

FFTW ARPACK.

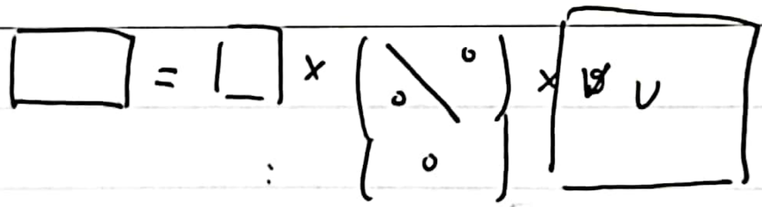
矩阵：类型

方阵 $A_{n\times n}$ $\begin{cases} A=A^+ \begin{cases} \mathbb{R}. \Rightarrow U\in O(N) \text{ 群.} \\ \mathbb{C} \Rightarrow U\in U(N) \text{ 幺正变换.} \end{cases} \\ U^+AU=\lambda \\ A\neq A^+ \to \text{相似变换. } P^{-1}xP=\lambda. \end{cases}$

$[P^{-1}AP, A]\neq 0$

非方阵 $A_{n\times m}$ $n\neq m$. ：奇异值分解 singular value decomposition

$A=UDV^T$



$\begin{cases} UU^T=I \\ VV^T=I \end{cases}$

MMA SingularValueDecomposition

分解具唯一的.

矩阵分解：方阵：

1) LU 分解.

mma: LUDecomposition.

找到 $A = (\quad) \times (\quad)$ L:下三角 U:上三角.

用处：求解 $Ax=b$ $LUX=b$ $\Rightarrow$ $LY=b$ $Y=UX$

2) QR分解. $(\text{unitary}) \times (\quad) = A$.

mma: QRDecomposition.

Grand-Schmidt 分解.

非方阵：SVD分解。对方阵和非方阵均适用.

特殊矩阵：

1. 范德蒙特行列式.

$$\det(A) = \prod_{i<j}(x_i - x_j) \Rightarrow \text{QHE}, \text{FQHE}.$$

2. Pfaffian $2n\times 2n$ skew matrix $A^T=-A$.

$$Pf^2(A) = \det(A)$$

3) 三对角阵，计算复杂度 $O(N^2)$. 占用内存小. $N=10^8$. 可解析计算

4) Toeplitz matrix

$$A = \begin{pmatrix} t_0 & t_{-1} & \cdots \\ t_1 & t_0 & \ddots \\ \vdots & & \ddots \end{pmatrix} = (t_{i-j})_{N\times N}.$$

Circular matrix

5). 随机矩阵. Random matrix.

$A = (\xi_{ij})_{N\times N}$ ξ_{ij}独立同分布的随机数.

作业：求解 google matrix 的本征值.
讨论其分布.（至少 $N \geq 2000$）

$G = cP + c(1-c)E$

矩阵乘法.

$$A_{n\times n} \times B_{n\times n} = \begin{pmatrix} \vdots \end{pmatrix} \times \begin{pmatrix} 1 & \cdots \end{pmatrix}$$

$O(n^3)$.

1969年 斯特拉森.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

七乘法 ~ 10 t加法.

$$AB = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

$P_1 = (A_{11} + A_{22})(B_{11} + B_{22})$.

$P_2 = (A_{21} + A_{22}) B_{11}$

$P_3 = A_{21}(B_{12} - B_{22})$

$P_4 = A_{22}(B_{21} - B_{22})$

$P_5 = (A_{11} + A_{12}) B_{22}$

$C_{11} = P_1 + P_4 - P_5 + P_7$

$C_{12} = P_3 + P_5$

$C_{21} = P_2 + P_4$

$C_{22} = P_1 - P_2 + P_3 + P_6$.

For $n \times n$.

$T(n) = 7\,T(\frac{n}{2}) + O(n^2)$

IF $T(n) \sim n^\alpha$

$$n^\alpha = 7(\tfrac{n}{2})^\alpha \Leftrightarrow \alpha = \log_2 7 \simeq 2.81.$$