NOTES ON PERIODICITY THEOREM

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Abstract. In this notes, we introduce the periodicity theorem and explain the proof.

Contents

1. Introduction

Definition 1.1. Let X be a $p-$local finite spectrum, and $n \geq 0$. A self-map $v : \Sigma^k X \to X$ is said to be a v_n self-map if

$$K(m)_* v = \begin{cases} \text{multiplication by a rational number,} & \text{if } m = n = 0, \\ \text{isomorphism,} & \text{if } m = n \neq 0, \\ \text{nilpotent,} & \text{otherwise.} \end{cases}$$

Let $\mathcal{C}_0$ be the homotopy category of $p-$local finite spectra, we denote $\mathcal{C}_n, n \geq 1$ the full subcategory of $\mathcal{C}_0$ with $K(n-1)_*(X) = 0, \forall X \in \mathcal{C}_n$.

The main reference is [3], and the main theorem is following,

Theorem 1.2. [3, Theorem 9] *A $p-$local finite spectrum X admits a v_n self-map iff $X \in \mathcal{C}_n$. And if X admits a v_n self-map, then exists $N \in \mathbb{N}$, and $v : \Sigma^{2(p^n-1)p^N} X \to X$, such that such that*

$$K(m)_* v = \begin{cases} v_n^{p^N}, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2.1}$$

Denote $\mathcal{V}_n$ to be the full subcategory of $\mathcal{C}_0$ admitting self-map v satisfies (1.2.1). So Theorem 1.2 is just saying $\mathcal{V}_n = \mathcal{C}_n$.

Proof of Theorem 1.2 (logically).

We need three facts.

(1) $\mathcal{V}_n$ is thick.
(2) $\mathcal{V}_n \subseteq \mathcal{C}_n$.
(3) $\exists x \; x \in \mathcal{C}_n \setminus \mathcal{C}_{n+1} \wedge x \in \mathcal{V}_n$.

Fact 1 says $\exists m \in \mathbb{N}, \mathcal{V}_n = \mathcal{C}_m$(thick subcategory theorem), fact 2 says $m \geq n$, fact 3 says $m = n$. Otherwise, $m \geq n+1$, then $X \in \mathcal{C}_m \setminus \mathcal{C}_{n+1} = \varnothing$, a contradiction. So, $\mathcal{V}_n = \mathcal{C}_n$. □

Now, we only need to prove those three facts. Basically fact 1 followed by some algebraic facts about $K(m)_*(X \wedge DX)$, and we will prove it in section 2. The proof of fact 3 involves constructing $X \in \mathcal{C}_n \backslash \mathcal{C}_{n+1}$ and an element in $\pi_*(X \wedge DX)$ such that the image in $K(m)_*(X \wedge DX)$ satisfies 1.2.1, we will use several Adams spectral sequences to "approximates" $v_n^{p^N}$, and we will prove fact 3 in section 3.

Now we present a proof of fact 2.

Proof of fact 2. It is equivalent to show if X is $p-$local finite spectra, admitting v satisfies 1.2.1, then $K(n-1)_*X = 0$. Suppose the converse holds, i.e $\exists X\ X \in \mathcal{V}_n \setminus \mathcal{C}_n$, then $K(n-1)_*X \neq 0$. We denote the cofiber of $v : \Sigma^{2(p^n-1)p^N} X \to X$ as Y. Apply $K(n-1)_*$ to the exact triangle, we have

$$0 \to K(n-1)_*X \to K(n-1)_*Y \to K(n-1)_{*-1}\Sigma^{2(p^n-1)p^N} X \to 0.$$

This gives $K(n-1)_*(Y) \neq 0$. Apply $K(n)_*$ to the exact triangle, we get $K(n)_*(Y) = 0$. These together gives $Y \in \mathcal{C}_{n+1} \setminus \mathcal{C}_n = \varnothing$. A contradiction. $\square$

2. $\mathcal{V}_n$ IS THICK

We may consider $n \geq 1$, because for any $p-$local finite spectra, the degree p map is a v_0-map.

By the theory of Spanier-Whitehead duality, a self map $\Sigma^k X \to X$ is dual to $S^k \to R = F \wedge DF$.

Definition 2.1. Let R be a finite ring spectrum, $n \geq 1$. An element

$$\alpha \in \pi_* R$$

is called v_n-element if

$$K(m)_*\alpha = \begin{cases} \text{unit}, & \text{if } m = n, \\ \text{nilpotent}, & \text{otherwise.} \end{cases}$$

Remark 2.2. $K(m)_*\alpha$ is the image of α under the map $\pi_*(S^0 \wedge R) \to \pi_*(K(m)_* \wedge R)$.

Claim. The definition 1.1 and definition 2.1 are equivalent in some sense.

Proof. We need to prove X has v_n-self map iff $R = X \wedge DX$ has v_n-element. Form Spanier-Whitehead theory we should have

$$\begin{array}{ccc} [X, Y] & \xrightarrow{\quad\simeq\quad} & \pi_*(DX \wedge Y) \\ \downarrow & & \downarrow{\scriptstyle \text{Hure}} \\ \mathrm{Hom}_{K(m)_*}(K(m)_*(X), K(m)_*(Y)) & \xrightarrow{\simeq} & K(m)_*(DX \wedge X). \end{array}$$

So, there is a v_n-self map if and only if the dual of that map is a v_n-element in $\pi_*(R)$. $\square$

We now need a lemma convert a v_n-element to satisfy (1.2.1) in some sense.

Lemma 2.3. *Let $R = X \wedge DX$, and $\alpha \in \pi_*(R)$ is a v_n-element, there exists $i, j \in \mathbb{N}$ such that*

$$K(m)_*(\alpha^i) = \begin{cases} v_n^j, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

We actually will need following facts.

Proposition 2.4. *Let X and Y be finite spectra. For $m \gg 0$, we have*

(1) $K(m)_*(X) \simeq \mathrm{H}\,\mathbb{F}_{p_*}X \otimes K(m)_*$,
(2) $K(m)_*(f) = \mathrm{H}\,\mathbb{F}_{p_*}f \otimes \mathrm{id}_{K(m)_*}$, *for any* $f : X \to Y$.

Proof of Lemma 2.3. By Proposition 2.4 (1) we have $\mathrm{H}\,\mathbb{F}_{p_*}\alpha$ is nilpotent, and since X is finite spectra, for the reason of dimension after raising α to a power, we get for $m \gg 0$, $\mathrm{H}\,\mathbb{F}_{p_*}\alpha = 0$. But that just gives $K(m)_*(\alpha) = 0$, for $m \gg 0$. Again, after raising α to a power, we get $K(m)_*(\alpha) = 0$, for $m \neq n$. $K(n)_*(\alpha^i)$ is a power of v_n meaning that $\alpha^i = 1 \in K(n)_*R/(v_n - 1)$. And $K(n)_*R/(v_n - 1) \simeq \oplus_{0\leq i<|v_n|}K(n)_iR$ generates $K(n)_*(R)$. And it is finite because X is finite spectrum, so $K(n)_*(R)$ is finite dimensional $K(n)_*-$vector space, thus $\oplus_{0\leq i<|v_n|}K(n)_iR$ can be finite dimensional $\mathbb{F}_p-$vector space. Then we take the order of group $K(n)_*(R)/(v_n - 1)$, we get $\alpha^i = 1 \in K(n)_*(R)/(v_n - 1)$, that just means $\alpha^i = v_n^j \in K(n)_*(R)$. □

There is an algebra fact about $\mathbb{Z}_{(p)}-$algebra.

Lemma 2.5. *Suppose x and y are commuting elements of $\mathbb{Z}_{(p)}-$algebra, and if $x - y$ is torsion and nilpotent, then there exists $N \in \mathbb{N}$, such that*

$$x^{p^N} = y^{p^N}.$$

Proof. Take $N \gg 0$, so that $x^{p^N} = (y + (x - y))^{p^N} = y^{p^N}$. □

Lemma 2.6. *Let $R = X \wedge DX$, $\alpha \in \pi_*(R)$ is a v_n-element. There exists $i > 0$, α^i is in the center of $\pi_*(R)$.*

Proof. By Lemma 2.3, raising α to a power, we may assume that $K(m)_*(\alpha)$ is in the center of $K(m)_*(R)$ for all m. Let $l(\alpha), r(\alpha) \in \pi_*(R)$ be the left multiplication and right multiplication by α. Since $R \in \mathcal{C}_1$ ($\mathrm{H}_*(R, \mathbb{Q}) = 0$), $l(\alpha)-r(\alpha)$ has finite order. And $K(m)_*(l(\alpha)-r(\alpha)) = 0$, then $l(\alpha) - r(\alpha)$ is nilpotent.(we actually used nilpotence theorem here). By Lemma 2.5, $l(\alpha)^{p^M} = r(\alpha^{p^M})$, for some $M \in \mathbb{N}$. □

In the proof above we actually have used following fact.

Proposition 2.7 (Nilpotence Theorem)**.** [3, Theorem 3 (i)] *Let R be a $p-$local ring spectrum. An element $\alpha \in \pi_*(R)$ is nilpotent if and only if $K(n)_*(\alpha)$ is nilpotent for all $n \in \mathbb{N}$.*

Lemma 2.8. *Let $x, y \in \pi_*(R)$ be v_n-elements. Then there are $i, j \in \mathbb{N}$, such that $x^i = y^j$.*

Proof. Similar to Lemma 2.6, after raising x, y to powers, we may assume $K(m)_*(x-y) = 0$, and x, y are commute. Then by Proposition 2.7, $x - y$ is nilpotent. And $x - y$ is torsion, because $R \in \mathcal{C}_1$. Then by Lemma 2.5, there are $i, j \in \mathbb{N}$, $x^i = y^j$. □

Corollary 2.9. *Suppose X and Y have v_n self-maps v_X and v_Y. There are $i, j \in \mathbb{N}$, such that for any Z and any*

$$f : X \to Y$$

the following diagram commutes.

$$\begin{array}{ccc} \Sigma^k X & \xrightarrow{\Sigma f} & \Sigma^k Y \\ \downarrow{\scriptstyle v_X^i} & & \downarrow{\scriptstyle v_Y^j} \\ X & \xrightarrow{f} & Y. \end{array}$$

Proof. The spectrum $W = DX \wedge Y$ has two v_n self maps: $Dv_X \wedge \mathrm{id}_Y, \mathrm{id}_{DX} \wedge v_Y$. By Lemma 2.8 and the claim in Remark 2.2, $Dv_X^i \wedge \mathrm{id}_Y$ are homotopic to $\mathrm{id}_{DX} \wedge v_Y^j$. Assume f is dual to $\hat{f} \in \pi_*(W)$, W is a module spectrum over $DX \wedge X$, the product $\hat{v_X^i}\hat{f} = (Dv_x^i \wedge \mathrm{id}_Y)\hat{f}$ is the adjoint of fv_X^i. And also $(\mathrm{id}_X \wedge v_Y^j)\hat{f}$ is adjoint to $v_Y^j f$, so $fv_X^i \simeq v_Y^j f$. □

We actually have used following fact.

Proposition 2.10. *For any two spectra X and Y, the natural map*

$$K(n)_*X \otimes_{K(n)_*} K(n)_*(Y) \to K(n)_*(X \wedge Y)$$

is an equivalence.

Corollary 2.11. *The full subcategory of $\mathcal{C}_1$ consisting of spectra admitting a v_n self-map is thick.*

Proof. Denote the category in the statement to be $\mathcal{C}$. Note that $X \in \mathcal{C}$ iff $\Sigma X \in \mathcal{C}$. To show $\mathcal{C}$ is closed under cofiber sequences, we only need to show if

$$X \to Y \to Z$$

is a cofiber sequence with $X, Y \in \mathcal{C}$, then $Z \in \mathcal{C}$. (Because of the triangulated structure of spectra category). Using Corollary 2.9, we can choose v_X, v_Y are v_n maps of X and Y, such that

$$\begin{array}{ccccc} \Sigma^k X & \longrightarrow & \Sigma^k Y & \longrightarrow & \Sigma^k Z \\ \downarrow{\scriptstyle v_X} & & \downarrow{\scriptstyle v_Y} & & \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

is commutative. And any map $v_Z : \Sigma^k Z \to Z$ making the diagram a map of cofiber sequences is a v_n map.(And such map exists by an axiom of triangulated category). Now we need to show $\mathcal{C}$ is closed under retracts. Let $i : Y \to X, p : X \to Y$, and $p \circ i$ is homotopic to id_Y. By Lemma 2.6 and claim in Remark 2.2, we can choose a v_n self map X of X, such that v commutes with $i \circ p$. Then the map

$$\Sigma^k Y \xrightarrow{\Sigma^k i} \Sigma^k X \xrightarrow{v} X \xrightarrow{p} Y$$

is a v_n map of Y. □

Remark 2.12. The fact that $p \circ v \circ \Sigma^k i$ is a basic algebra exercise. It is equivalent to the following question. Let A, B be $R-$modules, $f : B \to B$, $i : A \to B, p : B \to A$, f is an isomorphism, $p \circ i = \mathrm{id}_A$, f commutes with $i \circ p$, then $p \circ f \circ i$ is an isomorphism.

Corollary 2.13. *The full subcategory of $\mathcal{C}_1$ consisting of spectra admitting a v_n self-map satisfying (1.2.1) is thick, i.e. $\mathcal{V}_n$ is thick, $n \geq 1$.*

Proof. The proof is the same, the only trouble is the powers in Lemma 2.3 and Lemma 2.8 could be not a power of p. But actually the condition (1.2.1) says that cannot happen. □

3. Constructing a v_n-element

We need to prove fact 3, and the claim in Remark 2.2 tells us we only need to find an element in $\pi_*(R)$ satisfying Lemma 2.3.

The strategy is use several Adams spectral sequences to "approximates" $v_n \in K(n)_*$. So, we first recall some basic facts about Steenrod algebra.

Definition 3.1 (mod 2 Steenrod algebra)**.** [7, Chapter II §3]

Let V be the $\mathbb{F}_2-$vetor space with basis $\{Sq^0, Sq^1, \dots\}$. $T(V)$ be the tensor algebra, and I is the ideal of $T(V)$ generated by

$$\{Sq^aSq^b - \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j}Sq^j, Sq^0Sq^k - Sq^k, Sq^kSq^0 - Sq^k\}$$

for $0 < a < 2b, k \geq 0$. Then $\mathcal{A}_2 := T(V)/I$.

Definition 3.2 (mod p Steenrod algebra, p odd)**.** [7, Chapter VI §2]

Let V be the $\mathbb{F}_p-$vetor space with basis $\{\beta, P^0, P^1, \dots\}$. $T(V)$ be the tensor algebra, and I is the ideal of $T(V)$ generated by

$$\{P^aP^b - \sum_{j=0}^{[a/p]} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j}P^j, P^0P^k - P^k, P^kP^0 - P^k, \beta P^0 - \beta, P^0\beta - \beta\}$$

$\cup\{P^{a'}\beta P^{b'} - \sum_{j=0}^{[a'/p]}(-1)^{a'+j}\binom{(p-1)(b'-j)}{a'-pj}\beta P^{a'+b'-j}P^j - \sum_{j=0}^{(a'-1)p}(-1)^{a'+j-1}\binom{(p-1)(b'-j)-1}{a'-pj-1}P^{a'+b'-j}\beta P^j.\}$
for $0 < a < pb, 0 < a' \leq b', k \geq 0$. Then $\mathcal{A}_p := T(V)/I$.

Remark 3.3. We actually omit the notion $\otimes$ in the defining ideals I.

Definition 3.4 (The dual Steenrod algebra)**.**

$$\mathcal{A}_{2_*} \simeq \mathbb{F}_2[\xi_1, \xi_2, \dots], |\xi_i| = 2^i - 1.$$

$$\mathcal{A}_{p_*} \simeq \mathbb{F}_{p_*}[\xi_1, \xi_2, \dots] \otimes \Lambda[\tau_0, \tau_1, \dots], |\xi_i| = 2(p^i - 1), |\tau_i| = 2p^i - 1.$$

Remark 3.5. There are some conventions, we denote $P_t^s \in \mathcal{A}, s < t$ to be the element dual to $\xi_t^{p^s}$. $Q_n \in \mathcal{A}_p$ dual to τ_n if p is odd and ξ_{n+1} if $p = 2$. We have $(P_t^s)^p = 0, Q_n^2 = 0$.

The subalgebra of $\mathcal{A}$ generated by

$$Sq^1, Sq^2, \dots, Sq^{2^n}, \text{when } p = 2,$$

$$\beta, P^1, P^p, \dots, P^{p^{n-1}}, \text{when } p \text{ odd} n \neq 1,$$

$$\beta, \text{when } p \text{ odd } n = 0.$$

is denoted $\mathcal{A}_n$.

The subalgebra generated by Q_n is denoted by $E[Q_n] \subseteq \mathcal{A}$.

Definition 3.6. Given an $\mathcal{A}-$module M, the Margolis homology $\mathrm{H}(M, d)$ is the homology of the complex (M_*, d_*) with

$$M_n = M, n \in \mathbb{Z},$$

$$d_{2n} = d,$$

$$d_{2n+1} = \begin{cases} d^{p-1}, & \text{if } d = P_t^s, \\ d, & \text{if } d = Q_n. \end{cases}$$

And we denote $\mathrm{H}(X,d)$ to be $\mathrm{H}(\mathrm{H}^* X, d)$.

We mentioned that we need Adams spectral sequence.

Proposition 3.7. [1, Theorem 2.1] *There is a spectral sequence, with terms $E_r^{s,t} = E_r^{s,t}(X)$ which are zero if $s<0$ or if $t<s$, and with differentials*

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

satisfying the following conditions.

(i) There is a canonical isomorphism

$$E_2^{s,t} \cong \mathrm{Ext}_{\mathcal{A}}^{s,t}\left(H^*(X), Z_p\right).$$

(ii) There is a canonical isomorphism

$$E_{r+1}^{s,t} \cong H^{s,t}\left(E_r; d_r\right).$$

(iii) There is a canonical monomorphism from $E_R^{s,t}$ to $E_r^{s,t}$ for $s<r<R \leqslant \infty$.

(iv) If (using (iii)) we regard $E_r^{s,t}$ as a subgroup of $E_{s+1}^{s,t}$ for $s<r\leqslant\infty$, we have

$$E_\infty^{s,t} = \bigcap_{s<r<\infty} E_r^{s,t}$$

(v) There exist groups $B^{s,t}$ such that

$$B^{s,t} \subset B^{s-1,t-1} \subset \cdots \subset B^{0,t-s}, \quad B^{0,m} = \pi_m^S(X)$$

and

$$E_\infty^{s,t} \simeq B^{s,t}/B^{s+1,t+1}.$$

(vi) $\bigcap_{t-s=m} B^{s,t} = K^m$

Remark 3.8. In our case, X is a finite $p-$local spectrum, the Adams spectral sequence has filtration to 0, i.e. $K^m = 0$ above. Actually, in our case we have an Adams filtration (X_s, g_s) such that $\lim X_s = \mathrm{pt}$, and if $\lim X_s = \mathrm{pt}$, we have $K_m = 0$. See [5, Lemma 2.1.12, 2.1.16].

There is an ring spectrum $k(n)$ such that $k(n)_* = \mathbb{F}_p[v_n] \subset K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$.

Lemma 3.9. *The transformation $k(n)_*X \to K(n)_*X$ extend to a natrual isomorphism*

$$v_n^{-1} k(n)_* X \simeq K(n)_* X.$$

Corollary 3.10. *If $k(n)_*X$ is finite then $K(n)_*X = 0$*

Proof. $k(n)_*X$ is finite meaning for $j \gg 0$, $k(n)_j X = 0$. Therefore $x \in k(n)_*X$, take $m \gg 0$, $v_n^m x = 0$. □

The mod p cohomology $\mathrm{H}^* k(n)$ has been calculated by Bass and Madsen [2, Theorem A,B].

Proposition 3.11. *As a $\mathcal{A}-$module*

$$\mathrm{H}^* k(n) \simeq \mathcal{A}//E[Q_n] \simeq \mathcal{A} \otimes_{\mathbb{F}_p} E[Q_n].$$

So the E_2 page of the Adams spectral sequence of $\pi_*(k(n) \wedge X)$ is

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A}//E[Q_n] \otimes \mathrm{H}^* X, \mathbb{F}_p) \simeq \mathrm{Ext}_{E[Q_n]}^{s,t}(H^* X, \mathbb{F}_p).$$

Remark 3.12. In the above, $E[Q_n] \subseteq \mathcal{A}$ is a sub Hopf algebra, this can be showed directly by checking the dual algebra is a quotient Hopf algebra. Let $B \subseteq C$ are Hopf algebra over k. Then $\mathrm{Hom}_C(C \otimes_B M, N) \simeq \mathrm{Hom}_B(M, N)$. This gives $\mathrm{Ext}^*_C(C \otimes M, N) \simeq \mathrm{Ext}^*_B(M, N)$. And we also have $C \otimes_B M \simeq C//B \otimes_k M \simeq C \otimes_B k \otimes_k M$. So, these together give $\mathrm{Ext}_C(C//B \otimes M, N) \simeq \mathrm{Ext}^*_B(M, N)$.

Corollary 3.13. *If X is a finite spectrum and $\mathrm{H}(X, Q_n) = 0$, then $K(n)_*X = 0$.*

Proof. Note that we have a injective $E[Q_n]$ resolution

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{x} \Sigma^{2p^n-1}\mathbb{F}_p[x]/x^2 \xrightarrow{x} \Sigma^{2(2p^n-1)}\mathbb{F}_p[x]/x^2 \xrightarrow{x} \cdots . \tag{3.13.1}$$

Apply $\mathrm{Hom}^t(\mathrm{H}^*(X), -)$ to it, one can prove if $\mathrm{H}(X, Q_n) = 0$, then $\mathrm{Ext}^{s,t}(\mathrm{H}^*(X), \mathbb{F}_p) = 0$ for $s > 0$. Actually we have projective resloution

$$0 \longleftarrow \mathbb{F}_p \xleftarrow[x]{} \Sigma^{2p^n-1}\mathbb{F}_p[x]/x^2 \xleftarrow[x]{} \Sigma^{2(2p^n-1)}\mathbb{F}_p[x]/x^2 \xleftarrow[x]{} \cdots . \tag{3.13.2}$$

And because $\mathrm{H}^*(X)$ is finite type over $E[Q_n]$, so $\mathrm{Tor}^{E[Q_n]}_{s,t}(\mathrm{H}^* X, \mathbb{F}_p) = 0 \Rightarrow \mathrm{Ext}^{s,t}_{E[Q_n]}(\mathrm{H}^* X, \mathbb{F}_p) = 0$. And apply $\otimes \mathbb{F}_p$ to 3.13.2 we then have $\mathrm{Tor}^{E[Q_n]}_{s,t}(\mathrm{H}^* X, \mathbb{F}_p) = \mathrm{H}(X, d) = 0, s > 0$.

Therefore $k(n)_*X \simeq \mathrm{Ext}^{0,*}(\mathrm{H}^* X, \mathbb{F}_p)$, and because X is finite so when $* \gg 0$ we have $\mathrm{H}^* X = 0$, then $\mathrm{Ext}^{0,*}(\mathrm{H}^* X, \mathbb{F}_p) = 0$. By Lemma 3.10, $K(n)_*X = 0$. ☐

Theorem 3.14. *There is a finite spectrum X_n satisfies following.*

(i) all differentials in the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{E[Q_n]}(\mathrm{H}^* X_n \wedge DX_n, \mathbb{F}_p) \to k(n)_*X_n \wedge DX_n$$

are zero.

(ii) The Margolis homology groups $\mathrm{H}(X_n \wedge DX_n, d) = 0$ if $|d| < |Q_n|$.

We will sketch the proof later.

Theorem 3.15. *$\exists X_n \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$ and has a v_n self map satisfying Definition 1.2.1, i.e. $X_n \in \mathcal{V}_n$.*

Proof. Denote $R = X_n \wedge DX_n$, where X_n is obtain in Theorem 3.14. $X_n \in \mathcal{C}_{n-1}$ by condition (ii) in Theorem 3.14 and Corollary 3.13. And $K(n)_*(X_n) \neq 0$ by (i) of Theorem 3.14. So, $X_n \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$. Note that if $B \subset \mathcal{A}$ sub Hopf algebra, then the ring $\mathrm{Ext}_B(\mathrm{H}^* R, \mathbb{F}_p)$ is a central algebra over $\mathrm{Ext}_B(\mathbb{F}_p, \mathbb{F}_p)$. As said in 2.2, we only need to find an element $\alpha \in \pi_*(R)$ whose Hurewicz image in $K(m)_*$ satisfying 2.3. Note that we have a injective $E[Q_n]$ resolution

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{x} \Sigma^{2p^n-1}\mathbb{F}_p[x]/x^2 \xrightarrow{x} \Sigma^{2(2p^n-1)}\mathbb{F}_p[x]/x^2 \xrightarrow{x} \cdots .$$

So, $\mathrm{Ext}^{0,2p^n-2}_{E[Q_n]} \subset \mathrm{Hom}^{2p^n-2}(\mathbb{F}_p, \Sigma^{2p^n-1}\mathbb{F}_p[x]/x^2) = 0$. By (v) in Proposition 3.7, $v_n \in B^{0,2p^n-2} = B^{1,2p^n-1}$. Now take a class in $\mathrm{Ext}^{1,2p^n-1}_{E[Q_n]}$ reprensents v_n, by abuse notion we still denote as v_n. We need find a class $w \in \mathrm{Ext}^{p^N, p^N(2p^n-1)}_{\mathcal{A}}(\mathrm{H}^* R, \mathbb{F}_p)$ restricting to $v_n^{p^N} \cdot 1$, for $N \gg 0$.

Consider the following diagram

$$\begin{array}{ccccc}
 & & w \in \mathrm{Ext}_{\mathcal{A}}(\mathrm{H}^* R, \mathbb{F}_p) & \Longrightarrow & \pi_*(R) \ni v \\
 & & \big\downarrow{\scriptstyle \text{aprroximation}} & & \\
\tilde{w} \in \mathrm{Ext}_{\mathcal{A}_m}(\mathbb{F}_p, \mathbb{F}_p) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}_m}(\mathrm{H}^* R, \mathbb{F}_p) \ni \tilde{w} \cdot 1 & & \big\downarrow{\scriptstyle \text{Hurewicz}} \\
\big\downarrow{\scriptstyle \text{restriction}} & & \big\downarrow{\scriptstyle \text{restriction}} & & \\
v_n^{p^N} \in \mathrm{Ext}_{E[Q_n]}(\mathbb{F}_p, \mathbb{F}_p) & \longrightarrow & \mathrm{Ext}_{E[Q_n]}(\mathrm{H}^* R, \mathbb{F}_p) & \Longrightarrow & k(n)_* R
\end{array}$$

Claim.

(1) There is $b > 0$, $\mathrm{Ext}_{\mathcal{A}}(\mathrm{H}^* R, \mathbb{F}_p) = 0$ when $s > 1/(2p^n - 2)(t - s) + b$.
(2) The approximation map is isomorphism if $s > 1/(2p^n - 2)(t - s) + k$, where $k < 0$, and $k \to -\infty$ if $m \to \infty$.
(3) For $N \gg 0$, $v_n^{p^N}$ is in the image of the restriction map.

Now, claim 3 says we can find $\tilde{w} \in \mathrm{Ext}_{\mathcal{A}_m}(\mathbb{F}_p, \mathbb{F}_p)$ restricting to $v_n^{p^N}$. Claim 2 says we can find $w \in \mathrm{Ext}_{\mathcal{A}}(\mathrm{H}^* R, \mathbb{F}_p)$ restricting to $\tilde{w} \cdot 1$. And w is in the commutes with every $\alpha \in \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathrm{H}^* R, \mathbb{F}_p), s \geq 1/(2p^n - 2)(t - s) + b$. Since $d_2 w = 0$, choose $N \gg 0$, so that $d_r w^{p^N} = 0, r > N$. Because $d_r w^{p^N}$ lie in $s \geq 1/(2p^n - 2)(t - s) + b$. And $d_r(w^p) = p d_{r-1}(w), d_2(w) = 0$, so $d_N(w^{p^{N-1}}) = 0$. Therefore w^{p^N} is a permanent cycle. We need to show $k(n)_*(v) - v_n^{p^N} = 0$. Form the commutative diagram we get $k(n)_*(v) - v_n^{p^N}$ is represents by a class in $v' \in \mathrm{Ext}^{s,t}(\mathrm{H}^* X, \mathbb{F}_p), s = 1/(2p^n - 2)(t - s) + 1$. Therefore some power of it represented by a class above vanishing line. That means v' is represented by a class in $v'' \in \mathrm{Ext}^{s,t}(\mathrm{H}^* X, \mathbb{F}_p), s = 1/(2p^n - 2)(t - s) + 2$. After finitely times the class is $0 = \bigcap_{t'-s'=t-s} B^{s',t'} = K^{t-s} \in \pi_{t-s}(k(n) \wedge R)$. Therefore $k(n)_*(v) - v_n^{p^N} = 0$.

Now $k(m)_*(v) - v_n^{p^N} = 0$, if $m < n$ for trivial reason. And form (3.13.1), we have $\mathrm{Ext}_{E[Q_m]}^{s,t}(\mathrm{H}^* R, \mathbb{F}_p) = 0$, where $s = 1/(2p^m - 2)(t - s)$ for degree reason. So when $m > n$, $k(m)_*(v)$ is above this vanishing line, thus it is zero.

□

Claim 1 is:

Proposition 3.16. [4] *If M is a connective $\mathcal{A}$-module with*

$$\mathrm{H}(M, d) = 0 \quad \textit{for} \quad |d| \leq n,$$

then

$$\mathrm{Ext}_{\mathcal{A}}^{*,*}(M, \mathbb{F}_p)$$

has a vanishing line $y = x/n + b$, b is a constant depending only on n.

Remark 3.17. The proof of Proposition 3.16 actually using the duality of Ext and Tor, so we actually require that M is finite type. For more detail see [4, Lemma 1.1].

Proposition 3.18. *Suppose that M is a connective $\mathcal{A}$-module, and*

$$y = mx + b$$

is a vanishing line for $\mathrm{Ext}_{\mathcal{A}}^{*,*}(M, \mathbb{F}_p)$. If N is a $(c-1)$-connected $\mathcal{A}$-module, then

$$y = m(x - c) + b$$

is a vanishing line for

$$\operatorname{Ext}_{\mathcal{A}}^{*,*}\left(M \otimes N, \mathbb{F}_p\right).$$

Claim 2 is:

Proposition 3.19. *Let M be a connective $\mathcal{A}$-module, and suppose that $\operatorname{Ext}_{\mathcal{A}}^{*,*}\left(M, \mathbb{F}_p\right)$ has a vanishing line of slope m. For $n \gg 0$, there is $b < 0$ such that the restriction map*

$$\operatorname{Ext}_{\mathcal{A}}^{s,t}\left(M, \mathbb{F}_p\right) \to \operatorname{Ext}_{\mathcal{A}_n}^{s,t}\left(M, \mathbb{F}_p\right)$$

is an isomorphism when

$$s \geq m(t-s) + b.$$

Claim 3 is:

Proposition 3.20. *Suppose that $B \subset C$ are finite, connected, graded, cocommutative Hopf algebras over a field k of characteristic $p > 0$. If*

$$b \in \operatorname{Ext}_B^{*,*}(k,k),$$

then for $N \gg 0, b^{p^N}$ is in the image of the restriction map

$$\operatorname{Ext}_C^{*,*}(k,k) \to \operatorname{Ext}_B^{*,*}(k,k).$$

Sketch of proof. Reduce to the case when B is normal in C. Now we have the spectral sequence

$$\operatorname{Ext}_{C//B}^{*}(k, \operatorname{Ext}_B^{*} k, R) \Rightarrow \operatorname{Ext}_C^{*}(k, R).$$

Finiteness gives there is $M \gg 0$, b^{p^M} is invariant under $C//B$. This gives an class in the spectral sequence, and is a permanent cycle. So the class in $\operatorname{Ext}_C^{*}(k,k)$ represents b^{p^M} is the desired class. □

We now explain the proof of Theorem 3.14, for more detail see [6, Appendix C].

Definition 3.21. A $p-$local finite CW-complex Y is **strongly type** n if it satisfies the following conditions.

(1) Margolis homology group $\mathrm{H}(Y,d) = 0$, if $|d| < |Q_n|$.
(2) Q_n acts trivially on $\mathrm{H}^*(Y)$.
(3) $K(n)^*(Y)$ and $\mathrm{H}^*(Y)$ have the same rank.

Definition 3.22. A $p-$local finite CW-complex Y is **partial type** n if it satisfies (2) and (3) of Theorem 3.21. And each $Q_i, i < n$ and P_t^0 acts nontrivially on $\mathrm{H}^*(X)$.

Proof sketch of Theorem 3.14. Denote the sub-Hopf algebra generated by $P_n^s, n > 0, s < n$ by T_n.

(1) A strongly type n complex Y satisfying conditions in Theorem 3.14.
(2) A partial type n exists, namely $B_2^{2p^n} = B^{2p^n}/B^1$. Where B^k is the $k-$skeleton of $B\mathbb{Z}/p$.
(3) Consider $X^{(l)} = \underbrace{X \wedge X \wedge \ldots \wedge X}_{l \text{ times}}$, Σ_l acts on $X^{(l)}$. This gives an action on $\mathrm{H}^*(X^{(l)})$. And it gives an $\mathbb{Z}_{(p)}[\Sigma_l]-$module.

(4) Let V be a $\mathbb{F}_p$ vector space. There is $e_V \in \mathbb{Z}_{(p)}[\Sigma_{k_V}]$, e_V is an idempotent, k_V is a constant depending only on V.
(5) If V is a module over either $E[Q_n]$ or T_n, and $V = U \oplus F$, where F is a nontrivial free module. Then $e_V V^{\otimes k_V}$ is a free module over $E[Q_n]$ or T_n, [6, Theorem C.2.2].
(6) Now X is of partial type n, then $Q_i, i < n$ and P_t^0 acts nontrivially on $\mathrm{H}^*(X)$. This gives a nontrivial free direct summand of $\mathrm{H}^*(X^{(l)})$ for l sufficiently large. Note that in this step, we used that T_t is self injective, so a free sub module is always a direct summand.
(7) There is an operation of spectra, such $e \in \mathbb{Z}_{(p)}[\Sigma_k]$ idempotent, $\mathrm{H}^*(eX^{(k)}) \simeq e\,\mathrm{H}^*(X^{(k)})$. Namely the direct limit of the system

$$X^{(k)} \xrightarrow{e} X^{(k)} \xrightarrow{e} \cdots .$$

(8) Finally, we take $Y = e_V X^{(lk_V)}$, where $V = \mathrm{H}^*(X^{(l)})$. So (5),(6),(7) tell us the Margolis homology of Y is vanishing. So Y is strongly type n. Therefore Y satisfying the conditions of Theorem 3.14.

□

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