宏观－微观

1905 爱因斯坦 $\frac{\partial}{\partial t}\phi = D\frac{\partial^2}{\partial x^2}\phi$　D: 扩散系数;

$\int dx\, \phi(x,t) = 1$.

$\phi \geq 0$

$$\frac{\partial j}{\partial x} + \frac{\partial \rho}{\partial t} = 0$$

$\frac{ds}{dt} = (\gamma+\xi)s$. $\Leftrightarrow d\ln s = \gamma + \xi$

$$\Rightarrow \ln S - \ln S_0 = \gamma t + \int_0^t \xi(t')\,dt'$$

Stock price. $\Leftrightarrow S = S_0 e^{\gamma t + \int_0^t \xi(t')\,dt'}$.

$10^4(1+0.01)(1-0.01)(1+0.01)(1-0.01)$.

1) 问题在哪？ chain rule.

2) 正确描述. Ito lemma. (随机分析创始人之一)　大数定理.

$\langle x^2\rangle \propto t$.

Ito lemma.

$\frac{dx}{dt} = a(x,t) + b(x,t)\xi$　ill-defined.

$\cancel{\frac{dx}{dt} = a\,dt}$

$dx = a\,dt + b(x,t)\,dW$　　$dW = \xi\,dt$.

增量

默认: $\frac{dx}{dt} = \frac{x(t+dt)-x(t)}{dt}\Big|_{dt\to 0^+}$

Wiener过程 (process). ⇒ Brown运动 ⇒ $dw = \xi dt$.

$w(t) = \int_0^t \xi(t')\,dt'$.

1) $\overline{w(t)} = 0$.

2) $\overline{w^2(t)} = \int_0^t \overline{\xi(t_1)\xi(t_2)}\,dt_1 dt_2$

$= \sigma^2 \int_0^t dt = \sigma^2 t$.

3) $\overline{w(t)w(s)} = \sigma^2 \min(t,s)$.

$w(t) = w(s) + \int_s^t \xi(t')dt'$

$\overline{w(t)w(s)}$

$= \overline{w^2(s)} + \overline{(\int_s^t \xi(t')dt')w(s)}$

4) $\overline{w^2(t)} - \overline{w^2(s)} = \sigma^2(t-s)$

5) $\overline{dw^2} = dt\,\sigma^2$.

$dx = a\,dt + b\,dw$.

$f(x,t)$满足什么方程.

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial^2 f}{\partial x \partial t}dx\,dt$.

Chain rule: $\frac{d}{dx}f(g(x)) = f'g'$

$(dx)^2 = a^2 dt^2 + b^2 dw^2 + 2ab\,dt\,dw$.

$\overline{dw^2} = \sigma^2 dt$

$dw \sim \sqrt{dt}$.

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保值

$$\frac{df}{dt} = \cancel{(\ )\,dw + (\ )\,dt}$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}b^2 dt$$

$$= \frac{\partial f}{\partial x}dx + \left\{\frac{\partial f}{\partial t} + \frac{b^2}{2}\frac{\partial^2 f}{\partial x^2}\right\}dt.$$

—— Ito lemma.

大数 ⇒ 二阶.

~~ds = ( ) + ( ) s~~

取 $f = \ln s$. $ds = rs\,dt + \sigma s\,dW$.

$$df = \frac{\partial f}{\partial s}ds + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}ds^2.$$

$$= \frac{1}{s}ds - \frac{1}{2}\frac{1}{s^2}(ds)^2$$

$$= \frac{1}{s}(rs\,dt + \sigma s\,dw) - \frac{1}{2s^2}(rs\,dt + \sigma s\,dw)^2$$

$$= r\,dt + \sigma dW - \frac{1}{2}\sigma^2 dW^2 = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma\,dW.$$

$$\Rightarrow df = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma\,dw.$$

$$f(t) = \left(r - \frac{\sigma^2}{2}\right)t + \sigma\int_0^t dw + f(0)$$

$$= f(0) + \left(r - \frac{\sigma^2}{2}\right)t + \sigma w(t)$$

$$\Rightarrow S = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma w(t)}$$

Euler 算法.

$$dx = a\,dt + b\,dw$$

$$x_{n+1} = x_n + a(x_n, t_n)\,dt + b(x_n, t_n)(w_{n+1} - w_n).$$

$$\int_0^T w\,dw = \sum_n w_n(w_n - w_{n-1})$$
$$\sum_n w_n(w_{n+1} - w_n)$$

在微积分中，结果相同（1阶无穷小）。但在数值计算中不同

$$\sum_n \overline{w_n(w_n - w_{n-1})} = \sum_n(\overline{w_n^2} - \overline{w_n w_{n-1}})$$
$$= \sum_n [n\,dt - (n-1)\,dt] = \sum_n dt = T.$$

$$\sum_n \overline{w_n w_{n+1}} - \overline{w_n^2} = \sum_n n\,dt - n\,dt = 0.$$

$$\int_0^T w\,dw = \int_0^T d\left(\frac{w^2}{2} - c\right) = \frac{w_T^2}{2} - \text{const}$$

平均值 $= \frac{T}{2} - \text{const}.$

Ito lemma.

$$f = \frac{1}{2}w^2.$$

$$df = \frac{\partial f}{\partial w}dw + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(dw)^2 = w\,dw + \frac{1}{2}dt.$$

$$\int_0^T df = \int_0^T w\,dw + \frac{T}{2}. \Rightarrow \int_0^T w\,dw = \overline{\frac{w_T^2}{2}} - \frac{T}{2} \Rightarrow \frac{T}{2} - \frac{T}{2} = 0.$$

数值 $O(h^3) \to O(dt^2) \to O(dt^3)$.

Ref:

1). Timothy Sauer: Computational solution of stochastic differential equations.

2) 刘玉金. 随机微分方程的数值解. (导师[illegible]).

3). 111 years of Brownian motion. Xin Bian.

$dx = a\,dt + \sigma\,dW$

$x \sim p(x,t) \to \begin{cases} dx = v\,t \\ dv = -\alpha v\,dt + \sigma\,dW - f\,dt \end{cases}$

$m\ddot{x} = -\alpha\dot{x} - f + \xi \qquad \dot{x} = v.$

$$\frac{\partial\phi}{\partial t} = D\frac{\partial^2\phi}{\partial x^2}.$$

① Kramers

1940 Brownian motion in a field of force and the diffusion model of chemical reactions

Physica Volume 7, Issue 4, April 1940, Pages 284-3

② Schrödinger WHAT IS LIFE?

③ Ming chen Wang 王明贞.

Uhlenbeck (spin)

On the ~~motion~~ theory of Brownian motion.

RMP 1945.

⑧ 1) 如何构造.

2) 满足定律: 1) 连续性方程. 2) 概率 = 1

回顾：刘维尔方程

$\rho(x, v; t)$

$$\begin{cases} \int \rho(x, v; t)\, dx\, dv = 1 \\ \dfrac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0. \end{cases}$$

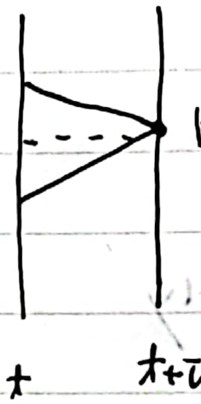
$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\dot{x} + \frac{\partial \rho}{\partial v}\dot{v} = 0$$ 准平衡.

$$= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}(\dot{x}\rho) + \frac{\partial \rho}{\partial v}(\dot{v}\rho) = 0$$

$$\nabla \cdot \vec{J} \Rightarrow \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial v}\right) \quad \vec{J} = (\dot{x}\rho, \dot{v}\rho)$$

求：Fokker-Planck eq.（Kramer eq.）

核心思想：Einstein.



$$p(x, t+\tau) = \int p(x+\Delta, t)\, \overbrace{p(\Delta)}^{\text{Brownian SDE.}}\, d\Delta.$$

$$= \int \left\{ p(x,t) + \frac{\partial p}{\partial x}\Delta + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}\Delta^2 \right\} p(\Delta)\, d\Delta$$

$$= p(x,t) + 0 + \frac{1}{2}\langle \Delta^2 \rangle \frac{\partial^2 p}{\partial x^2}$$

$$= p(x,t) + \frac{\partial p}{\partial t}\tau$$

$\langle \Delta^2 \rangle = 2D\tau$ ⇒ 得到扩散方程.

一般性地.

$$p(x, t+\tau) = \int \underbrace{p(x+\Delta, t)}_{\text{[illegible]}}\, \rho(\Delta, x, t, \tau)\, d\Delta.$$

$$= p(x,t) + \sum_{n=1}^{+\infty} \left(\frac{\partial}{\partial x}\right)^n p(x,t) \underbrace{\int \Delta^n \rho(\Delta, x, t, \tau)\, d\Delta}_{M_n(x,t,\tau)}$$

$$= p(x,t) + \sum_{n=1}^{+\infty} \left(\frac{\partial}{\partial x}\right)^n p(x,t) M_n(x,t,\tau)$$

$$\frac{\partial p}{\partial t}\tau = \sum_{n=1}^{+\infty} \left(\frac{\partial}{\partial x}\right)^n p(x,t) \underbrace{M_n(x,t,\tau)}$$

$M_n(x,t,\tau)$

①不能很简单求解.

②和 SDE 形式有关.

$$\frac{\partial p}{\partial t} = \sum_{n=1}^{+\infty} \left(\frac{\partial}{\partial x}\right)^n p(x,t) D_n^{(1)}(x,t)$$

下面理解 $\rho(\Delta, x, t, \tau)$ 的物理意义.

贝叶斯定理:

$$P(A) = \sum_B P(A|B) P(B)$$

$$p(x, t+\tau) = \int p(y,t)\, p(x, t+\tau | y, t)\, dy$$

$\rho(\Delta, x, t, \tau)$

$= p(x+\Delta, t+\tau)$ 条件概率.

传播子.

$\downarrow$

抽象的 $(\vec{x}, \vec{t})$