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Freeman J. Dyson

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A Brownian-Motion Model for the Eigenvalues of a Random Matrix

FREEMAN J. DYSON
Institute for Advanced Study, Princeton, New Jersey
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A new type of Coulomb gas is defined, consisting of n point charges executing Brownian motions under the influence of their mutual electrostatic repulsions. It is proved that this gas gives an exact mathematical description of the behavior of the eigenvalues of an $(n \times n)$ Hermitian matrix, when the elements of the matrix execute independent Brownian motions without mutual interaction. By a suitable choice of initial conditions, the Brownian motion leads to an ensemble of random matrices which is a good statistical model for the Hamiltonian of a complex system possessing approximate conservation laws. The development with time of the Coulomb gas represents the statistical behavior of the eigenvalues of a complex system as the strength of conservation-destroying interactions is gradually increased. A "virial theorem" is proved for the Brownian-motion gas, and various properties of the stationary Coulomb gas are deduced as corollaries.

I. STATIONARY MATRIX ENSEMBLES

THIS introductory section recapitulates known facts about ensembles of matrices. Consider an $(n \times n)$ square matrix M whose elements are of the form

$$M_{ij} = \sum_{\alpha=0}^{\beta-1} M_{ij\alpha} e_\alpha, \qquad i, j = 1, \cdots, n. \tag{1}$$

The $M_{ij\alpha}$ are real coefficients, and the e_α ($\alpha = 0, 1, \cdots, \beta - 1$) are units of a certain algebra Φ. The three possible choices for Φ are: (i) $\Phi = R$, the algebra of real numbers, for which $\beta = 1$, $e_0 = 1$; (ii) $\Phi = C$, the complex numbers, for which $\beta = 2$, $e_0 = 1$, $e_1 = i$; (iii) $\Phi = Q$, the algebra of real quaternions, for which $\beta = 4$, $e_0 = 1$, $e_1 = i$, $e_2 = j$, $e_3 = k$. We assume that M is Hermitian, which means $M = M^D$, where M^D is the matrix dual to M with respect to Φ. The coefficients of M^D are derived from those of M by the relations

$$M^D_{ij0} = M_{ji0}, \tag{2}$$

$$M^D_{ij\alpha} = -M_{ji\alpha} \quad \text{for} \quad \alpha \neq 0. \tag{3}$$

The number of independent coefficients for a Hermitian M is

$$N = n + \tfrac{1}{2}n(n-1)\beta. \tag{4}$$

It is convenient to label these independent coefficients M_μ, where μ is a single index running from 1 to N and replacing the triple index $(ij\alpha)$. We write

$$d_\mu = d_{ij\alpha} = 2 - \delta_{ij}. \tag{5}$$

Then

$$\text{spur } M^2 = \sum_{ij\alpha} M^2_{ij\alpha} = \sum_\mu d_\mu M^2_\mu. \tag{6}$$

The eigenvalues of M are n real numbers $(x_1, \cdots, x_n)$.

The properties of random matrices have often been discussed[1] on the basis of the so-called Gaussian ensemble E_G. E_G consists of the set of all Hermitian matrices M satisfying Eq. (1), with a probability distribution assigned as follows. The probability for finding each coefficient within a given small interval $[M_\mu, M_\mu + dM_\mu]$ is

$$P(M_1, \cdots, M_N)\, dM_1 \cdots dM_N, \tag{7}$$

$$P(M_1, \cdots, M_N) = c \exp\left[-\beta\, (\text{spur } M^2)/2a^2\right], \tag{8}$$

where c and a are constants. In E_G, the M_μ are independent Gaussian random variables, each having mean value zero and variance

$$\langle M^2_\mu \rangle = [a^2/2\beta] g_\mu, \tag{9}$$

$$g_\mu = g_{ij\alpha} = 1 + \delta_{ij}. \tag{10}$$

This ensemble has been found useful in many investigations of the statistical behavior of random matrices of large order.

The main theorem concerning the ensemble E_G is the following.[2]

Theorem I. If M is chosen at random in E_G, the probability for finding an eigenvalue within each of the small intervals $[x_i, x_i + dx_i]$ is

$$F(x_1, \cdots, x_n)\, dx_1 \cdots dx_n, \tag{11}$$

$$F(x_1, \cdots, x_n) = C\Big[\prod_{i<j} |x_i - x_j|^\beta\Big] \times \exp\Big[-\beta\Big(\sum_i x_i^2\Big)/2a^2\Big], \tag{12}$$

[1] E. P. Wigner, *Proceedings of the 4th Canadian Mathematics Congress* (Toronto University Press, Toronto, Canada, 1959), p. 174; C. E. Porter and N. Rosenzweig, Suomalaisen Tiedeakat. Toimituksia A VI, No. 44 (1960); Phys. Rev. **120**, 1698 (1960).

[2] A proof of this theorem for $\beta = 1$ is given by Porter and Rosenzweig (reference 1). Their argument can easily be extended to the cases $\beta = 2, 4$.

where C is a normalization constant depending on a, n, and β.

When $\beta = 1$, the distribution function (12) is known as the Wishart distribution.[3]

For any value of β, Eq. (12) can be written

$$F = C \exp [-\beta W], \tag{13}$$

$$W(x_1, \cdots, x_n) = -\sum_{i<j} \ln |x_i - x_j| + \sum_i (x_i^2/2a^2). \tag{14}$$

The distribution (12) is identical with the probability distribution of the positions of n point charges, free to move on the infinite line $[-\infty < x < \infty]$ under the influence of forces derived from the potential energy (14) according to the laws of classical mechanics, in a state of thermodynamical equilibrium at a temperature given by

$$kT = \beta^{-1}. \tag{15}$$

This system of point charges in thermodynamical equilibrium is called the Coulomb gas model[4] corresponding to the ensemble E_a.

In all previous discussions of random matrices, the matrix ensembles and the Coulomb gas models were assumed to be stationary. No time-dependence of the distribution was allowed, and the velocities of the charges x_i played no role in the calculations of thermodynamic properties of the Coulomb gas.

II. THEORY OF BROWNIAN MOTION

The idea of the present paper is to generalize the notion of matrix ensemble in such a way that the Coulomb gas model acquires a meaning, not only as a static model in timeless thermodynamical equilibrium, but as a dynamical system which may be in an arbitrary nonequilibrium state changing with time. The word "time" in this paper will always refer to a fictitious time which is a property of the mathematical model and has nothing to do with real physical time.

When one tries to interpret the Coulomb gas as a dynamical system, one naturally thinks of it first as an ordinary conservative system in which the charges move as Newtonian particles and exchange energy with one another only through the electric forces arising from the potential (14). One has then to give a meaning to the velocity of each particle, and to regulate the behavior of the matrix M in such a way that the eigenvalues x_i possess the normal Newtonian property of inertia. There does not seem to be any reasonable way of doing this. The program of interpreting the Coulomb gas as a conservative Newtonian system has therefore failed.

After considerable and fruitless efforts to develop a Newtonian theory of ensembles, we discovered that the correct procedure is quite different and much simpler. The x_i should be interpreted as positions of particles in Brownian motion.[5] This means that the particles do not have well-defined velocities, nor do they possess inertia. Instead, they feel frictional forces resisting their motion. The gas is not a conservative system, since it is constantly exchanging energy with its surroundings through these frictional forces. The potential (14) still operates on the particles in the following way. The particle at x_i feels an external electric force

$$E(x_i) = -\frac{\partial W}{\partial x_i} = \sum_{i \neq j} \left[\frac{1}{x_i - x_j}\right] - \frac{x_i}{a^2}, \tag{16}$$

in addition to the local frictional force and the constantly fluctuating forces which give rise to the Brownian motion.

A precise description of the Brownian motion of the Coulomb gas is the following.[6] Let the positions of the particles be $[x_1, \cdots, x_n]$ at time t. Then the positions at time $(t + \delta t)$ are $[x_1 + \delta x_1, \cdots, x_n + \delta x_n]$, where the δx_i are random variables. To the first order in the small quantity δt, the variables δx_i are independent and have first and second moments given by

$$f\langle \delta x_i \rangle = E(x_i)\ \delta t, \tag{17}$$

$$f\langle (\delta x_i)^2 \rangle = 2kT\ \delta t, \tag{18}$$

all higher moments being zero to this order. The constant f is the friction coefficient which fixes the rate of diffusion, $E(x_i)$ is the external force (16), and kT is the temperature in energy units.

An equivalent description of the motion is obtained by considering the time-dependent probability density $F(x_1, \cdots, x_n; t)$ for finding the particles at the positions x_i at time t. In consequence of Eqs. (17) and (18), F satisfies the Smoluchowski equation

[3] J. Wishart, Biometrika 20, 32 (1928).
[4] F. J. Dyson, J. Math. Phys. 3, 140 (1962).
[5] A convenient summary of the theory of Brownian motion is contained in G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. 36, 823 (1930), and in M. C. Wang and G. E. Uhlenbeck, Revs. Modern Phys. 17, 323 (1945). These two papers are reprinted in *Noise and Stochastic Processes*, edited by N. Wax (Dover Publications, New York, 1954).
[6] See Wang and Uhlenbeck, reference 5, Sec. 8.

$$f\frac{\partial F}{\partial t} = \sum_i \left[kT\frac{\partial^2 F}{\partial x_i^2} - \frac{\partial}{\partial x_i}(E(x_i)F)\right]. \quad (19)$$

Equation (19) describes the development with time of the Coulomb gas. Starting from an arbitrary initial probability distribution F at $t = t_0$, a unique solution of Eq. (19) will exist for all $t > t_0$. Any such solution we will call a *time-dependent Coulomb gas model.*

It is easy to verify that Eq. (19) implies Eqs. (17) and (18), so that the descriptions of the motion by Eqs. (17) and (18) and by Eq. (19) are equivalent. Also, there exists a unique solution of Eq. (19) which is independent of time, and this time-independent solution is given by Eqs. (13) and (14). So the stationary Coulomb gas is a special case of the more general time-dependent model.

A Brownian-motion model can also be constructed for the matrix M of which the x_i are the eigenvalues. Suppose that the coefficients of the matrix have the values $[M_1, \cdots, M_N]$ at time t, and the values $[M_1 + \delta M_1, \cdots, M_N + \delta M_N]$ at time $(t + \delta t)$. A Brownian motion of M is defined by requiring that each δM_μ is a random variable with the moments

$$f\langle \delta M_\mu \rangle = -[M_\mu/a^2]\,\delta t, \quad (20)$$

$$f\langle (\delta M_\mu)^2 \rangle = g_\mu kT\,\delta t, \quad (21)$$

with g_μ defined by Eq. (10). This is a Brownian motion of the simplest type, the various components M_μ being completely uncoupled, and each being subject only to a fixed simple harmonic force. The Smoluchowski equation corresponding to Eqs. (20) and (21) is

$$f\frac{\partial P}{\partial t} = \sum_\mu \left[\frac{1}{2} g_\mu kT \frac{\partial^2 P}{\partial M_\mu^2} + \frac{1}{a^2}\frac{\partial}{\partial M_\mu}(M_\mu P)\right], \quad (22)$$

where $P(M_1, \cdots, M_N; t)$ is the time-dependent probability density of the M_μ. The solution of Eq. (22) corresponding to a given initial condition $M = M'$ at $t = 0$ is known explicitly.[7] It is

$$P(M; t) = c[1 - q^2]^{-N/2}$$

$$\times \exp\{-\text{spur}\,(M - qM')^2/[2a^2kT(1 - q^2)]\}, \quad (23)$$

$$q = \exp[-t/a^2 f]. \quad (24)$$

The solution shows that the Brownian process is invariant under symmetry-preserving unitary transformations of the matrix M; in fact the awkward-looking factor g_μ in Eq. (21) is put in just in order to assure this invariance. When $t \to \infty$, $q \to 0$, and the distribution (23) tends to the stationary form (8), which is the unique time-independent solution of Eq. (22).

We are now ready to state our main result.

Theorem II. When the matrix M executes a Brownian motion according to the simple harmonic law (20), (21), starting from any initial condition whatever, its eigenvalues $(x_1, \cdots, x_n)$ execute a Brownian motion obeying the equations of motion (17), (18), (19) of the time-dependent Coulomb gas.

Note that the temperature kT is still related to the basic algebra Φ by Eq. (15).

To prove the theorem, we need only show that Eqs. (17) and (18) follow from Eqs. (20) and (21). Suppose then that Eqs. (20) and (21) hold. We have seen that the process described by Eqs. (20) and (21) is independent of the representation of M. Therefore we may choose the representation so that M is diagonal at the time t. The instantaneous values of the $M_{ij\alpha}$ at time t are then

$$M_{jj0} = x_j, \qquad j = 1, \cdots, n, \quad (25)$$

with all other components zero. At the later time $(t + \delta t)$, the matrix $(M + \delta M)$ is no longer diagonal, and its eigenvalues $(x_j + \delta x_j)$ must be calculated by perturbation theory. We have, to second order in δM,

$$\delta x_j = \delta M_{jj0} + \sum_{i \neq j} \sum_{\alpha=0}^{\beta-1} \frac{(\delta M_{ij\alpha})^2}{(x_j - x_i)}. \quad (26)$$

Higher terms in the perturbation series will give no contribution to first order in δt. When we take the expectation value on each side of Eq. (26), using Eqs. (20), (21), (15), and (16), the result is Eq. (17). When we take the expectation value of $(\delta x_j)^2$, only the first term on the right of Eq. (26) contributes to order δt, and this term gives Eq. (18) by virtue of Eq. (21). The theorem is thus proved.

The proof of Theorem II incidentally provides a new proof of the old Theorem I. The new proof is simpler than the standard proof[2] of Theorem I, and is in some respects more illuminating. The new proof shows how the repulsive Coulomb potential (14), pushing apart each pair of eigenvalues, arises directly from the perturbation formula (26). It has long been known that perturbations generally split levels which are degenerate in an unperturbed system. We now see that this splitting effect of perturbations is quantitatively identical with the repulsive force of the Coulomb gas model.

Theorem II is a much stronger statement than Theorem I. It shows that the electric force (16) acting upon the eigenvalue x_j has a concrete meaning

[7] Except for a misprint, this is Eq. (15) of Uhlenbeck and Ornstein (reference 5).

for any matrix M whatever, not only for an ensemble of matrices in stationary thermal equilibrium. The force $E(x_i)$ is precisely proportional to the mean rate of drift of x_i which occurs when the matrix M is subjected to a random perturbation.

III. UNITARY BROWNIAN MOTION MODEL

In Sec. II we constructed a Brownian motion model for a random Hermitian matrix. We now construct a similar model for a random matrix U which is unitary and self-dual with respect to Φ. Such a matrix has elements of the form

$$U_{ij} = \sum_{\alpha=0}^{\beta-1} U_{ij\alpha} e_\alpha, \qquad i, j = 1, \cdots, n, \tag{27}$$

where the $U_{ij\alpha}$ are now complex coefficients but still satisfy the same symmetry condition $U = U^D$. The eigenvalues of U are n complex numbers $[\exp(i\theta_j)]$, $j = 1, \cdots, n$, all lying on the unit circle.

An isotropic and representation-independent Brownian motion of U is defined in the following way. Every unitary self-dual U can be represented in the form

$$U = VV^D, \tag{28}$$

with V unitary. A permissible small change in U is then given by

$$\delta U = iV\, \delta M\, V^D, \tag{29}$$

where δM is an infinitesimal Hermitian self-dual matrix. Suppose that U satisfies Eq. (28) at the time t. We assume that U moves by Brownian motion to the position $(U + \delta U)$ at time $(t + \delta t)$, where δU is given by Eq. (29), and the matrix δM has real coefficients δM_μ which are independent random variables with the moments

$$\langle \delta M_\mu \rangle = 0, \tag{30}$$

$$f\langle (\delta M_\mu)^2 \rangle = g_\mu kT\, \delta t. \tag{31}$$

This Brownian motion of U is a pure diffusion without any restoring force, since the harmonic force term which appeared in Eq. (20) has been omitted from Eq. (30). The rate of diffusion given by Eq. (21) remains the same as before.

The process of diffusion will spread the probability distribution of U more and more evenly as time goes on. The unique time-independent configuration for U is one in which the probability density is constant (in the sense of invariant group measures) over the whole space of unitary self-dual matrices. This uniform distribution of U is the stationary ensemble E_β defined in an earlier paper.[4]

Now we consider the effect of the Brownian motion of U on the eigenvalues $[\exp(i\theta_j)]$. We may use a representation in which U is diagonal at time t. The perturbation-theory formula analogous to Eq. (26) is then

$$\delta\theta_i = \delta M_{ii0} + \sum_{i \neq j} \sum_{\alpha=0}^{\beta-1} \{(\delta M_{ij\alpha})^2 \tfrac{1}{2} \cot [\tfrac{1}{2}(\theta_i - \theta_i)]\}. \tag{32}$$

Equations (30) and (31) then imply that the angles θ_i execute a Brownian motion with

$$f\langle \delta\theta_i \rangle = E(\theta_i)\, \delta t, \tag{33}$$

$$f\langle (\delta\theta_i)^2 \rangle = 2kT\, \delta t, \tag{34}$$

$$E(\theta_i) = \sum_{i \neq j} \tfrac{1}{2} \cot [\tfrac{1}{2}(\theta_i - \theta_i)]. \tag{35}$$

This force $E(\theta_i)$ is exactly the component, tangential to the circle, of the electric field produced at $\exp(i\theta_i)$ by unit charges at all the other points $\exp(i\theta_i)$ at which U has eigenvalues. Thus

$$E(\theta_i) = -(\partial W/\partial\theta_i), \tag{36}$$

$$W = -\sum_{i<j} \ln |\exp(i\theta_i) - \exp(i\theta_j)|. \tag{37}$$

So the eigenvalue angles θ_i behave like a gas of n unit charges on the unit circle, executing Brownian motions with repulsive Coulomb forces derived from the potential (37). Every free diffusion of the matrix U gives rise to a corresponding Brownian motion of the Coulomb gas formed by its eigenvalues on the unit circle.

In particular, the uniform probability distribution of U corresponds to the unique stationary probability density,

$$F(\theta_1, \cdots, \theta_n) = c \exp[-\beta W] = c \prod_{i<j} |\exp(i\theta_i) - \exp(i\theta_j)|^\beta, \tag{38}$$

for the eigenvalue angles. This is again a new and simple proof of an old result.[4]

IV. APPLICATION TO SYSTEMS WITH SEMI-CONSERVED QUANTUM NUMBERS

The stationary random-matrix models which have been studied in the past have always had an all-or-nothing character. That is to say, they represented situations in which a certain set of quantities (for example, total spin, charge, or isotopic spin) was exactly conserved, while no other integral of the motion existed even approximately. The total set

of states of the system could then be divided into subsets, each subset corresponding to a particular set of values for the conserved quantities. The energy levels of states belonging to different subsets would be completely uncorrelated, while the level distributions within each subset would be described by a separate random-matrix model.

The time-dependent Brownian motion models discussed in Sec. II provide a basis for a statistical theory of energy levels in systems possessing approximate conservation laws. Such systems occur in practice more frequently than those of the all-or-nothing type. For example, in complex atomic spectra it is usually the case that either an LS or a JJ coupling scheme is approximately valid, so that it is incorrect to treat all matrix elements of the Hamiltonian as having the same *a priori* probability distribution. We will now illustrate with a simple model how such situations can be quantitatively described in terms of the Brownian-motion picture.

Suppose that a system has an approximately conserved two-valued quantum number g, taking the values 1 and 2. Suppose that the manifolds of quantum states with $g = 1, 2$ have dimensions n_1, n_2, respectively. We write the Hamiltonian H in a representation with g diagonal. Then H splits into four blocks

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}. \tag{39}$$

For simplicity we suppose that we are in the case $\Phi = R$, $\beta = 1$ of Sec. I, so that the matrix H is real and symmetric. A reasonable statistical hypothesis for the elements of H is then the following. Each element is an independent random variable, having a Gaussian distribution with mean value zero and variance given by

$$\langle (H_{11,ij})^2 \rangle = \tfrac{1}{2}(1 + \delta_{ij})a^2\xi, \tag{40}$$

$$\langle (H_{22,ij})^2 \rangle = \tfrac{1}{2}(1 + \delta_{ij})a^2\eta, \tag{41}$$

$$\langle (H_{12,ij})^2 \rangle = \tfrac{1}{2}a^2\epsilon. \tag{42}$$

The parameter ϵ measures the ratio between the strengths of non-g-conserving and g-conserving interactions. It is convenient to choose ξ and η given by

$$\xi = 1 + [n_2/(1 + n_1)](1 - \epsilon), \tag{43}$$

$$\eta = 1 + [n_1/(1 + n_2)](1 - \epsilon). \tag{44}$$

Then Eqs. (40) to (42), with $\epsilon = 1$, describe the situation in which g is completely unconserved; the probability distribution of H is then identical with the stationary ensemble E_G in which all the elements of H are treated alike. When $\epsilon = 0$, Eqs. (40) to (42) describe two independent Gaussian ensembles E_{G1}, E_{G2}, referring to the states with $g = 1$ and $g = 2$ separately; this is the case of exact g conservation. The advantage of the choice (43) and (44) for ξ and η is that it makes the variance of any eigenvalue

$$\langle x_i^2 \rangle = \tfrac{1}{2}a^2(n_1 + n_2 + 1) \tag{45}$$

independent of ϵ and the same for E_{G1}, E_{G2}, E_G. Thus the over-all spread of the eigenvalue distributions does not vary as the parameter ϵ is changed.

We denote by $E(\epsilon)$ the ensemble of matrices H whose elements are Gaussian variables satisfying Eqs. (40) to (44). It is easy to verify by referring to Eqs. (23) and (24) that $E(\epsilon)$ is identical with the time-dependent Brownian motion model defined by Eqs. (20) and (21), provided we choose

$$kT = 1, \quad \epsilon = 1 - q^2 = 1 - \exp[-2t/a^2f]. \tag{46}$$

The initial condition at $t = \epsilon = 0$ is

$$E(0) = E_{G1} \times E_{G2}, \tag{47}$$

while in the limit $t \to \infty$ we have

$$E(\epsilon) \to E(1) = E_G. \tag{48}$$

The distribution of eigenvalues for a matrix in $E(\epsilon)$ is determined by Theorem II. We have thus proved

Theorem III. Let H be a matrix chosen at random in the ensemble $E(\epsilon)$. Its eigenvalues $(x_1, \cdots, x_n)$ are then distributed according to the time-dependent Coulomb gas model defined by Eq. (19), taken at the time

$$t = -\tfrac{1}{2}a^2f[\ln(1 - \epsilon)], \tag{49}$$

the initial condition at $t = 0$ being a superposition of two uncorrelated stationary Coulomb gases containing n_1 and n_2 charges, respectively.

The statistical distribution of eigenvalues of a system with an approximate conservation law is thus determined in principle. The eigenvalue distribution is the solution of the Smoluchowski equation (19) with the initial condition that F should be at $t = 0$ a product of two Wishart distributions. This solution has a very simple physical interpretation. The situation with exact g conservation is represented by two superposed noninteracting stationary Coulomb gases. At the instant $t = 0$ a repulsive Coulomb interaction between the charges in one gas and those in the other is suddenly switched on. The resulting nonstationary Coulomb gas is

allowed to adjust itself to the sudden change in the forces by Brownian motion for a time given by Eq. (49). At the end of this time the gas will represent the eigenvalue distribution for the situation with partial g conservation.

It is an interesting and deep mathematical problem to describe accurately the approach of the Coulomb gas to equilibrium as t or ϵ increases. Since we were careful to choose the parameters ξ and η to make the over-all shape of the charge distribution independent of t, the approach to equilibrium involves only local adjustments. We make the conjecture that the approach to equilibrium proceeds exponentially with a time scale which is of the order of

$$\tau = [fS/E], \tag{50}$$

where S is an average level-spacing and E an average Coulomb force between nearest-neighbor charges. This τ is the time taken for a single level to respond to the changed interaction between itself and its neighbors. In the ensemble E_G, Eq. (45) gives

$$S \cong a(n_1 + n_2)^{-1/2}, \qquad E \cong S^{-1}, \tag{51}$$

and therefore

$$\tau \cong [fa^2/(n_1 + n_2)]. \tag{52}$$

Comparing this with Eq. (49), we see that the eigenvalues already approach their final distribution when ϵ is of the order

$$\epsilon \sim (n_1 + n_2)^{-1}. \tag{53}$$

So for matrices whose order $n = n_1 + n_2$ is large, the g-violating matrix elements $H_{12,ij}$ need only to be of the order of $(n^{-1/2})$ times the g-conserving elements in order to bring the eigenvalue distribution close to the state of complete g-nonconservation. In random matrices of large order, a conservation law must be almost exact in order to produce a noticeable effect on eigenvalue distributions.

A different type of system with partially conserved quantum numbers can arise in the following way. Suppose that there are two almost-conserved quantities g, g' which do not commute with each other. Suppose that g takes the values 1, 2, while g' has matrix elements linking states with $g = 1$ to states with $g = 2$. A familiar example of such a situation occurs when g and g' are two different components of angular momentum. The Hamiltonian H again splits into four blocks as in Eq. (39). However, the states of the system must now become degenerate in pairs in the limit of exact g and g' conservation. A reasonable statistical hypothesis for H is that its elements are independent Gaussian variables satisfying the Brownian motion conditions (20) and (21), but now starting from the initial condition

$$H = \begin{bmatrix} H' & 0 \\ 0 & H' \end{bmatrix} \tag{54}$$

at $t = 0$, with H' distributed according to the Gaussian ensemble E_{G1}. In this case we must have $n_1 = n_2 = \frac{1}{2}n$. The distribution of eigenvalues in the system with partial g and g' conservation will be obtained from the time-dependent Coulomb gas as before, but now the initial condition has the positions of the charges coinciding in pairs. Instead of two noninteracting Coulomb gases, we have in the initial state a single Coulomb gas of $\frac{1}{2}n$ charges, each charge being suddenly replaced by two independently moving charges at the instant when the Brownian motion begins. The approach to equilibrium should again occur in a time of the order of Eq. (52).

Many other statistical ensembles, describing random matrices with approximate conservation laws, could be constructed from the time-dependent Brownian motion model with suitable initial conditions. In some cases it will be appropriate to use ensembles with $\beta = 2$ or 4 instead of $\beta = 1$, depending on the type of group representation which the symmetry of the problem requires.[8] We will not attempt here any systematic discussion of the possible alternatives. The two simple examples which we described above show well enough the general principles to be followed.

V. PROPERTIES OF THE BROWNIAN MOTION MODEL

In studying the approach to equilibrium of the time-dependent Coulomb gas defined by Eqs. (16) to (18), we have derived a few general properties of the gas which may be worthy of record. Let $G = G(x_1, \cdots, x_n)$ be any function of the positions of the charges, not depending explicitly on time. Then $\langle G \rangle$, the ensemble average of G, varies with time according to the equation

$$f\frac{d}{dt}\langle G\rangle = -\sum_i \left\langle \frac{\partial W}{\partial x_i}\frac{\partial G}{\partial x_i}\right\rangle + kT\sum_i \left\langle \frac{\partial^2 G}{\partial x_i^2}\right\rangle, \tag{55}$$

with W given by Eq. (14).

Take in this equation

$$R = \sum_k \langle x_k^2 \rangle \tag{56}$$

for $\langle G \rangle$. The equation becomes

$$\tfrac{1}{2}a^2 f\dot{R} = R_\infty - R, \tag{57}$$

[8] F. J. Dyson, J. Math. Phys. 3, 1199 (1962), following paper.

with

$$R_\infty = a^2[\tfrac{1}{2}(n^2 - n) + nkT]. \tag{58}$$

Equation (57) has the solution

$$R = R_0 q^2 + R_\infty(1 - q^2), \tag{59}$$

where q is given by Eq. (24) and R_0 is the value of R at $t = 0$. Equation (59) shows rigorously that at least the ensemble average R approaches its equilibrium value R_∞ with exponential speed as $t \to \infty$.

Next take $G = W$ in Eq. (55). After some algebra we find

$$f(d\langle W\rangle/dt) = (kT - 1)\sum_{i\neq j}\langle(x_i - x_i)^{-2}\rangle + (n^2 - n + nkT)a^{-2} - \sum_i \langle x_i^2\rangle a^{-4}. \tag{60}$$

For the stationary Coulomb gas at temperature T, the left side of Eq. (60) vanishes, and Eq. (58) may be used on the right. We thus find a "virial theorem" for the stationary gas,

$$\sum_{i\neq j}\langle(x_i - x_i)^{-2}\rangle = \frac{n^2 - n}{2a^2(1 - kT)}. \tag{61}$$

The probability density of eigenvalues becomes proportional to $|x_i - x_i|^\beta$ when two eigenvalues (x_i, x_i) come close together. The ensemble average of $(x_i - x_i)^{-2}$ is therefore defined only for $\beta > 1$, and Eqs. (60), (61) hold only for $kT < 1$.

We are especially interested in the case $kT = 1$, which requires a passage to the limit in Eq. (60). As $kT \to 1$, we have for any fixed value of Δ

$$\lim (kT - 1)\int_{-\Delta}^{\Delta} |y|^{\beta-2}\, dy = \lim (kT - 1)(\beta - 1)^{-1}2\Delta^{\beta-1} = -2. \tag{62}$$

We obtain the correct limit in Eq. (60) if we replace

$$(kT - 1)(x_i - x_i)^{-2}$$

by

$$-2\,|x_i - x_i|^{-1}\,\delta(x_i - x_i), \tag{63}$$

which has a well-defined meaning as an ensemble average when $kT = 1$, since the probability density then contains a factor $|x_i - x_i|$. Equation (60) thus becomes, in the limit,

$$f(d\langle W\rangle/dt) = -2\sum_{i\neq j}\langle|x_i - x_i|^{-1}\,\delta(x_i - x_i)\rangle + n^2a^{-2} - \sum_i \langle x_i^2\rangle a^{-4}, \qquad kT = 1. \tag{64}$$

The corresponding "virial theorem" is

$$\sum_{i\neq j}\langle|x_i - x_i|^{-1}\,\delta(x_i - x_i)\rangle = \frac{n^2 - n}{4a^2}, \qquad kT = 1, \tag{65}$$

for the stationary gas.

Equation (64) suggests very forcibly the following picture of the approach of the gas to equilibrium. The first term on the right is a "collision term" measuring the frequency with which two charges come into coincidence. This term is mainly sensitive to the local (microscopic) configuration of the gas particles. By means of this term the gas will come into local thermodynamic equilibrium in a microscopic time scale

$$t \sim fa^2n^{-1}. \tag{66}$$

After local equilibrium is established, the gas will still not be in a stationary state, because the third term on the right of Eq. (64) will not in general have its stationary value. The gas must adjust itself by macroscopic motion on the time scale

$$t \sim fa^2, \tag{67}$$

until the over-all charge distribution reaches its stationary shape.

We chose the parameters ξ and η in Eqs. (40) to (42) so that the Coulomb gas representing the eigenvalues of a system with semiconserved quantum numbers should have the stationary macroscopic shape from the beginning. In this case the entire process of reaching equilibrium should occur with the microscopic time scale (52).

Of course the picture of the gas coming into equilibrium in two well-separated stages, with microscopic and macroscopic time scales, is only suggested by Eq. (64) with the help of physical intuition. A rigorous proof that this picture is accurate would require a much deeper mathematical analysis.

The equations of this section all have analogs for the time-dependent Coulomb gas on the unit circle, with the Brownian motions defined by Eqs. (33) to (35). The potential energy W is now given by Eq. (37). The macroscopic mass center of the gas is

$$R = n^{-1}\sum_k \langle\exp(i\theta_k)\rangle, \tag{68}$$

and the analog of Eq. (57) is

$$f\dot{R} = -[\tfrac{1}{2}(n - 1) + kT]R. \tag{69}$$

The analogs to Eqs. (60) and (64) are[9]

[9]Here use is made of the identity

$$\cot a \cot b + \cot b \cot c + \cot c \cot a = 1,$$

which holds when $a + b + c = 0$.

$$f(d\langle W\rangle/dt) = \tfrac{1}{12}(n^3 - n) + (kT - 1)\sum_{i\neq j} \times \langle|\exp(i\theta_i) - \exp(i\theta_j)|^{-2}\rangle, \quad kT < 1, \tag{70}$$

$$f(d\langle W\rangle/dt) = \tfrac{1}{12}(n^3 - n) - 2\sum_{i\neq j} \times \langle|\theta_i - \theta_j|^{-1}\,\delta(\theta_i - \theta_j)\rangle, \quad kT = 1. \tag{71}$$

These give virial theorems analogous to Eqs. (61) and (65). We state the results only for the limiting case $n \to \infty$ which is most important in practice.

Let x_j, $(-\infty < j < +\infty)$, be the positions of charges in an infinite Coulomb gas in thermodynamic equilibrium at temperature T. Let D be the mean spacing between nearest neighbors. The virial theorems are

$$\sum_{j\neq 0}\langle(x_j - x_0)^{-2}\rangle = \frac{\pi^2}{3D^2(1 - kT)}, \quad kT < 1, \tag{72}$$

$$\sum_{j\neq 0}\langle|x_j - x_0|^{-1}\,\delta(x_j - x_0)\rangle = \frac{\pi^2}{6D^2}, \quad kT = 1. \tag{73}$$

Equation (73) is a known result, giving the slope of the level-spacing distribution function at zero spacing.[10] The above derivation of it seems to be the simplest yet found.

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[10] M. L. Mehta, Nuclear Phys. 18, 395 (1960).

[11] A. Lenard, J. Math. Phys. 2, 682 (1961); S. F. Edwards and A. Lenard, *ibid.* 3, 778 (1962). The Edwards-Lenard paper describes a Brownian motion model which has some similarity to ours; however, their model differs fundamentally from ours in identifying the fictitious time variable t with the space coordinate x.