MOTIVIC HOMOTOPY THEORY

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Abstract. This note is for a talk in the upcoming IWOAT winter school. We introduce motivic homotopy theory using the language of ∞-category, and the classical construction will be briefly mentioned. We will explain how to construct the element τ using Milnor–Witt K-theory, and we introduce the main results in [GWX21]. In the end, we talk about connections between algebraic K-theory and t-structure.

1. Introduction

A classical question asked by Serre is whether every finitely generated projective $k[x_1, \dots, x_n]$ module is free when k is a field. This question can be reformulated as follows, whether

$$\mathrm{Vect}_r(\mathrm{Spec}\, k) \to \mathrm{Vect}_r(\mathbb{A}^n_k)$$

is a bijection, for any $n \geq 1$. Here, $\mathrm{Vect}_r(X)$ is the set of isomorphism classes of rank r vector bundles over X. The answer is yes, and it is proved by Quillen and Suslin. This can be viewed as an $\mathbb{A}^1$-invariant phenomenon, and there are more $\mathbb{A}^1$-invariant phenomena. For smooth scheme X over a field k, we have isomorphisms

$$\mathrm{CH}^*(X) \to \mathrm{CH}^*(X \times_k \mathbb{A}^1),$$

$$\mathrm{K}_*(X) \to \mathrm{K}_*(X \times_k \mathbb{A}^1),$$

$$\mathrm{H}^*_{\text{ét}}(X, \mu_l) \to \mathrm{H}^*_{\text{ét}}(X \times_k \mathbb{A}^1, \mu_l).$$

In the above, l is a prime number and coprime to the $\mathrm{char}\, k$. In the topological setting, we consider a topological space (or CW complex) X, we have bijection

$$\mathrm{Vect}_r(X) \to \mathrm{Vect}_r(X \times I).$$

And also for reasonable coefficients (celluar) cohomology, we have isomorphism

$$\mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{H}^*(X \times I, \mathbb{Q}).$$

The above phenomena suggest that $\mathbb{A}^1$ is an analogue of interval $I = [0, 1]$. And there should exist homotopy theory for (smooth) schemes.

Theorem 1.1 (Atiyah–Hirzebruch spectral sequence)**.** *Let E be a generalized cohomology theory, we may take (complex) topological K-theory, then for any space (CW complex) X we have a spectral sequence as follows*

$$E_2^{p,q} = \mathrm{H}^p(X, E^q(*)) \Longrightarrow E^{p+q}(X).$$

There should be a (Weil) cohomology theory for smooth schemes such that there is a spectral sequence from (Weil) cohomology to algebraic K-theory.

2. Motivic homotopty theory and motivic stable homotopy theory

If a category $\mathcal{C}$ equipped with model structure (in the sense of Quillen), there is a homotopy theory for $\mathcal{C}$. But nowadays, it seems that the most convenient language is the ∞-category. If we want to use homological algebra method to study a scheme, we can consider the coherent sheaves over the scheme (or abelian sheaves). We denote $\mathcal{S}$ to be the category of simplicial sets.

Theorem 2.1 ([Lur09, Proposition A.2.8.2])**.** *Let $\mathcal{C}$ be a small category, then the functor category* $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ *has a model structure, called projective model structure. The weak equivalence is pointwise weak equivalence, and the fibration is pointwise fibration.*

The above theorem suggests that we should consider simplicial presheaves over schemes if we want to do homotopy theory over schemes. Let Sm_S denote the category of smooth finite type schemes over S, and the base scheme S is noetherian. Denote $\mathcal{P}(\mathrm{Sm}_S)$ to be the category of presheaves of simplicial sets over Sm_S, therefore by Theorem 2.1, $\mathcal{P}(\mathrm{Sm}_S)$ is a model category. Usually we consider a certain (Grothendieck) topology over Sm_S, and finally the motivic spaces over S is defined to be $\mathrm{Shv}(\mathrm{Sm}_S)[\mathbb{A}^1\text{-equivalence}^{-1}]$. To be more precisely, we recall some basic definitions.

Definition 2.2. A morphism of schemes $f : X \to Y$ is said to be étale if f is flat and unramified. A morphism f is flat if the induced map of local rings is flat. A morphism f is called unramified if f is locally finite type, and for any $x \in X, y = f(x)$, the residue field $k(y)$ is a separable field extension of $k(x)$.

There are some other ways to define étale morphism, e.g. formally étale (the uniqueness lifting property for square zero extension) and locally finite presentation; smooth and relative dimension 0. The main point is that étale is an algebraic analogue of local homeomorphism in topology. Note that algebraic K-theory does not satisfies étale descent, but it satisfies Nisnevich descent, thus we need to consider Nisnevich topology.

Definition 2.3 ([Lur09, Definition 6.2.2.1])**.** Let $\mathcal{C}$ be an ∞-category. A sieve on $\mathcal{C}$ is a full subcategory $\mathcal{C}^{(0)} \subseteq \mathcal{C}$ having the property that if $f : C \to D$ is a morphism in $\mathcal{C}$ and D belongs to $\mathcal{C}^{(0)}$, then C also belongs to $\mathcal{C}^{(0)}$. Let $C \in \mathrm{Obj}(\mathcal{C})$, then a sieve on C is a sieve on $\mathcal{C}_{/C}$.

A Grothendieck topology on an ∞-category $\mathcal{C}$ consists of a specification, for each object $C \in \mathrm{Obj}\,\mathcal{C}$ a collection of sieves on C which we called covering sieves, such that the following conditions hold.

1. If $C \in \mathrm{Obj}\,\mathcal{C}$, the maximal sieve on C is a covering sieve, i.e. $\mathcal{C}_{/C}$ is a covering sieve.
2. If $f : C \to D \in \mathrm{Mor}\,\mathcal{C}$, and $\mathcal{C}^{(0)}_{/D}$ is a covering sieve on D then $f^*\mathcal{C}^{(0)}_{/D}$ is a covering sieve on C.
3. Let $C \in \mathrm{Obj}\,\mathcal{C}$, $\mathcal{C}^{(0)}_{/C}$ is a covering sieve on C and $\mathcal{C}^{(1)}_{C}$ is an arbitrary sieve on C. Suppose for each morphism $f : D \to C \in \mathrm{Mor}\,\mathcal{C}^{(0)}_{/C}$, the pullback $f^*(\mathcal{C}^{(1)}_{/C})$ is a covering sieve on D, then $\mathcal{C}^{(1)}_{/C}$ is a covering sieve on C.

Remark 2.4. A Grothendieck topology on $\mathcal{C}$ can be also defined as the homotopy category $\mathrm{h}\mathcal{C}$ equipped with a Grothendieck topology.

Definition 2.5 ([Lur09, Definition 6.2.2.6, 6.2.2.7])**.** A presheaf (of simplicial sets) F on a category $\mathcal{C}$ with Grothendieck topology is a sheaf if and only if for any $C \in \mathcal{C}$ and any covering sieve $\mathcal{C}^{(0)}_{/C}$ on C

$$F(C) \simeq \lim_{C' \in \mathcal{C}^{(0)}_{/C}} F(C').$$

Definition 2.6. Let $X \in \mathrm{Sm}_S$, a finite family of maps $\{U_i \to X\}_{i \in I}$ is called Nisnevich covering, if it is étale covering, and for any $x \in X$ there is $i \in I, y \in U_i$ such that $k(x) \simeq k(y)$. The Nisnevich topology is the Grothendieck topology generated by these covering families.

The sheaf condition (Čech descent) on site with a cd topology is closed related to Mayer-Vietoris property.

Theorem 2.7 (See [Voe10]). *Let* Sm_S *equipped with a cd topology, then* $F \in \mathcal{P}(\mathrm{Sm}_S)$ *is a sheaf if and only if* $F(\emptyset) \simeq *$ *and for every distinguished square*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

we have $F(X) \simeq F(U) \times_{F(W)} F(V)$.

For Zariski topology, the distinguished square is

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

where p and i are open immersions.

Definition 2.8. Let X be a scheme. A Nisnevich distinguished square is a Cartesian diagram of schemes

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

where i is an open immersion, p is étale, and $p : p^{-1}(X \setminus U) \to X \setminus U$ is isomorphism.

Remark 2.9. For cdh topology, we need to add abstract blowup squares.

Theorem 2.10 ([Lur09, Proposition 6.5.2.14]). *Let* $\mathcal{C}$ *be a small category equipped with a Grothendieck topology and let* $\mathbf{A}$ *denote the category of simplicial presheaves on* $\mathcal{C}$ *endowed with the local model structure (local projective model structure, see* [Lur09, Remark 4.2.4.5]*). The full subcategory* $\mathbf{A}^\circ$ *consisting with fibrant and cofibrant objects. Then* $\mathrm{N}(\mathbf{A}^\circ) \simeq \mathrm{Shv}(\mathcal{C})^\wedge$.

The above discussion suggest that we should let $\mathcal{C} = \mathrm{N}(\mathrm{Sm}_S)$ and consider $\mathrm{Shv}_{\mathrm{Nis}}(\mathcal{C})^\wedge$. Since we are considering Nisnevich topology, descent is equivalent to hyperdescent (when S has finite Krull dimension, see [Rob15, Theorem 2.30]). We know that $\mathcal{P}(\mathrm{Sm}_S)$ is presentable, and according to [Lur09, Proposition 5.5.4.15] we find that $\mathrm{Shv}_{\mathrm{Nis}}(\mathcal{C})^\wedge \simeq \mathrm{Shv}_{\mathrm{Nis}}(\mathcal{C})^{\mathrm{hyp}}$ is presentable. And Nisnevich topology is subcanonical, therefore we still have Yoneda embedding. We write $\mathcal{H}(S)$ to be the localization of $\mathrm{Shv}_{\mathrm{Nis}}(\mathcal{C})^\wedge$ with respect to the family of maps $\{X \times \mathbb{A}^1 \to X\}_{X \in \mathrm{Obj}\,\mathrm{Sm}_S}$. This $\mathcal{H}(S)$ is the $\mathbb{A}^1$ homotopy category that we want to construct (cf. [Rob15, Section 2.4]).

Note that $\mathcal{H}(S)$ has a final object, namely S.

Definition 2.11. Let $p : S \to \mathcal{H}(S)$ be the inclusion, where S viewed as one point set. The pointed motivic homotopy category $\mathcal{H}(S)_*$ is defined to be $\mathcal{H}(S)_{p/}$ which is a presentable category (see [Lur09, Proposition 5.5.3.11]). An object of $\mathcal{H}(S)_*$ consists of $X \in \operatorname{Obj} \mathcal{H}(S)$ and a morphism $S \to X \in \operatorname{Mor}(\mathcal{H}(S))$.

The forgetful functor $\mathcal{H}(S)_* \to \mathcal{H}(S)$ admits left adjoint $(-)_+ : \mathcal{H}(S) \to \mathcal{H}(S)_*$, it sends any X to $X \coprod S$.

Since the underlying model category $\mathbf{M} = \operatorname{Shv}(\operatorname{Sm}_S)[(\mathbb{A}^1\text{equivalence})^{-1}]$ admits symmetric monoidal structure, and hence for $\mathbf{M}_*$. According to Theorem 2.10, it would be not surprised that $\mathcal{H}(S)_*$ is a symmetric monoidal presentable ∞-category.

Theorem 2.12. *The category $\mathcal{H}(S)$ admits a symmetrical monoidal structure (basically induced by product) and by* [Rob15, Corollary 2.32] *$\mathcal{H}(S)_*$ admits a monoidal structure, given by $X \otimes Y := X \times Y / X \coprod_* Y$. And we have an equivalence of presentable symmetric monoidal ∞-categories*

$$\mathcal{H}(S)_*^{\otimes} \to \mathrm{N}^{\otimes}(((\mathbf{M})_*^{\circ})^{\wedge}),$$

cf. [Rob15, Proposition 2.37].

From now, let us restrict to the situation that $S = \operatorname{Spec} k$, where k is a field. In this situation, we have $\operatorname{Spec} k_+ \simeq S^0 \in \mathcal{H}(k)_*$ which is the unit of $\mathcal{H}(k)_*$. Let us consider the following pullback square

$$\begin{tikzcd}
(\mathbb{A}^1 \setminus \{0\}, 1) \arrow[r] \arrow[d] & (\mathbb{A}^1, 1) \arrow[d, "p"] \\
(\mathbb{A}^1, 1) \arrow[r, "i"] & (\mathbb{P}^1, 1)
\end{tikzcd}$$

which is a Nisnevich square, basically because i, p are open immersions, assume $\mathbb{P}^1 = \operatorname{Proj} k[x, y]$, then $\mathbb{P}^1 \setminus \mathbb{A}^1 = V(y) \subset D_+(x) = \mathbb{A}^1 \simeq \operatorname{Spec} k[y/x]$. Thus $p : p^{-1}(\mathbb{P}^1 \setminus \mathbb{A}^1) \to \mathbb{P}^1 \setminus \mathbb{A}^1$ is isomorphism. By Theorem 2.7 and the definition of colimit in the ∞-category we arrive at a push-out square

$$\begin{tikzcd}
(\mathbb{A}^1 \setminus \{0\}, 1) \arrow[r] \arrow[d] & (\mathbb{A}^1, 1) \arrow[d, "p"] \\
(\mathbb{A}^1, 1) \arrow[r, "i"] & (\mathbb{P}^1, 1)
\end{tikzcd}$$

in $\mathcal{H}(k)_*$. By the consideration that $(\mathbb{A}^1, 1) \simeq \operatorname{Spec} k_+$ in $\mathcal{H}(k)_*$ and $(\mathbb{P}^1, 1) \simeq (\mathbb{P}^1, \infty)$ (using the map $x \to [1 : x]$), therefore we have $S^1 \wedge (\mathbb{G}_m, 1) \simeq (\mathbb{P}^1, \infty)$ where $(S^1, *)$ is the pointed simplicial sphere. And also $(\mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}, 1) \simeq (\mathbb{P}^1, \infty)$.

Definition 2.13. We define the motivic stable homotopy category as the formal inversion $\mathcal{H}(k)_*^{\otimes}[(\mathbb{P}^1, \infty)^{-1}]$, which is a presentable stable symmetric monoidal ∞-category (cf. [Rob15, Definition 2.6]). We denote the category to be $\mathcal{SH}(k)$, and we denote the canonical functor as $\Sigma^\infty : \mathcal{H}(k)_* \to \mathcal{SH}(k)$.

It is clear that $\Sigma^\infty \operatorname{Spec} k_+$ is the unit of $\mathcal{SH}(k)$, we denoted it by $\mathbb{1}$. Note that an object in $\mathcal{SH}(k)$ is nothing but a $\mathbb{P}^1$-spectrum, i.e. a sequence of motivic spaces $(X_n)_{n \in \mathbb{N}} \subseteq \operatorname{Obj} \mathcal{H}(k)_*$, with bonding maps $X_i \simeq \Omega_{\mathbb{P}^1} X_{i+1}$, where $\Omega_{\mathbb{P}}$ is the right adjoint of $(-) \otimes \mathbb{P}^1$. The objects in $\mathcal{SH}(k)$ can be called as k-motivic spectra.

3. Motivic spectra and the element tau

We denote $\Sigma^{1,1} := \Sigma_{\mathbb{G}_m} := (-) \wedge \Sigma^\infty \mathbb{G}_m : \mathcal{SH}(k) \to \mathcal{SH}(k)$ and $\Sigma^{1,0} := \Sigma_{S^1} := (-) \wedge \Sigma^\infty S^1 : \mathcal{SH}(k) \to \mathcal{SH}(k)$. For $p, q \in \mathbb{Z}$, we set $\Sigma^{p,q} := (\Sigma^{1,1})^{\circ p}(\Sigma^{1,0})^{\circ(q-p)}$.

Definition 3.1. For $E \in \mathcal{SH}(k)$, we define the motivic homotopy group to be

$$\pi_{i,j}(E) := [\Sigma^{i,j}\,\mathbb{1}, E] = \pi_0 \mathrm{Map}(\Sigma^{i,j}\,\mathbb{1}, E).$$

we set

$$\pi_i(E)_j := \pi_{i-j,-j}(E).$$

From the definition of motivic homotopy groups and Definition 2.13, we find that if $A \to B \to C$ is a cofiber sequence, then for any $q \in \mathbb{Z}$ we have a long exact sequence

$$\cdots \longrightarrow \pi_{*+1,q}(C) \longrightarrow \pi_{*,q}(A) \longrightarrow \pi_{*,q}(B) \longrightarrow \pi_{*,q}(C) \longrightarrow \cdots$$

Let $\mathrm{Cor}(\mathrm{Sm}_S)$ be the category with same object as Sm_S but whose morphisms are finite S-correspondences. For any $X, Y \in \mathrm{Obj}\,\mathrm{Sm}_S$, the set $\mathrm{Mor}_{\mathrm{Cor}(\mathrm{Sm}_S)}(X, Y) := \mathrm{Cor}_S(X, Y)$ is the free abelian group generated by closed integral subschemes $Z \subseteq X \times_S Y$ such that the induced morphism $Z \to X$ is finite and dominates an irreducible component of X. The category $\mathrm{Cor}(\mathrm{Sm}_S)$ is additive and admits direct sum and has symmetric monoidal structure. We denote $\mathcal{P}^{\mathrm{tr}}(\mathrm{Sm}_S, R)$ to be the category of additive presheaves of simplicial R-modules over $\mathrm{Cor}(\mathrm{Sm}_S)$. Let R be a commutative ring, we have adjunction:

$$R^{\mathrm{tr}} : \mathcal{P}(\mathrm{Sm}_S) \leftrightarrows \mathcal{P}^{\mathrm{tr}}(\mathrm{Sm}_S, R) : u^{\mathrm{tr}}.$$

For $S = \mathrm{Spec}\, k$, $R^{\mathrm{tr}}(X) := C_\bullet R(X)$, i.e. $C_n(R(X))(U) = R \otimes_{\mathbb{Z}} \mathrm{Cor}_k(U \times \Delta^n, X)$, where $\Delta^n := \mathrm{Spec}\, k[x_0, \ldots x_n]/(\sum_{i=0}^n x_i - 1), X \in \mathrm{Obj}\,\mathrm{Sm}_k$. For general objects in $\mathcal{P}(\mathrm{Sm}_S)$, we define R^{tr} using left Kan extension. See [MVW06, Definition 2.14]. We still have the adjunction pair after localization (along Nisnevich topology and $\mathbb{A}^1$-equivalence) and some compatible properties cf. [HKOsr17, Section 2.1], meaning that we have adjunction:

$$\mathbf{L}R^{\mathrm{tr}} : \mathcal{H}(k) \leftrightarrows \mathcal{H}^{\mathrm{tr}}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k, R) : u^{\mathrm{tr}}.$$

Definition 3.2. For any $p \geq q \geq 0$, and an R-module A, the motivic Eilenberg-Maclane space $K(A(q), p) \in \mathcal{H}(k)$ is defined to be

$$K(A(q), p) := u^{\mathrm{tr}}(\mathbf{L}R^{\mathrm{tr}} S^{p,q} \otimes^{\mathbf{L}}_R A).$$

Note that this space does not depend on R, since we have

$$K(A(q), p) \simeq u^{\mathrm{tr}}(\mathbf{L}\mathbb{Z}^{\mathrm{tr}} S^{p,q} \otimes^{\mathbf{L}}_{\mathbb{Z}} A).$$

This is exactly the motivic complex constructed in [MVW06, Definition 3.1], and note that motivic cohomology can be computed by Nisnevich hyper cohomology [MVW06, Remark 13.11]. The Elienberg-Maclane spectrum $HA \in \mathcal{SH}(k)$ is defined by

$$HA := u^{\mathrm{tr}}(\mathbf{L}R^{\mathrm{tr}}\,\mathbb{1} \otimes^{\mathbf{L}}_R A).$$

By Dold-Kan correspondence, we have an adjunction

$$\mathbf{L}R^{\mathrm{tr}} : \mathcal{SH}(k) \leftrightarrows \mathbf{DM}(k, R) : u^{\mathrm{tr}},$$

where $\mathbf{DM}(k, R)$ is the derived category of $\mathrm{Fun}_{\mathrm{Nis},\mathbb{A}^1}((\mathrm{Cor}(\mathrm{Sm}_k))^{\mathrm{op}}, \mathrm{Ch}^-(R))$, i.e. Voevodsky's derived category of motives.

Definition 3.3. We define the cohomology theory induced by $HA \in \mathcal{SH}(k)$ to be

$$HA^{p,q}(X) := [X, \Sigma^{p,q} HA],$$

for any $X \in \mathcal{SH}(k)$. And note that for $X \in \mathrm{Obj}\,\mathrm{Sm}_k$ we have

$$\mathrm{H}^{p,q}(X, A) \simeq [\Sigma^\infty X_+, \Sigma^{p,q} HA]$$

by [HKOsr17, Theorem 2.13]. Then combining with Definition 3.1, we have

$$\pi_{p,q} HA \simeq HA^{-p,-q}(\mathbb{1}) \simeq \mathrm{H}^{-p,-q}(\mathrm{Spec}\, k, A).$$

The computation of motivic cohomology sometimes may reduce the computation of étale cohomology, due to [HW19, Theorem B,C].

Theorem 3.4. *Let $X \in \mathrm{Obj}\,\mathrm{Sm}_k$, and l is a prime number and co-prime to* $\mathrm{char}\, k$. *Then we have isomorphism*

$$\mathrm{H}^{p,q}(X, \mathbb{Z}/l) \simeq \mathrm{H}^p_{\text{ét}}(X, \mu_l^{\otimes q})$$

for $p \leq q$.

And basically for degree reason, we have the vanishing result.

Theorem 3.5 ([MVW06, Theorem 3.6])**.** *For any $X \in \mathrm{Obj}\,\mathrm{Sm}_k$, we have*

$$\mathrm{H}^{p,q}(X, A) = 0$$

for any abelian group A, and $p > q + \dim X$.

By the virtue of Theorem 3.4, 3.5, we have

$$\mathrm{H}^{*,*}(\mathrm{Spec}\, k, \mathbb{Z}/l) \simeq \mathbb{Z}/l[\tau],$$

where k is an algebraic closed field, and τ is a primitive l-th root of unity, the degree $|\tau| = (0, 1)$.

The homotopy groups of motivic sphere spectrum turns out related to Milnor–Witt K-theory.

Definition 3.6. Let k be a field. The graded ring $\mathrm{K}^{MW}_*(k)$ called Milnor–Witt K-theory of k is defined to be the quotient of the free non-commutative algebra on generators $[a]$ in degree 1 for $a \in k^\times$ and a generator η in degree -1, subject to the following relations

(1) $\eta[a] = [a]\eta$;
(2) $[a][1 - a] = 0$, for $a \in k \setminus \{0, 1\}$;
(3) $[ab] = [a] + [b] + \eta[a][b]$;
(4) $\eta(2 + \eta[-1]) = 0$.

Remark 3.7. Note that the quotient $\mathrm{K}^{MW}_*(k)/(\eta) \simeq \mathrm{K}^M_*(k)$. For $u \in k^\times$ we define $\langle u\rangle := 1+\eta[u] \in \pi_{0,0}(\mathbb{1})$. We then have $\eta h = \eta(\langle -1\rangle + 1) = \eta(2 + \eta[-1])$. The h corresponds to hyperbolic plane over k, and it is zero in $GW(k)$. And we actually have isomorphism $GW(k) \simeq \pi_{0,0}(\mathbb{1})$.

Theorem 3.8. *The graded ring $\pi_0(\mathbb{1})_* = \pi_{-*,-*}(\mathbb{1}) \simeq \mathrm{K}^{MW}_*(k)$, where $\mathbb{1}$ is the unit of $\mathcal{SH}(k)$. And the elements $[a] \in \mathrm{K}^{MW}_1(k)$ corresponds to $a \in \mathbb{G}_m(k)$, i.e. $a : \mathrm{Spec}\, k \to \mathbb{G}_m$. The map $H : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$, corresponds to $\eta : \Sigma^{1,1}\,\mathbb{1} \to \mathbb{1}$, by noticing that $\mathbb{A}^2 \setminus \{0\} \simeq \mathbb{G}_m \wedge \mathbb{P}^1$.*

Now for $k = \mathbb{C}$, take a sequence of compatible primitive p-power roots of unity $\{\zeta_{p^k}\}_{k=1}^\infty$. Note that $[\zeta_{p^k}] \in \pi_{-1,-1}(\mathbb{1})$, set $\alpha_k := (2 + [-1]\eta)[\zeta_{p^k}]$. By the relations of Milnor–Witt K-theory, we have

$$p^k[\zeta_{p^k}] \equiv [1] \mod \eta,$$
$$[1] = -\eta([1])^2.$$

Therefore,

$$p^k \alpha_k = (2 + [-1]\eta) p^k [\zeta_{p^k}] = 0.$$

Recall that we have

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{0,-1}(\mathbb{1}/p^{k+1}) \arrow[r] \arrow[d] & \pi_{-1,-1}\mathbb{1} \arrow[r, "p^{k+1}"] \arrow[d, "p"] & \pi_{-1,-1}\mathbb{1} \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & \pi_{0,-1}(\mathbb{1}/p^{k+1}) \arrow[r] & \pi_{-1,-1}\mathbb{1} \arrow[r, "p^{k}"] & \pi_{-1,-1}\mathbb{1} \arrow[r] & \cdots
\end{tikzcd}$$

Each row is exact in the above diagram and each square is commutative. Using the fact that $p\alpha_{k+1} - \alpha_k \in \operatorname{Ker}(p^k)$, the standard diagram chasing eventually arrive at a compatible sequence of classes $\{\theta_k\}$, where $\theta_k \in \pi_{0,-1}(\mathbb{1}/p^k)$ is a lift of α_k. By passing to limit, we get a class $\tau \in \pi_{0,-1}(\mathbb{1}_p^\wedge)$, this is exactly the element we want to construct. By passing to p-completion we view τ as $(\Sigma^{0,1}\mathbb{1})_p^\wedge \to \mathbb{1}_p^\wedge$. The cofiber of τ will denoted by $\mathbb{1}_p^\wedge/\tau$, it is an E_∞-algebra object in $\mathcal{SH}(\mathbb{C})$ as stated in [GWX21].

Remark 3.9. Note that under suitable choice, $\tau \in \pi_{0,-1}(\mathbb{1}_p)$ sends to $\tau \in \pi_{0,-1}(H\mathbb{Z}/p) \simeq \mathbb{Z}/p$, this can be proved from following facts.

(1) For $d \geq 0$, we have isomorphisms
$$\mathrm{CH}^d(\operatorname{Spec}\mathbb{C}, d) \simeq K_d^M(\mathbb{C}) \simeq \mathrm{H}^{d,d}(\operatorname{Spec}\mathbb{C}, \mathbb{Z}),$$
$$K_d^{MW}(\mathbb{C})/(\eta) \simeq K_d^M(\mathbb{C}).$$
Or, the spectrum $H\mathbb{Z}$ is weakly orientable, meaning that $\eta \in \pi_{1,1}(\mathbb{1})$ sends to $0 \in \pi_{1,1}(H\mathbb{Z})$.

(2) We have isomorphisms $\pi_{-1,-1}(H\mathbb{Z}) \simeq \mathrm{H}^{1,1}(\operatorname{Spec}\mathbb{C}, \mathbb{Z}) \simeq \mathrm{CH}^1(\mathbb{C}, 1) \simeq K_1^M(\mathbb{C}) \simeq \mathbb{C}^\times$.

For every $X \in \operatorname{Obj}\mathrm{Sm}_k$, it is clear that we have K-theory space $K(X)$ (consider the category of finite rank of vector bundles on X, and using the group completion). This gives an object in $\mathcal{P}(\mathrm{Sm}_k)$, and it is $\mathbb{A}^1$-invariant (by the smoothness), and satisfies Nisnevich descent (Thomason–Trobaugh). It is not obvious that we have a $\mathbb{P}^1$-spectrum, but we actually have.

Definition 3.10. There is an object KGL $\in \operatorname{Obj}\mathcal{SH}(k)$, such that for any $X \in \operatorname{Obj}\mathrm{Sm}_k$ we have
$$\mathrm{K}_i(X) \simeq [\Sigma^{i,0}\Sigma^\infty X_+, \mathrm{KGL}].$$

As discussed in Section 1, we have Atiyah–Hirzebruch type spectral sequence.

Theorem 3.11 (Motivic spectral sequence)**.** *Let $X \in \operatorname{Obj}\mathrm{Sm}_k$, we have a spectral sequence*
$$E_2^{p,q} \simeq \mathrm{H}^{p-q,-q}(X, \mathbb{Z}) \Longrightarrow \mathrm{K}_{-p-q}(X).$$

And there is motivic analogue of cobordism.

Definition 3.12. The algebraic cobordism spectrum MGL $\in \operatorname{Obj}\mathcal{SH}(k)$ is the $\mathbb{P}^1$-spectrum
$$(\mathrm{Th}(\gamma_0), \mathrm{Th}(\gamma_1), \ldots),$$
where γ_n is the universal vector bundle
$$\gamma_n \to \mathrm{BGL}_n.$$

4. Motivic Adams spectral sequence

The similarity of algebraic Novikov spectral sequence and motivic Adams spectral sequence for $\mathbb{1}_p^\wedge/\tau$ suggests that there should have deep connections between $\mathbb{1}_p^\wedge/\tau$ and BP-theory. This was discovered by the authors [GWX21], more precisely they proved Theorem 4.7, 4.8.

One can define the motivic Steenrod algebra as the algebra of *bistable* motivic cohomology operations (typically, this is incorrect, but when we consider $\mathcal{SH}(\mathbb{C})$, it turns out to be true, cf. [HKOsr17, Theorem 1.1]). We denote $\mathcal{A}_p^{*,*} \simeq (H\mathbb{Z}/p)^{*,*}(H\mathbb{Z}/p)$ to be the mod p motivic Steenrod algebra.

Definition 4.1 ([HKOsr17, Theorem 5.1])**.** For p odd, $\mathcal{A}_p^{*,*}$ is the $\mathrm{H}^{*,*}(\operatorname{Spec}\mathbb{C}, \mathbb{Z}/p) \simeq \mathbb{F}_p[\tau]$-algebra generated by $\{\beta, P^0, P^1, \dots\}$ satisfies the usual Adem relations.

Definition 4.2 ([HKOsr17, Theorem 5.1])**.** For $p = 2$, the Steenrod algebra $\mathcal{A}_2^{*,*}$ is the $\mathbb{F}_2[\tau]$-algebra generated by $\{\mathrm{Sq}^1, \mathrm{Sq}^2, \dots\}$ satisfies the following *homogeneous* relations for $a < 2b$:

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_c \binom{b-1-c}{a-2c} \tau^? \mathrm{Sq}^{a+b-c}\mathrm{Sq}^c,$$

where ? is 0 or 1. And $|\mathrm{Sq}^{2k}| = (2k, k), |\mathrm{Sq}^{2k-1}| = (2k-1, k-1)$, for $k \geq 1$.

In the following, we fixed a prime number p, and denote $H\mathbb{Z}/p \in \mathcal{SH}(\mathbb{C})$ by H. We will consider H cellular modules $H\text{-Mod}_{\text{cell}}$, which is the smallest stable full subcategory of $\mathcal{SH}(\mathbb{C})$ satisfies following properties.

(1) contains $\Sigma^{p,q}\mathbb{1}\wedge H$, for all $p, q \in \mathbb{Z}$;
(2) closed under arbitrary small colimits.

Thus $(\mathbb{1}/p^n)/\tau$ belongs to $H\text{-Mod}_{\text{cell}}$ for any $n \geq 1$.

Now, for any $X \in \mathcal{SH}(\mathbb{C})$, one can form a cosimplicial object in $\mathcal{SH}(\mathbb{C})$:

$$H \wedge X \rightrightarrows H \wedge H \wedge X \Rrightarrow H \wedge H \wedge H \wedge X \quad \cdots$$

If X is H-cellular, then we have Künneth isomorphism (cf. [DI10, Lemma 7.6]),

$$\mathrm{H}_{*,*}(X) \otimes_{\mathbb{F}_p[\tau]} \mathrm{H}_{*,*}(H)^{\otimes s} \simeq \mathrm{H}_{*,*}(X \wedge H^{\wedge(s)}).$$

Here, $\mathrm{H}_{*,*}$ is the H-homology. Let $\overline{H}$ to be the homotopy fiber of $\mathbb{1} \to H$, we know that $\overline{H}$ is H-cellular. Consider the standard Adams tower $\{X_s, W_s\}$, $X_s := \overline{H}^{\wedge(s)} \wedge X$, $W_s = H \wedge \overline{H}^{\wedge(s)} \wedge X$. From the tower, and by standard method we arrive at a spectral sequence

$$E_1^{*,*} = \pi_{*,*}(W_s) \simeq (\mathrm{H}_{*,*}(\overline{H}))^{\otimes s} \otimes_{\mathbb{F}_p[\tau]} \mathrm{H}_{*,*}(X).$$

Note that

$$\mathrm{H}_{*,*}(W_s) \simeq \mathrm{H}_{*,*}H \otimes_{\mathbb{F}_p[\tau]} \otimes (\mathrm{H}_{*,*}(\overline{H}))^{\otimes s} \otimes_{\mathbb{F}_p[\tau]} \mathrm{H}_{*,*}(X).$$

Therefore the cobar complex

$$0 \to \mathrm{H}_{*,*}(X) \to \mathrm{H}_{*,*}(W_0) \to \mathrm{H}_{*,*}(W_1) \to \cdots$$

is a resolution. And combining with following two facts, we can identify E_2 page as $E_2^{*,*} \simeq \mathrm{Ext}_{\mathrm{H}_{*,*}H}(\mathbb{F}_p[\tau], \mathrm{H}_{*,*}(X))$.

(1) $(\mathrm{H}_{*,*} \simeq \mathbb{F}_p[\tau], \mathrm{H}_{*,*}H)$ is a Hopf algebroid (cf. [HKOsr17, Proposition 5.5], and note that for $\mathcal{SH}(\mathbb{C})$, $\rho = 0$).
(2) For any Hopf algebroid (A, Γ), and M is an arbitrary A-module, we have isomorphisms

$$M \simeq \mathrm{Hom}_A(A, M) \simeq \mathrm{Hom}_\Gamma(A, \Gamma \otimes_A M).$$

Theorem 4.3 ([DI10, Proposition 7.10])**.** *For any X is H-cellular, we have (homological) motivic Adams spectral sequence as follows:*

$$E_2^{s,(t+s,u)} \simeq \mathrm{Ext}_{\mathrm{H}_{*,*}H}(\mathbb{F}_p[\tau], \mathrm{H}_{*,*}(X)) \Longrightarrow \pi_{t,u}(X_H^\wedge).$$

It would not be surprised that we have cohomological motivic Adams spectral sequence.

$$E_2^{s,(t+s,u)} \simeq \mathrm{Ext}_{\mathcal{A}^{*,*}}(\mathrm{H}^{*,*}(X), \mathbb{F}_p[\tau]) \Longrightarrow \pi_{t,u}(X_H^\wedge).$$

But actually, it requires that $\mathrm{H}^{*,*}(X)$ is free $\mathbb{F}_p[\tau]$-module, cf. [DI10, Proposition 7.14].

In the following, we mainly live in the p-complete world. We denote $\mathbb{1}_p$ to be the p-completed motivic sphere spectrum, and $\mathbb{1}_p/\tau$ is the cofiber of $\tau \in \pi_{0,-1}(\mathbb{1}_p)$. And also p-complete suspension

$\Sigma^{s,w} := (\Sigma^{s,w}\,\mathbb{1})^{\wedge}_{p} \wedge_{\mathbb{1}_p} -$. We denote $\mathbb{1}_p\,/\tau\text{-Mod}_{\text{cell}}$ to be the smallest stable subcategory of $\mathbb{1}_p$-Mod satisfies following properties.

(1) contains $\Sigma^{s,w}(\mathbb{1}_p\,/\tau)$, for all $s, w \in \mathbb{Z}$;
(2) closed under arbitrary small colimits.

Definition 4.4. Let X be a motivic spectrum in $\mathbb{1}_p\,/\tau$-Mod, X is said to be harmonic if X is $\mathbb{1}_p\,/\tau$-cellular, and the map

$$X \to X^{\wedge}_{\mathrm{MGL}}$$

induces isomorphism on $\pi_{*,*}$.

Definition 4.5. For a motivic spectrum X, the Chow–Novikov degree of an element in $\pi_{s,w}(X)$ is $s-2w$. If $I \subseteq \mathbb{Z}$, we say X has Chow–Novikov degree concentrated on I if any nonzero element whose Chow–Novikov degree belongs to I.

Remark 4.6. For a $\mathbb{P}^1$-spectrum E, if we set $E_i(X) := [\Sigma^{i,0}\,\Sigma^{\infty} X_+, E]$ for any $X \in \operatorname{Obj}\mathrm{Sm}_k$, then $\pi_{s,w}(E) = [\mathbb{1}^{s,w}, E] \simeq [\mathbb{1}^{s,w}, E \wedge \mathbb{P}^w] \simeq [\mathbb{1}^{s,w}, E \wedge \mathbb{1}^{2w,w}] \simeq [\mathbb{1}^{s-2w,0}, E] = E_{s-2w}(\operatorname{Spec} k)$. In paritular, if we take $E = \mathrm{KGL}$, we get $\mathrm{K}_{s-2w}(\operatorname{Spec} k) \simeq \pi_{s,w}(\mathrm{KGL})$.

We now arriving at the main result in [GWX21].

Theorem 4.7 ([GWX21, Theorem 1.1])**.** *There is an equivalence of stable ∞-categories equipped with t-structures at each prime p,*

$$\mathbf{D}^{\mathrm{b}}(\mathrm{BP}_*\,\mathrm{BP}\text{ - }\mathrm{CoMod}^{\mathrm{ev}}) \to \mathbb{1}_p\,/\tau\text{- }\mathrm{Mod}^{\mathrm{b}}_{\mathrm{harm}}\,.$$

And the t-structure of $\mathbb{1}_p\,/\tau\text{- }\mathrm{Mod}^{\mathrm{b}}_{\mathrm{harm}}$ is induced by Chow–Novikov degree. It means that we have following.

Theorem 4.8 ([GWX21, Theorem 1.13, Proposition 1.10])**.** *Let $\mathbb{1}_p\,/\tau\text{- }\mathrm{Mod}^{\geq 0}_{\mathrm{harm}}$ be the full subcategory of $\mathbb{1}_p\,/\tau\text{- }\mathrm{Mod}_{\mathrm{harm}}$ whose objects have Chow–Novikov degree concentrated on $\mathbb{Z}_{\geq 0}$.*

Let $\mathbb{1}_p\,/\tau\text{- }\mathrm{Mod}^{\leq 0}_{\mathrm{harm}}$ be the full subcategory of $\mathbb{1}_p\,/\tau\text{- }\mathrm{Mod}_{\mathrm{harm}}$ whose objects have Chow–Novikov degree concentrated on $\mathbb{Z}_{\leq 0}$.

The pair $(\mathbb{1}_p\,/\tau\text{- }\mathrm{Mod}^{\geq 0}_{\mathrm{harm}}, \mathbb{1}_p\,/\tau\text{- }\mathrm{Mod}^{\leq 0}_{\mathrm{harm}})$ determines a t-structure on $\mathbb{1}_p\,/\tau\text{- }\mathrm{Mod}_{\mathrm{harm}}$, and the heart is equivalent to

$$\mathrm{BP}_*\,\mathrm{BP}\text{ - }\mathrm{CoMod}^{\mathrm{ev}}\,.$$

5. Algebraic K-theory and t-structure

In this section, we explore the localization sequence concerning algebraic K-theory of motivic spectrum $\mathbb{1}^{\wedge}_{p}$ follow the idea of [ABG18] and [Lur17]. Since $\mathbb{1}^{\wedge}_{p}$ is an E_∞-algebra object in $\mathcal{SH}(\mathbb{C})$, so the set $S = \{1, \tau, \tau^2, \dots\} \subseteq \pi_{*,*}(\mathbb{1}^{\wedge}_{p})$ satisfies Ore condition. By [Lur17, Proposition 7.2.3.17], and a merely same argument as in [ABG18, Theorem 1.11], we have localization sequence of stable ∞-categories,

$$\mathrm{Mod}^{S-nil}_{\mathbb{1}^{\wedge}_{p}} \to \mathrm{Mod}_{\mathbb{1}^{\wedge}_{p}} \to \mathrm{Mod}_{S^{-1}\,\mathbb{1}^{\wedge}_{p}}\,.$$

And actually,

$$\mathrm{Mod}_{(\mathrm{End}(\mathbb{1}^{\wedge}_{p}\,/\tau))^{\mathrm{op}}} \to \mathrm{Mod}_{\mathbb{1}^{\wedge}_{p}} \to \mathrm{Mod}_{\mathbb{1}^{\wedge}_{p}[\tau^{-1}]}\,.$$

By [Pst23, Theorem 4.37], we actually have

$$\begin{tikzcd}
\mathrm{Mod}_{(\mathrm{End}_{\mathbb{1}^{\wedge}_{p}}(\mathbb{1}^{\wedge}_{p}\,/\tau))^{\mathrm{op}}} \arrow[r] & \mathrm{Mod}_{\mathbb{1}^{\wedge}_{p}} \arrow[rr] \arrow[dr, "\mathrm{Re}"] & & \mathrm{Mod}_{\mathbb{1}^{\wedge}_{p}[\tau^{-1}]} \\
& & \mathrm{Mod}_{\mathbb{S}^{\wedge}_{p}} \arrow[ur, "\simeq"] &
\end{tikzcd}$$

By taking compact objects, and applying non-connective algebraic K-theory, we have a fiber sequence:

$$\mathbb{K}((\mathrm{End}_{\mathbb{1}_p^\wedge}(\mathbb{1}_p^\wedge/\tau))^{\mathrm{op}}) \to \mathbb{K}(\mathbb{1}_p^\wedge) \to \mathbb{K}(\mathbb{S}_p^\wedge).$$

Thus it is natural to ask whether

$$\mathbb{K}(\mathbb{1}_p^\wedge/\tau) \to \mathbb{K}((\mathrm{End}_{\mathbb{1}_p^\wedge}(\mathbb{1}_p^\wedge/\tau))^{\mathrm{op}})$$

an isomorphism?

If this is true, then by [GWX21, Theorem 1.1] and [Bar15, Theorem 6.1] we have

$$\mathrm{K}_*(\mathbb{1}_p^\wedge) \simeq \mathrm{K}_*(\mathbb{S}_p^\wedge) \oplus \mathrm{K}_*(\mathcal{M}_{FG}).$$

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