Topics around Synthetic Spectra

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Abstract

We hope to learn some basics and techniques related to synthetic spectra, such as the application of synthetic spectra method in the computation of stable homotopy groups of spheres, obstruction theory of multiplicative structures on spectra, even filtration, etc...

1 Introduction

There are many applications of the framework of synthetic spectra, see [HRW22, Bur22, BX23, BIX24, LWX24]. It seems that the word "synthetic spectra" first appeared in [HL20, Definition 4.1.1] according to [AR24, Remark 2.14], and the theory was developed in [Pa23]. In [HL20], the category of Esynthetic spectra is defined to be category of spherical sheaves (finite products preserving presheaves) of a certain subcategory $\operatorname{Mod}_E^{\operatorname{mol}}$ of Mod_E . The category $\operatorname{Mod}_E^{\operatorname{mol}}$ should be viewed as cellular Emodules satisfying some conditions, this will coincide with [Pa23] in some sense. And there are three models for synthetic spectra, magically, 1. spherical sheaves [Pa23], 2. filtered spectra [GIKR22], 3. cellular motivic modules [GIKR22]. In some scenes, the most proper language is filtered spectra, e.g. motivic filtration, Hochschild–May spectral sequence for THH, and things related to localizing invariant. Basically because the input of localizing invariant requires some algebra structure (in the sense of [Lur17]), and the algebra can be viewed as filtered algebra easily via Postnikov tower or Whitehead tower. And for the computation of spectral sequences, it is also not surprised for the powerful of filtered spectra, because basically the theory of spectral sequence is equivalent to the theory of filtration. However, for the obstruction theory of multiplicative structure, the theory of synthetic spectra comes more naturally (and equivalently for filtered spectra), see [HL20, PV19]. For the motivic part, it comes more subtly, as a beautiful accident, in my current impression.

2 Hope and some candidate topics

There are some hopes for the talks, namely

- Ideally, the speaker should present the main idea, main results, motivations in the first 5-10 minus.
- If there is no proof presented in the talk, then the speaker should carry out some computational examples.
- The speaker is encouraged to write a note, not necessarily.

We hope to learn something related to synthetic spectra, in the sense that every talk should related to the theory of synthetic spectra, more concrete one could choose following topics.

- 1. Applications in the computations of homotopy groups of sphere.
 - (a) Computation of Adams differentials. See [BX23, BIX24, LWX24].
 - (b) Detecting v_n -periodicity elements, see [CD24].
- 2. Obstruction theory for \mathbb{E}_n -structures on Moore spectra. See [Bur22, Theorem 1.2, 1.4, 1.5], and also [PV19].

- 3. Motivic filtration (even filtration) for THH, and cyclotomic synthetic spectra.
 - (a) Motivic filtration (even filtration), and computation of syntomic cohomology for Adams summand, [HRW22].
 - (b) Cyclotomic synthetic spectra, [AR24].
- 4. Hochschild–May spectral sequence for THH.
 - (a) Computation of THH of image of J, see [AK21, LL23].
 - (b) Proof of Segal conjecture for THH(BP $\langle n \rangle$), see [HW22, §4].
- 5. Comparison of three models, spherical sheaves, filtered spectra, cellular motivic modules. See [GIKR22, Pa23].

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3 Some rules

Although this should be a freedom seminar, in the sense that anyone could choose any topic that you like, however there are still some unusual points I would like point out.

- A talk should longer than (\geq) 50 minus, and there is no upper-bound for the time principally.
- If the talk is less than 50 minus, then the speaker must come up with some specific questions, problems, conjectures, related to your talk. Or, you could present some computational examples unless you haven't present them in the talk.
- There is principally no restriction to the time of discussion, it should longer than 5 minus.
- There could be **more than** one talk in a seminar week, either by one person or different persons.
- If there is nobody will give the talk some week, then we still have to meet (virtually) for at least 10 minus (except for the period of Spring festival, concretely, 1/27/2025-2/4/2025). And everyone (participate) should come up with one question, or problem, conjecture.

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