NOTES ON MODULI PROBLEMS

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Abstract. In this note, we mainly concerned about some examples of moduli problems, and some of examples come from the exercises mentioned in the course.

1. Examples

In any area of mathematics, we always care about classification problem. In algebraic geometry, we intended to classify algebraic varieties. But actually, there are two different ways to study the problem. Namely, the first strategy is to study the representative element of a class of varieties, the second strategy is to study the equivalent class of varieties via algebraic geometry method. The first strategy is called *birational geometry*, while the second strategy is called *moduli space method*.

In the first case, we always study varieties up to *birational equivalent*, so the equivalence relation is fixed. But in the second case, the problem indeed depends on the equivalence relation that you defined. Let us see some basic examples.

First, let us see a naive example.(May not related to algebraic geometry.)

Example 1.1. Consider all invertible matrix over $\mathbb{C}$ of order n, may view as $\mathrm{GL}_n(\mathbb{C})$. Define $A \sim_1 B$ if there is a $P \in \mathrm{GL}_n(\mathbb{C})$ such that $A = P^{-1}BP$. Define $A \sim_2 B$ if there are $P, Q \in \mathrm{GL}_n(\mathbb{C})$, such that $A = PBQ$. It is clear that $\sim_1$ and $\sim_2$ are different equivalence relations, and from basic linear algebraic facts, the *moduli space* under $\sim_2$ is one point set $\{I_n\}$, while the *moduli space* under $\sim_1$ is not. $\mathcal{M}_{\sim_1}$ consists of all diagonal matrixes with Jordan type.

We actually already know so called moduli problem, namely:

Definition 1.2. A moduli problem is a set A of a class of objects (comes from algebraic geometry), with an equivalence relation $\sim$ over A. The moduli set of this problem is $A/\sim$.

Remark 1.3. The terminology above is not standard, but it is acceptable.

Example 1.4. Consider 1−dimensional projective space $\mathbb{P}^1_{\mathbb{C}}$. All vector bundles over $\mathbb{P}^1_{\mathbb{C}}$ is a collection of objects we interested. The equivalence relation is isomorphism between two vector bundles. The moduli set of this problem is all the tuples $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ with $a_1 \geq a_2 \geq \ldots \geq a_n$. Actually, it corresponds to the vector bundle $\mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_n)$.

Remark 1.5. Before give a proof of 1.4 we remark that 1.4 only classifying vector bundles in a weak sense. We have

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}(1), \mathcal{O}(-1)) \simeq \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(-1) \otimes \mathcal{O}(1)^\vee) \simeq \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(-2)) \simeq \mathbb{C}.$$

The first isomorphism is followed from the fact that Hom and $\otimes$ are adjoint and $\mathrm{Ext}^i, \mathrm{H}^i$ are universal δ functors. The last isomorphism is followed by Serre duality. There is a family of vector bundle $\mathbb{V}$ over $\mathbb{P}^1 \times \operatorname{Spec}(Sym(\operatorname{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)))) \simeq \mathbb{P}^1 \times \operatorname{Spec}\mathbb{C}[t] \simeq \mathbb{P}^1 \times \mathbb{A}^1$ is an extension of bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. In other words, $\mathbb{V}_{|\mathbb{P}^1 \times 0} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$. For $a \neq 0 \in \mathbb{C}$, $\mathbb{V}_{|\mathbb{P}^1 \times a}$ is a vector bundle over $\mathbb{P}^1$ corresponds to the extension $a \in \operatorname{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) \simeq \mathbb{C}$. Since this vector bundle must be $\mathcal{O} \oplus \mathcal{O}$, so $\mathbb{V}_{|\mathbb{P}^1 \times a} \simeq \mathcal{O} \oplus \mathcal{O}$. This is a *jump phenomena*, namely for any $a \in \mathbb{A}^1 \setminus \{0\}$ the family are isomorphic, but when $a = 0$, $\mathbb{V}_{\mathbb{P}^1 \times 0}$ is not isomorphic to $\mathbb{V}_{\mathbb{P}^1 \times b}, b \neq 0$.

In the above, we actually have used a fact.

Fact 1. If X is scheme, G, H are vector bundles over X, then there is a family of bundles(see 1.6) $\mathbb{V}$ over $X \times \operatorname{Spec}(\operatorname{Sym}\operatorname{Ext}^1(G, H)^\vee)$ such that for any $e \in \operatorname{Ext}^1(G, H)$, the restriction of $\mathbb{V}|_{X \times \{e\}}$ is a vector bundle over X and corresponds to the extension $e \in \operatorname{Ext}^1(G, H)$.

We now present a proof of 1.4, one can see [2].

Proof of 1.4. $\mathbb{P}^1$ is covered by two affine space. Namely $U_0 = \{x_0 \neq 0\}, U_1 = \{x_1 \neq 0\}$. $U_0 = \operatorname{Spec}\mathbb{C}[x_1/x_0], U_1 = \operatorname{Spec}\mathbb{C}[x_0/x_1]$. So $U_0 \setminus \{0\} \simeq \operatorname{Spec}\mathbb{C}[x_1/x_0, x_0/x_1] \simeq U_1 \setminus \{0\} \simeq \operatorname{Spec}\mathbb{C}[x_0/x_1, x_1/x_0]$. So actually $\mathbb{P}^1$ is obtained by gluing two affine space $U_0 = \operatorname{Spec}\mathbb{C}[x], U_1 = \operatorname{Spec}\mathbb{C}[y]$ via isomorphism

$$U_0 \setminus \{0\} = \operatorname{Spec}\mathbb{C}[x, x^{-1}] \simeq \operatorname{Spec}\mathbb{C}[y, y^{-1}]$$
$$x \to y^{-1}$$

Let E be a vector bundle of rank n over $\mathbb{P}^1$, so $E|_{U_0} \simeq U_0 \times \mathbb{A}^n, E|_{U_1} \simeq U_1 \times \mathbb{A}^n$ are trivial, because vector bundle over affine space are trivial. So E is obtained by gluing $U_0 \times \mathbb{A}^n$ and $U_1 \times \mathbb{A}^n$ via a linear map over $U_0 \setminus \{0\} \simeq \operatorname{Spec}\mathbb{C}[x, x^{-1}]$, this gives a matrix $g \in \operatorname{GL}(\mathbb{C}[x, x^{-1}])$. And if E is isomorphic to E', then $Pg_E Q = g_{E'}$, where $P \in \operatorname{GL}_n(\mathbb{C}[x]) = \operatorname{Aut}(E|_{U_0}), Q \in \operatorname{GL}_n(\mathbb{C}[x^{-1}]) = \operatorname{Aut}(E|_{U_1})$. And from the definition of $\mathcal{O}(n)$ we know that $g_{\mathcal{O}(n)} = x^n$. So $\mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \cdots \mathcal{O}(a_n)$ corresponds to the matrix

$$\begin{pmatrix} x^{a_1} & 0 & \cdots & 0 \\ 0 & x^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x^{a_n}. \end{pmatrix} \tag{1.0.1}$$

We denoted (1.0.1) by $\operatorname{diag}(a_1, a_2, \cdots, a_n)$.

Now, we only need to show $\forall g \in \operatorname{GL}_n\mathbb{C}[x, x^{-1}]$, there is $P \in \operatorname{GL}_n\mathbb{C}[x], Q \in \operatorname{GL}_n\mathbb{C}[x^{-1}]$ such that PgQ is of form in (1.0.1). And formula in (1.0.1) is determined by the tuple $(a_1, a_2, \ldots, a_n), a_1 \geq a_2 \geq \cdots \geq a_n$.

1) After multiple $x^s I, s \gg 0$, we actually may assume $g \in \operatorname{GL}_n\mathbb{C}[x]$.
2) After changing rows we may assume $\deg g_{11} = \min\{\deg b_{j1} | 1 \leq j \leq n\}$.
3) Let $g_{j1} = K_j g_{11} + r_j$, where $K_j \in \mathbb{C}[x],\ \deg r_j < \deg g_1 1$. So we can do a collection of row transformations so that the matrix is like

$$\begin{pmatrix} g'_{11} & \cdots & * \\ 0 & \cdots & * \\ \vdots & \cdots & * \\ 0 & \cdots & * \end{pmatrix}.$$

4) Now $g'_{11} = x^{k_1}$, because $\det(g) = x^p$, and $g'_{11} | \det(g)$.

5) By induction on n we can assume the matrix transform to the following form:

$$\begin{pmatrix} x^{k_1} & * & * & * & * \\ 0 & x^{k_2} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \cdots & x^{k_n} \end{pmatrix}.$$

6) Consider all the matrix equivalent to the matrix in 5) such that k_1 is maximal. This is capable, because $k_1 \leq \deg \det(g)$. Then $k_1 \geq \max\{k_2, \ldots, k_n\}$. Otherwise, suppose $k_j > k_1$, for some $j \geq 2$. Then we exchange first column and jth column and exchange first row and jth row, so that x^{k_j} is in the (1,1) position but this matrix contradicts to the choice of k_1.
7) Now $g_{1j} = P_j x^{k_j} + R_j, j \geq 2$ where $P_j \in \mathbb{C}[x], \deg R_j < k_j$. So we can do a collection of row transformations so that $\deg g_{1j} < k_j,\ j \geq 2$. In particular, $k_1 \leq g_{1j},\ j \geq 2$.
8) Now, subtracting $g_{1j}/x^{k_1} \in \mathbb{C}[x^{-1}]$ multiples of first column from jth column we get a diagonal matrix, because $g_{1j} - g_{1j}/x^{k_1} \cdot x^{k_1} = 0$. In other words we get a matrix $\operatorname{diag}(k_1, k_2, \ldots, k_n)$.
9) We have to show the standard matrix is unique, i.e. if we have $P \operatorname{diag}(a_1, a_2, \cdots, a_n) Q = P' \operatorname{diag}(b_1, b_2, \cdots, b_n) Q', \{P, P\} \subseteq \mathrm{GL}_n \mathbb{C}[x], \{Q, Q'\} \subseteq \mathrm{GL}_n \mathbb{C}[x^{-1}]$. So we actually have $A \operatorname{diag}(a_1, a_2, \cdots, a_n) = \operatorname{diag}(b_1, b_2, \cdots, b_n) B, A \in \mathrm{GL}_n \mathbb{C}[x], B \in \mathrm{GL}_n \mathbb{C}[x^{-1}]$.
10) From the property of minors we have

$$A^{1,2,\ldots,k}_{i_1,\ldots,i_k} x^{a_{i_1}+\cdots+a_{i_k}} = x^{b_1+\cdots+b_k} B^{1,2,\ldots,k}_{i_1,\ldots,i_k}$$

for all $i_1 < i_2 < \cdots < i_k$. Because $A \in \mathrm{GL}_n \mathbb{C}[x], B \in \mathrm{GL}_n \mathbb{C}[x^{-1}]$, so there is some $i_1, \ldots, i_k$ such

$$A^{1,2,\ldots,k}_{i_1,\ldots,i_k} \neq 0.$$

So $b_1 + \cdots b_k \geq a_{i_1} + \cdots + a_{i_k} \geq a_1 + \cdots + a_k$, for all $1 \leq k \leq n$. By symmetry $a_1 + \cdots + a_k \leq b_1 + \cdots b_k$, for all $1 \leq k \leq n$. So we have $a_1 = b_1$, which implies $b_2 \geq a_2$ and $a_2 \geq b_2$, so $b_2 = a_2$. So after $n-1$ steps we get $a_i = b_i, \forall 1 \leq i \leq n$. □

As said in Remark 1.5, we actually want a moduli problem omitted *jump phenomena*. So instead of define a moduli problem in set level we need to defined a moduli problem in the category level, so that the good behavior is preserved by good morphism.

Example 1.6 (Moduli of vector bundles on X)**.** Suppose X is an algebraic variety over $\mathbb{C}$. For a fixed rank n, we define a functor

$$\begin{aligned} F : \text{Schemes}/\mathbb{C} &\to \text{Sets} \\ S &\mapsto \{\text{A family of vector bundle over } S\}/\sim \end{aligned}$$

A family vector bundle over S is a $\mathbb{V}$,a vector bundle over $X \times S$, $\forall s \in S$, s is a closed point. And $\mathbb{V}|_{X \times \kappa(s)}$ is a vector bundle of rank n. $\mathbb{V}_1 \sim \mathbb{V}_2$ iff $\mathbb{V}_1 \simeq \mathbb{V}_2$ as vector bundles over S.

We expect that the moduli space is an algebraic geometry object, i.e. the space could have a scheme structure. A very basic example is all the hypersurface of degree m in $\mathbb{P}^n_k$, it just corresponds to $V(f), f$is homogenuous, $\deg f = k$, And $V(f) = V(\lambda f), \lambda \in k$. So the moduli set actually is $\mathbb{P}_k^{\binom{m+n}{n}-1}$. This set has an algebraic structure so that we can use algebraic geometry to study it.

Definition 1.7. A (contravariant)functor

$$F : \text{Schemes} \to \text{Sets}$$

is said to be *representable* if $F \simeq h_X$, for some scheme X.

$$h_X = \text{Hom}(-, X).$$

And a moduli problem F like 1.6 has a *fine moduli space* if $F \simeq h_X$, for some X, X is called *fine moduli space* of F.

But actually, many moduli problems that we concern have no fine moduli space.

Example 1.8. If $F : \text{Schemes}/k \to \text{Sets}$ is a moduli functor parameterizing a class of objects. Suppose $F(k)$ has non-trivial automorphism $g \in \text{Aut}(F(k)$ we might construct a nontrivial family such that every fiber is isomorphic. Then F is not representable. The first example one might get is vector bundles over curves. Let $X = \text{Spec}\,\mathbb{C}$ in 1.6, so we have a moduli functor, $F(\mathbb{C}) = \{\text{vector space over } \mathbb{C}\}$. It is obviously that there is a nontrivial family that are isomorphic on any fiber, namely $\mathcal{O}_{\mathbb{P}^1}(-1)$ is vector bundles over $\mathbb{P}^1$. $\mathcal{O}(-1)$ is nontrivial (by 1.4, i.e. $\mathcal{O}_{\mathbb{P}^1}(-1) \not\simeq \mathcal{O}_{\mathbb{P}^1}$).But any fiber $\mathcal{O}(-1)_{|p} \simeq \mathbb{C}, \forall p \in \mathbb{P}^1$, because $\mathcal{O}(-1)$ is a vector bundle. So F is not representable. We need to prove following fact.

Fact 2. If $F : \text{Schemes}/k \to \text{Sets}$ is a moduli functor, There is a nontrivial family i.e. an element $\alpha \in F(S)$ such that α is not obtained by base change over $S \to k$. In other words, $\forall \beta \in F(k)$, $\alpha \not\simeq \beta \times_k S$.(This notation is clear if we know the definition of *stack*, i.e. β is in the fiber category $F(k)$, so that we are free to talk about base change), and any fiber $\forall s \in S$, all α_s are isomorphic. Then F is not representable.

Proof. Suppose the converse is true, i.e. F satisfies the condition and F is representable. So there is $M \in \text{Schemes}/k$, such that $\Psi : F \simeq h_M$. Then $\alpha \in \text{Hom}(S, M)$, and any fiber α_s sends to same element in $\text{Hom}(\text{Spec}\,k, M)$. It means following diagram is commutative.

$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[rr, bend right, "\Psi(\alpha)"'] & \text{Spec}\,k \arrow[r, "\Psi(\alpha_s)"] & M. \end{tikzcd}$$

But the above diagram actually says $\Psi(\alpha)$ is the image of $\Psi(\alpha_s)$ under the map $h_M(\eta) : h_M(k) \to h_M(S)$. And $h_M(\eta)$ is obtained by base change, namely we have $\Psi(\alpha) = \Psi(\alpha_s) \times_k S \in h_M(S)$, and because Ψ is an isomorphism between two functors, so after apply Ψ^{-1} we have $\alpha = \alpha_s \times_k S \in F(S)$, i.e. $\alpha \simeq \alpha_s \times_k S$. But that contradicts to the hypothesis that α is nontrivial family. Therefore F is not representable. $\square$

There is an example about nontrivial family of elliptic curves.

Example 1.9. Consider a family of elliptic curves V over $\mathbb{C}$ defined by equation

$$y^2 = x^3 - t.$$

This is a family of elliptic curves over $\text{Spec}\,\mathbb{C}[t, t^{-1}] = \mathbb{A}^1 \setminus \{0\}$, it is smooth when $t \neq 0$, because $\Delta = -16 \cdot 27 t^2 \neq 0$. (See [3, Chapter 3 Proposition 1.4]). And for any $t \in \mathbb{C} \setminus \{0\}$, the *j-invariant* $j = 0$, so by [3, Chapter 3 Proposition 1.4(b)] all the fiber of V are isomorphic.

But V is a nontrivial family, if it is trivial family then V isomorphic to $y^2 = x^3 - 1$ over $\mathbb{P}^2_{\operatorname{Spec}\mathbb{C}[t,t^{-1}]}$. By [3, Chapter 3], the only change variable preserve that form of equation is

$$x = u^2 x', \quad y = u^3 y', \text{for some } 0 \neq u \in \mathbb{C}(t).$$

So, all the possible forms of V are

$$u^6 Y^2 = u^6 X^3 - t.$$

If V is isomorphic to $y^2 = x^3 - 1$, that will give $u^6 t = 1$, for some u. But $t^{1/6}$ is not in $\mathbb{C}(t)$. Therefore V is a nontrivial family, as a consequence the moduli functor of elliptic curves is not representable(by the fact 2).

There are several strategies to solve the non-representable problem.

(1) We define so called *coarse moduli space*.
(2) We expend the category of schemes to the category of *stacks*, and also category of *algebraic stacks*, *Artin stacks* and *Deligne–Mumford stacks* etc.
(3) We modify our moduli problems so that the objects have small automorphism groups, namely we defined *moduli of stable curves*, *moduli of stable sheaves over curves*.

2. Coarse moduli and Stacks

Definition 2.1. A *coarse moduli space* of a moduli functor $F : \text{Schemes}/k \to \text{Sets}$ is a scheme X/k together with a natural transformation $\eta : F \to h_X$, satisfies following conditions.

- $\eta(k)$ is a bijection.
- For any scheme $T/k, \varphi : F \to h_T$, if $\varphi(k)$ is a bijection, then there is an unique morphism $\alpha : X \to T$ such that the following diagram is commutative.

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & h_X \arrow[d, "h_\alpha"] \\ & h_T \end{tikzcd}$$

Example 2.2. Consider the moduli functor F of elliptic curves over $\mathbb{C}$.

$F : \text{Schemes}/\mathbb{C} \to \text{Sets}$

$$S \mapsto \{\mathcal{C} \to S \text{ family of elliptic curves and } \sigma : S \to \mathcal{C} \text{ is a section}\}/\sim$$

A (smooth) family of elliptic curves $\mathcal{C} \to S$ is a smooth and proper morphism such that $\forall s \in \operatorname{cl}(S)$, $\mathcal{C}_s$ is an elliptic curve. $\operatorname{cl}(S)$ is the set of closed points of S.

$$(\mathcal{C}_1 \to S) \sim (\mathcal{C}_2 \to S)$$

iff there is an isomorphism $f : \mathcal{C}_1 \to \mathcal{C}_2$ such that

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "f"] \arrow[d] & \mathcal{C}_2 \arrow[d] \\ S \arrow[r, "\mathrm{id}_S"] & S \end{tikzcd}$$

$\mathbb{A}^1$ is coarse moduli space of F.
Sketch of Proof.

- The first condition is clear by [3, Chapter 3 Proposition 1.4], because if E, E' are elliptic curves then $E \simeq E' \iff j(E) = j(E')$.
- Now, for any $\mathcal{C} \to S$ we need to define a point in $\mathbb{A}^1(S)$. For any open affine $U = \operatorname{Spec} R \subseteq S$, using the section σ we can define an embedding $\mathcal{C}_U \to \mathbb{P}^2_U$. By changing variable (see [3, Chapter 3 §1]) we may assume the equation is of form
$$y^2 = x^3 + Ax^2 + B, A, B \in R.$$
We now define $j(\mathcal{C}_U) = -1728(4A)^3/\Delta, \Delta = -16(4A^3 + 27B^2)$. $j(\mathcal{C}_U) \in \mathbb{A}^1(U)$. Now, since F is a *stack*, so morphisms are able to glued. Finally this gives an element in $\mathbb{A}^1(S)$. And this map is functorial, because the morphisms gluing in a unique way.(An axiom of stack).
- Assume there is another scheme T satisfies the first condition of coarse moduli space of F. Then for an affine scheme $Y = \operatorname{Spec} \mathbb{C}[\lambda, \lambda^{-1}, (\lambda-1)^{-1}]$, there is a family of elliptic curve over Y defined by $y^2 = x(x-1)(x-\lambda)$. So there is a morphism form $Y \to T$, S_3 acting on Y. Namely by permuting those three elements. And actually $Y \to T$ factor though $Y/S_3 \to T$, basically because S_3 acting trivially on T. And $j = \frac{2^8(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$, one can show $Y/S_3 \simeq \operatorname{Spec} \mathbb{C}[j]$. So we get a morphism $\mathbb{A}^1_j = \operatorname{Spec} \mathbb{C}[j] \to T$ satisfies the commutative diagram. □

We have seen examples that objects has nontrivial automorphism group, but it actually depends on the description of the moduli problem.

Example 2.3. Consider moduli of genus g curves in $\mathbb{P}^5$. The moduli functor should be $F : \text{Schemes}/\mathbb{C} \to \text{Sets}$, $F(S) = \{C \subseteq \mathbb{P}^5_S, C \text{ is a smooth curve of genus } g\}/\sim$. And
$$[C \subseteq \mathbb{P}^5_S] \sim [C' \subseteq \mathbb{P}^5_S]$$
if there is an isomorphism $f : C \to C'$ such that the following diagram is commutative.
$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "f"] & \mathbb{P}^5_S \arrow[d, "\mathrm{id}"] \\ C' \arrow[r, hook] & \mathbb{P}^5_S \end{tikzcd}$$
So in this case object in $F(\mathbb{C})$ has $\operatorname{Aut} = \{\mathrm{id}\}$.

As seen before, we need to define *stack* so that we can think moduli problem strictly.

Definition 2.4. Let C be a category and let $p : \mathcal{S} \to C$ be a functor. Category $\mathcal{S}$ is *fibered in groupoids* over C if the following conditions hold.

- For any $\psi : U \to V$ in C, and any $y \in \operatorname{Obj} \mathcal{S}$ is over T, there is an $x \in \operatorname{Obj} \mathcal{S}$, and $f : x \to y$, such that $p(f) = \psi$.
- Given maps $f : x \to z$, $g : y \to z$. For any $\alpha : p(x) \to p(y)$ such that $p(f) = p(g)\alpha$ then there is a unique $h : x \to y$, such that $f = gh, p(h) = \alpha$.

Remark 2.5. We call $\mathcal{S}$ is fibered in groupoids because the fiber category $\mathcal{S}(T)$ for any $T \in \operatorname{Obj} C$ is a groupoid. The objects in $\mathcal{S}(T)$ are objects in $\mathcal{S}$ which are over T. For any $x, y \in \operatorname{Obj} \mathcal{S}(T)$, we define $\operatorname{Hom}_{\mathcal{S}(T)}(x, y)$ to be the morphisms over id_T .(This is similar to the equivalence relation in Example 2.2). In other words $\forall f \in \operatorname{Hom}_{\mathcal{S}(T)}(x, y), p(f) = \mathrm{id}_T$. By the second condition in Definition 2.4, any $f \in \operatorname{Hom}_{\mathcal{S}(T)}(x, y)$ is invertible. Therefore

$\mathcal{S}(T)$ is a groupoid. The intuition is that we should think the objects of $\mathcal{S}(T)$ as family of objects parameterized by T, and morphisms of $\mathcal{S}(T)$ should be understood as isomorphism between two family of objects. From Definition 2.4 we know what is so called base change mentioned in proof of 2, the "base change" actually is the first condition in 2.4, and it is unique by the second condition.

Definition 2.6. Let $\mathcal{S}, \mathcal{T}$ be categories over C fibered in groupoids. A morphism $f : \mathcal{S} \to \mathcal{T}$ is a functor between these two categories ans for any $x \in \mathcal{S}, p_{\mathcal{S}}(x) = p_{\mathcal{T}}(f(a))$.

Definition 2.7. Let C be a site.(A category with a Grothendieck topology). A category $\mathcal{S}$ fibered in groupoids over C is called a *stack* if the following condition holds. Let $\{T_i \to T\}$ be a covering of $T \in \operatorname{Obj} C$.

- For any $x, y \in \mathcal{S}(T)$, and morphisms $f_i : x|_{T_i} \to y$ such that $f_i|_{T_{ij}} = f_j|_{T_{ij}}$ then there is $f : x \to y$ such that $fr_i = f_i$ where $r_i : x|_{T_i} \to y$ such that $p(r_i) = S_i \to S$.
- For objects $x_i \in \operatorname{Obj} \mathcal{S}(T_i)$ and isomorphisms $f_{ij} : x_i|_{T_{ij}} \to x_j|_{T_{ij}}$. If the isomorphisms satisfies cocycle condition $f_{jk}|_{T_{ijk}} f_{ij}|_{T_{ijk}} = f_{ik}|_{T_{ijk}}$, then there exists $x \in \mathcal{S}(T)$ and isomorphisms $\varphi_i : x|_{T_i} \to x_i$ such that $f_{ij}\varphi_i|_{T_{ij}} = \varphi_j|_{T_{ij}}$.

Definition 2.8. Let $\mathcal{S}, \mathcal{T}$ be stacks over site C, morphism $f : \mathcal{S} \to \mathcal{T}$ is morphism between categories fibered in groupoids.

Remark 2.9. The first condition says we could gluing morphisms, the second condition says we could gluing objects. So first condition says the second step in sketch of proof of Example 2.2 is workable.

Remark 2.10. For any scheme X, $h_X : \text{Schemes} \to \text{Sets}$. Consider the category $\mathcal{X}$ with objects of pairs $(a, S), a \in h_X(S)$. $\operatorname{Hom}_{\mathcal{X}}((a, S), (b, T)) = \{f : S \to T | h_X(f)(b) = a\}$. It is clear that $\mathcal{X}$ is a category over Schemes fibered in groupoids. But it is not clear that $\mathcal{X}$ is a stack over $\text{Schemes}_{\text{ét}}$, this is actually a fact follows form the *decent theory*.

Fact 3. $\mathcal{X}$ is a stack over $\text{Schemes}_{\text{ét}}$. See [1, Theorem 4.29, Theorem 4.31].

Definition 2.11. Let $g \geq 2$. Consider the category $\mathcal{M}_g$ with objects $(f : \mathcal{C} \to S)$. f is smooth and proper morphism and for any $s \in \operatorname{cl}(S)$, the fiber $\mathcal{C}_s$ is a smooth connected curve with genus g. A morphism $(f_1 : \mathcal{C}_1 \to S_1) \to (f_2 : \mathcal{C}_2 \to S_2)$ is maps $\alpha : \mathcal{C}_1 \to \mathcal{C}_2, \beta : S_1 \to S_2$ such that $f_2 \circ \alpha = \beta \circ f_1$. And $p : \mathcal{M}_g \to \text{Schemes}$ is defined by $p((\mathcal{C} \to S)) = S$. It is clear that $\mathcal{M}_g$ is a category over Schemes fibered in groupoids, because we have pull back. From the fact that there is étale decent for morphisms we know $\mathcal{M}_g$ satisfies first condition in Definition 2.7.

Fact 4. $\mathcal{M}_g$ is a stack over $\text{Schemes}_{\text{ét}}$. See [4, Tag 0E83].

Definition 2.12. Let $k[\epsilon] = k[\epsilon]/\epsilon^2$. Let $i : \operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]$. $\mathcal{X}$ is an algebraic stack and $x : \operatorname{Spec} k \to \mathcal{X}$ the tangent space of $\mathcal{X}$ at x is the set

$$\{(y, \tau) | y : \operatorname{Spec} k[\epsilon] \to \mathcal{X}, \tau \text{ is a 2 morphism form } x \text{ to } y \circ i\} / \sim .$$

A 2 morphism τ is a natural transformation form x to $i \circ y$ such that $\tau(T)$ is a morphism in the fiber category $\mathcal{X}(T)$ for all $T \in \operatorname{Obj}(\operatorname{Spec} k)$. $(y, \tau) \sim (y', \tau')$ iff there is an isomorphism $\alpha : y \simeq y'$ such that $\tau' = \alpha \circ \tau$. So for any $T \in \operatorname{Obj}(\operatorname{Spec} k)$, we have $\tau'(T) = \alpha \circ \tau(T)$, therefore α is an isomorphism which is identity restrict to $\operatorname{Obj}(\operatorname{Spec} k)$. This is similar with the definition of equivalence relation of two *deformations* over $\operatorname{Spec} k[\epsilon] \to \operatorname{Spec} k$.

We recall a basic fact mentioned in the course.

Fact 5. If X is smooth over k then $\mathrm{Def}(X) \simeq \mathrm{H}^1(X, T_X)$.

Example 2.13. Now, take a $\mathbb{C}$-point of $\mathcal{M}_g$, i.e. a smooth proper curve over $\mathbb{C}$ of genus g, $[C] : \operatorname{Spec} \mathbb{C} \to \mathcal{M}_g$. From the definition of tangent space and the fact any deformation D of C over $\mathbb{C}[\epsilon]$ is a family of genus g curves, because any such deformation D would have only closed fiber $D_{\mathbb{C}} \simeq C$. And the equivalence relations are same. By fact 5 So $T_{\mathcal{M}_g,[C]} \simeq \mathrm{Def}(C) \simeq \mathrm{H}^1(C, T_C)$. By Riemann–Roch theorem $\chi(T_c) = \deg(T_C) + 1 - g = -\deg(K_C) + 1 - g = 2 - 2g + 1 - g = 3 - 3g$. Now $\dim_{\mathbb{C}}(\mathrm{H}^0(C, T_C)) - \dim_{\mathbb{C}}(\mathrm{H}^1(C, T_C)) = 0 - \dim_{\mathbb{C}}(\mathrm{H}^1(C, T_C))$. Because $\deg(T_C) = 2 - 2g \leq 2 - 4 = -2 < 0$, therefore $\dim_{\mathbb{C}}(\mathrm{H}^0(C, T_C)) = 0$, otherwise the linear system of it is nonempty but that contradicts to $\deg(T_C) < 0$. Therefore $\dim_{\mathbb{C}}(T_{\mathcal{M},[C]}) = \dim_{\mathbb{C}}(\mathrm{H}^1(C, T_C)) = -(3 - 3g) = 3g - 3$. And actually, using a general deformation theory(generalization of fact 5) we can prove $\mathcal{M}_g$ is smooth over $\operatorname{Spec} \mathbb{Z}$ [4, Tag 0E84], so we actually have $\dim(\mathcal{M}_g) = \dim_{\mathbb{C}}(T_{\mathcal{M}_g,[C]}) = 3g - 3$.

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