LAMBDA ALGEBRA AND ADAMS SPECTRAL SEQUENCE

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Abstract. In this notes, we mainly introduce the construction of lambda algebra and its relation with Adams spectral sequence.

Contents

1. Introduction

1.1. Original definition of Λ.

Let V be the $\mathbb{Z}/2$ vector space with basis $\{\lambda_p, p \geq 0\}$. $T(V)$ be the tensor algebra, i.e.

$$T(V) = \oplus_k V^{\otimes k}.$$

I be the ideal of $T(V)$ generated by

$$\sum_{i+j=n} \binom{n}{i} \lambda_{p+i} \otimes \lambda_{2p+1+j} \in V^{\otimes 2}, p \geq 0, n \geq 0.$$

Then we define $\Lambda = T(V)/I$. Clearly, Λ is a bigraded $\mathbb{Z}/2$ algebra. $\Lambda = \oplus_{s,t\geq 0} \Lambda^{s,t}$, s represents length, $t-s$ is the degree in the usual sense, i.e. for a fixed (s,t), $\Lambda^{s,t}$ is the $\mathbb{Z}/2$ subspace of Λ generated by $\lambda(a_1, \dots, a_s) = \lambda_{a_1} \dots \lambda_{a_s}$, $t = a_1 + \cdots + a_s + s$. And the differential of Λ is given by

$$d(\lambda_{n-1}) = \sum_{i+j=n} \binom{n}{i} \lambda_{i-1} \otimes \lambda_{j-1}, n \geq 1.$$

1.2. **Another definition of Λ .**

Λ is a bigraded $\mathbb{Z}/2$- algebra with generators $\lambda_n \in \Lambda^{1,n+1}, n \geq 0$ and relations

$$\lambda_i \lambda_{2i+1+n} = \sum_{j\geq 0} \binom{n-j-1}{j} \lambda_{i+n-j}\lambda_{2i+1+j}, \text{ for } i, n \geq 0$$

with differential

$$d(\lambda_n) = \sum_{j\geq 1} \binom{n-j}{j} \lambda_{n-j}\lambda_{j-1}.$$

The above definitions are equivalent, due to [11, Theorem 1.5.4, 1.6.5]. We only sketch the proof. For a combinatorial proof one can see [7, Lemma 6.3].

Definition 1.1. A monomial $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s} \in \Lambda$ is admissible if $2i_r \geq i_{r+1}$, for $1 \leq r < s$. $\Lambda(n)$ is the subcomplex spanned by admissible monomials with $i_1 < n$.

Sketch of proof. 1. Notice $x_i = \lambda_i\lambda_{2i+1} = 0 \in \Lambda$. And there is a derivation map $D : \Lambda \to \Lambda$, such that $D(\lambda_n) = \lambda_{n+1}, n \geq 0$. This is because there exists $\overline{D} : T(V) \to T(V)$, such that $D(\lambda_n) = \lambda_{n+1}$, and $D(I) \subseteq I$. This is basically because $I =< x_i, D(x_i), \ldots, D^n(x_i), \cdots >$. (By Leibniz formula).

2. Then,

$$D^n x_i = \lambda_i\lambda_{2i+1+n} + \sum_{j\geq 0} a_{n-j,j}\lambda_{i+n-j}\lambda_{2i+1+j}$$

by using admissible basis and operator D, we can get $a_{n-j,j} = \binom{n-j-1}{j}$.
Therefore, the relations can be reformulated as follow

$$\lambda_i \lambda_{2i+1+n} = \sum_{j\geq 0} \binom{n-j-1}{j} \lambda_{i+n-j}\lambda_{2i+1+j}.$$

3. For the differential formula, setting

$$d(\lambda_n) = \sum_{j\geq 0} b_{n-j,j}\lambda_{n-j-1}\lambda_j.$$

By using admissble basis and comparing coefficients we can get $b_{n-j,j} = \binom{n-j}{j}$.

□

Remark 1.2. Actually, there is a topological way to simplify the relations, we will explain it later. But I do not know how to caculate it.

One might ask, why should we consider this werid algebra. Actually, lambda algebra is E_1 page of a spectral sequence, and it has close relation with Adams spectral sequence.

Definition 1.3. (The lower p-central series)
Let G be a group and p a prime. The p-central series of G is the filtration

$$\cdots \subseteq \Gamma_r G \subseteq \cdots \subseteq \Gamma_2 G \subseteq \Gamma_1 G = G,$$

where $\Gamma_r G$ is the subgroup generated by all elements

$$[a_1, \ldots, a_k]^{p^i}$$

for which $k \geq 1, kp^i \geq r$, and each $a_j \in G$, the symbol $[\,, \ldots, \,]$ denotes the commutator$[\ldots[\,,\,], \ldots],]$.

For any semisimplicial spectrum X, we have a free group spectrum FX, $\pi_q FX = \pi_q X$. Then we consider the lower 2-central series of FX,

$$\cdots \subseteq \Gamma_{2^{r+1}} FX \subseteq \Gamma_{2^r} FX \subseteq \cdots \subseteq \Gamma_1 G = FX,$$

We denote $(E_r X, d_r X)$ to be the derived spectral sequence of the homotopy exact couple of this filtration.

i.e.,

$$\begin{aligned}
Z_r^{s,t} &= im\{\pi_{t-s}(\Gamma_{2^s}/\Gamma_{2^{s+r}} FX) \to \pi_{t-s}(\Gamma_{2^s}/\Gamma_{2^{s+1}} FX)\}, \\
B_r^{s,t} &= im\{\pi_{t-s}(\Gamma_{2^{s-r+1}}/\Gamma_{2^s} FX) \to \pi_{t-s}(\Gamma_{2^s}/\Gamma_{2^{s+1}} FX)\}, \\
E_r^{s,t} &= Z_r^{s,t}/B_r^{s,t}.
\end{aligned}$$

And d_r defined in a natrual way.

Remark 1.4. A priori, we should take

$$\cdots \subseteq \Gamma_{r+1} FX \subseteq \Gamma_r FX \subseteq \cdots \subseteq \Gamma_1 G = FX.$$

But we later will prove $\pi_*(\Gamma_r/\Gamma_{r+1} FX) = 0$, if $r \neq 2^k, \forall k \geq 0$.

Then, by standard argument we have $\pi_*(\Gamma_{2^r}/\Gamma_{2^r+1} FX) \simeq \pi_*(\Gamma_{2^r}/\Gamma_{2^{r+1}} FX)$. After reindexing subscripts, we will get the real lambda algebra.

Especially, if we take X to be the sphere simplicial spectrum S, we will get the main result of the paper.

Theorem 1.5. [1, 2.6] *$(E_1 S, d_1 S)$ is the Λ we described before, and $(E_2 X, d_2 X)$ is the E_2 page of Adams spectral seqence of X.*

In the next sections, we will give necessary background material and prove Theorem 1.5 in detail.

2. Simplicial Homotopy Theory

The main material are [3],[2],[4],[5],[12, 8.1-8.4], [9, 14.1-14.3 and 14.24].

Definition 2.1. A simplicial set K is a sequence of sets, $K = \{K_0, K_1, \ldots, K_n, \ldots\}$, together with functions

$$\begin{aligned}
d_i &: K_n \to K_{n-1}, \\
s_i &: K_n \to K_{n+1},
\end{aligned}$$

for each $0 \leqslant i \leqslant n$. These functions are required to satisfy the simplicial identities

$$d_i d_j = d_{j-1} d_i \qquad \text{for } i < j,$$

$$d_i s_j = \begin{array}{ll} s_{j-1} d_i, & \text{for } i < j \\ \text{identity}, & \text{for } i = j, j+1 \\ s_j d_{i-1}, & \text{for } i > j+1. \end{array}$$

$$s_i s_j = s_{j+1} s_i \quad \text{for} \quad \text{for} \quad i \leqslant j+1,$$

Remark 2.2. The standard n simplex $\Delta[n]$ is the simplicial set with vertices $0, 1, 2, \ldots, n$, where

$$(\Delta[n])_q = \{\langle v_0, \ldots, v_q \rangle : 0 \leqslant v_0 \leqslant \cdots < v_q \leqslant n\}.$$

Let $i_n = \langle 0, i, \ldots, n\rangle \in (\Delta[n])_n$. The "boundary of $\Delta[n]$ " is $\partial\Delta[n] = \Delta[n]^{(n-1)}$ = the $n-1$ skeleton.

The n sphere S^n is the quotient simplicial set $\Delta[n]/\partial\Delta[n]$. Thus S^n has two nondegenerate simplices, a vertex which we call $*$, and σ_n in dimension n, which is the image of i_n. In dimensions $n+q, S^n$ has the iterated degeneracy of $*$, and simplices $s_{i_q} \cdots s_{i_1}\sigma_n$, where $n+q > i_q > \cdots > i_1 \geqslant 0$.

Definition 2.3. (Extension condition)

Let K be a simplicial set. Then K satisfies the extension condition $\Leftrightarrow$ for every collection $y_0, \ldots, \hat{y}_k, \ldots, y_n$ of simplices in K_{n-1}, with $d_i y_j = d_{j-1} y_i$ for $i < j, i \neq k, j \neq k$, and there is a simplex $y \in K_n$ with $d_i y = y_i, i \neq k$.

Definition 2.4. A simplicial set satisfying extension condition is called Kan complex.

Definition 2.5. Let K be a Kan complex, $x, y \in K_n$. we define $x \simeq y$, if $d_i x = d_i y$ for all i, and for some $0 \leqslant k \leqslant n$ there is $w \in K_{n+1}$ with $d_k w = x, d_{k+1} w = y$, and $d_i w = d_i s_k x = d_i s_k y, k \neq i \neq k+1$.

Remark 2.6. If K is a Kan complex, then the relation above is indeed a equivalence relation. [3, proposition 2.4]

Definition 2.7. Let (K, ϕ) be a simplicial complex with base point, and K is a Kan complex. For every integer $n \geq 0$ we define a set $\pi_n(K, \phi)$ as follows. Ler Γ_n be the set consisting of those n simplices $\sigma \in K$ such that

$$d_i\sigma = s_{n-2} \cdots s_1 s_0 \phi, \quad 0 \leq i \leq n.$$

The equivalence relation $\sim$ divides Γ_n into classes.

We define

$$\pi_n(K, \phi) = \Gamma_n/(\sim).$$

Remark 2.8. For $n > 0$ let $a, b \in \pi_n(K, \phi)$, $\sigma \in a, \tau \in b$, since K is Kan comlex, there is a $(n+1)$ simplex $\rho \in K$ such that

$$d_{n-1}\rho = \sigma, d_{n+1} = \tau, d_i\rho = s_{n-1} \cdots s_1 s_0 \phi.$$

We then define product of $ab = [d_n\rho]$.

This product defined a group structure on $\pi_n(K, \phi)$.

Definition 2.9. A simplicial group is a simplicial set $\{G_n\}$, each G_n is a group. And those face maps and degeneracy maps are group homomorphism.

Remark 2.10. Simpilicial groups are Kan complexes. [3, proposition 5.2]

We now focus on simplicial abelian group.

Definition 2.11. (Moore complex) Let A_* be a simplicial abelian group For each $n \geq 1$, we define a group homomorphism $\partial : A_n \to A_{n-1}$ by the formula

$$\partial(\sigma) = \sum_{i=0}^{n} d_i(\sigma).$$

It's easy to cheek $\partial \circ \partial = 0$.

We set

$$\mathrm{C}_n(A) = \begin{cases} A_n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

$C_*(A)$ is called the moore complex of A.

Definition 2.12. Let $A_\bullet$ be a simplicial abelian group. For each $n \geq 0$, let $\mathrm{D}_n(A)$ denote the subgroup of $\mathrm{C}_n(A) = A_n$ generated by the images of the degeneracy operators $\{s_i : A_{n-1} \to A_n\}_{0 \leq i \leq n-1}$. By convention, we set $\mathrm{D}_n(A) = 0$ for $n < 0$.

Proposition 2.13. *Let $A.$ be a simplicial abelian group. For every positive integer n, the boundary operator $\partial : \mathrm{C}_n(A) \to \mathrm{C}_{n-1}(A)$ carries the subgroup $\mathrm{D}_n(A)$ into $\mathrm{D}_{n-1}(A)$. Consequently, we can regard $\mathrm{D}_*(A)$ as a subcomplex of the Moore complex $\mathrm{C}_*(A)$.*

Definition 2.14. (The Normalized Moore Complex: First Construction)
Let $A.$ be a simplicial abelian group. We let $\mathrm{N}_*(A)$ denote the chain complex given by the quotient $\mathrm{C}_*(A)/\mathrm{D}_*(A)$, where $\mathrm{C}_*(A)$ is the Moore complex of 2.11 and $\mathrm{D}_*(A) \subseteq \mathrm{C}_*(A)$ is the subcomplex of Proposition 2.13 We will refer to $\mathrm{N}_*(A)$ as the normalized Moore complex of the simplicial abelian group $A_\bullet$.

Definition 2.15. (The Normalized Moore Complex: Second Construction)
Let $A.$ be a simplicial abelian group. For each $n \geq 0$, we let $\widetilde{\mathrm{N}}_n(A)$ denote the subgroup of $\mathrm{C}_n(A) = A_n$ consisting of those elements x which satisfy $d_i(x) = 0$ for $1 \leq i \leq n$. Note that if x satisfies this condition, then we have

$$\partial(x) = \sum_{i=0}^{n} (-1)^i d_i(x) = d_0(x).$$

Moreover, the identity $d_i d_0(x) = d_0 d_{i+1}(x) = 0$ shows that $\partial(x) = d_0$ belongs to the subgroup $\widetilde{\mathrm{N}}_{n-1}(A) \subseteq \mathrm{C}_{n-1} = A_{n-1}$. We can therefore regard $\widetilde{N}_*(A)$ as a subcomplex of the Moore complex $\mathrm{C}_*(A)$.

Proposition 2.16. *Let $A_\bullet$ be a simplicial abelian group. Then the composite map $\widetilde{\mathrm{N}}_*(A) \hookrightarrow \mathrm{C}_*(A) \to \mathrm{N}_*(A)$ is an isomorphism of chain complexes. In other words, the Moore complex $\mathrm{C}_*(A)$ splits as a direct sum of the subcomplex $\widetilde{\mathrm{N}}_*(A)$ of 2.15 and the subcomplex $\mathrm{D}_*(A)$ of Proposition 2.13.*

Proposition 2.17.

$$\mathrm{H}_n(N_*(A)) \simeq \pi_n(A, e), n > 0$$

Theorem 2.18. *(The Dold-Kan Correspondence)*
The normalized Moore complex functor determines an equivalence of categories

$$\mathrm{N}_* : \mathrm{Ab}_\Delta \to \mathrm{Ch}(\mathrm{Z})_{\geq 0}.$$

Definition 2.19. A semisimplicial spectrum (or set spectrum or simply spectrum) X consists of
(1) for every integer q a set $X_{(q)}$ with a distinguished element $*$ (called base point); the elements of $X_{(q)}$ will be called simplices of degree q.

(2) for every integer $i \geq 0$ and every integer q a function $d_i : X_{(q)} \to X_{(q-1)}$ such that $d_i * = *$ (the i-face operator) and a function $s_i : X_{(q)} \to X_{(q+1)}$ such that $s_i * = *$ (the i-degeneracy operator). These operatirs are required to satisfy the following axioms: I.

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i && \text{for } i < j \\
d_i s_j &= s_{j-1} d_i && \text{for } i < j \\
&= \text{ identity} && \text{for } i = j, j+1 \\
&= s_j d_{i-1} && \text{for } i > j+1 \\
s_i s_j &= s_j s_{i-1} && \text{for } i > j
\end{aligned}$$

II. For every simplex $\alpha \in X$ all but a finite number of its faces are "at the base point," i.e. $d_i \alpha = *$ for all but a finite number of i 's.

Definition 2.20. A subspectrum of a spectrum X is a subset of X which is closed under all (face and degeneracy) operators.

Definition 2.21. A map $w : X \to Y$ between two spectra is a degree preserving function which commutes with all operators. An equivalence is a map which is $1-1$ onto.

Remark 2.22. A useful spectrum is the sphere spectrum S, i.e. the spectrum all of whose simplices are degenerate (i.e. of the form $s_i\alpha$ for some i and α) except one simplex ϕ of degree 0 .

Definition 2.23. For every simplicial spectrum X and integer q we define a complex $X_q \in \mathscr{S}_*$ as follows. An n-simplex of X_q is any $\alpha \in X$ such that degree $\alpha = n - q, d_0 \ldots d_n \alpha = *$ and $d_j \alpha = *$ for $j > n$. The base point will be the simplex $* \in X_{(-q)}$ The face and degeneracy operators of X_q are those induced by the corresponding operators of X. It is not difficult to verify that X_q is indeed a set complex, i.e. that for every n-simplex $\alpha \in X_q$ the simplices $d_j\alpha$ and $s_j\alpha$ are also in X_q for $0 \leq j \leq n$.

Remark 2.24. For the sphere spectrum 2.22 and every integer $q \geq 0, S_q$ is the semisimplicial q-sphere. In this case, the natural map $u : \Sigma S_q \to S_{q+1}$ is an equivalence.

3. Construction

For a simplicial spectrum X, consider the free group spectrum FX.

Denote$(E_i X, d_i X)$ the spectral sequence associated with the homotopy exact couple of the following filtration of FX

$$\cdots \subseteq \Gamma_{2^{j+1}} FX \subseteq \Gamma_{2^j} FX \subseteq \cdots \subseteq \Gamma_2 FX \subseteq \Gamma_1 FX = FX.$$

Theorem 3.1. *(The structure of $(E_1 S, d_1 S)$). Let S denote the sphere spectrum. Then*

(1)$(E_1 S, d_1 S)$ is the graded associative differential $\mathbb{Z}/2$ algebra.

(2)With generators λ_i (of degree i) for every integer $i > 0$.

(3)For $m \geq 1, n \geq 0$ a relation

$$\sum_{i+j=n} \binom{n}{i} \lambda_{i-1+m} \lambda_{j-1+2m} = 0.$$

(4)Differential is given by

$$d(\lambda_{n-1}) = \sum_{i+j=n} \binom{n}{i} \lambda_{i-1}\lambda_{j-1}.$$

Theorem 3.2. *(The structure of (E_1X, d_1X)).*
(1) (E_1X, d_1X) is a differential right (E_1S, d_1S) module, where
(2) the module structure is given by

$$E_1X = \mathrm{H}_* X \otimes E_1S.$$

(3) The differential is given by

$$d_1(a \otimes 1) = \sum_{i>0}(aSq^i \otimes \lambda_{i-1}), a \in \mathrm{H}_* X.$$

Since we only consider $\mathbb{Z}/2$ coefficients (co)homology , then $\mathrm{H}_n(X) = \mathrm{Hom}(H^n(X), \mathbb{Z}/2)$. There steenrod algebra acting on $\mathrm{H}_* X$ in an obvious way, which decreasing the homological degree.

In order to see the E_1S module structure of E_1X, we have to introduce free restricted Lie algebra.

We recall the definition of the free restricted Lie algebra on a $\mathbb{Z}/p\mathbb{Z}$-module M.

Let TM be the tensor algebra $TM = \oplus_{r\geqq 0} M^{\otimes r}$, where $M^{\otimes r} = M \otimes \ldots \otimes M$ r-times. For $a, b \in TM$, define $[a, b] = ab - ba$ and $a^{[p]} = a^p$; then the free unrestricted Lie algebra LM on M is the smallest sub $\mathbb{Z}/p\mathbb{Z}$-module of TM containing M. Denote by

$$L^uM = \sum_{r>0} L^u_r M, L^uM = L^uM \cap M^{\otimes r}.$$

And the free restricted Lie algebra LM on M is the smallest sub $\mathbb{Z}/p\mathbb{Z}$-module of TM containing M and closed under the operations [,] and ($)^{[p]}$. Put

$$L_rM = LM \cap M^r$$

so that

$$LM = \sum_{r\geqq 1} L_rM.$$

For each r, L_rM is a functor of M. A result of Zassenhaus is

Proposition 3.3. [6, Prop 3.3] *If G is a free group, there is for each r a natrual isomorphism*

$$\Gamma_r G/\Gamma_{r+1} G \simeq L_r(G/\Gamma_2 G).$$

Since the isomorphism is natrual , and $\Gamma/\Gamma_2 = A$, by applying to FX, we get

$$\pi_*\Gamma_r/\Gamma_{r+1}FX \simeq \pi_* L_r AX, r > 0.$$

We now can define the composition operation on homotopy groups of $\pi_*(\Gamma_r/\Gamma_{r+1})FX \simeq \pi_* L_r AX$.

Definition 3.4. For $x \in \pi_t L_r AS$, we have $x \in \pi_{t+s} L_r AS_s = \pi_t L_r K(s)$, by Dold-Kan correspondence, it corresponds to a map

$$AS_{t+s} = K(t + s) \to L_r K(s) = L_r AS_s.$$

And $y \in \pi_s L_q AX$, it corresponds to

$$AS_s = K(s) \to L_q AX.$$

We therefore define their composition xy to be

$$AS_{t+s} = K(t+s) \xrightarrow{x} L_r K(s) = L_r AS_s \xrightarrow{L_q(y)} L_q L_r AX \longrightarrow L_{qr} AX \ .$$

This is an element of $\pi_{t+s} LAX$.

Now, we see the composition operation turns $\pi_* LAS$ into an associative graded algebra with unit and $\pi_* LAX$ into a right module over it.

For Theorem 3.2 (2), there is an isomorphism

$$\mathrm{H}_* X \otimes \pi_* LAS \to \pi_* LAX$$

given by $a \otimes 1 \mapsto a$.

Proof. (Sketch) This is trivial if X has homotopy type of q-fold suspension $\Sigma^q S$ of sphere spectrum. For arbitary X, this is basically follows from AX is $\mathbb{Z}/2$ simplicial vector space, which must be direct sum of $K(\mathbb{Z}/2, n) = A\Sigma^n S$. □

3.1. **The additive structure of $E_1 S$.**

The groups $\pi_* LAS_n$ are connected by the suspension homomorphisms

$$\pi_* LAS_n \xrightarrow{\Sigma} \pi_{*+1} LAS_{n+1} \quad n \geq 0$$

and for each element $\alpha \in \pi_q L_r AS_n$ by the composition homomorphism (defined as above)

$$\pi_* L_s AS_q \xrightarrow{\alpha} \pi_* L_{rs} AS_n$$

We will often use the same symbol for an element $\alpha \in \pi_q LAS_n$ and its suspensions as well as for the corresponding element of $\pi_{q-n} LAS$. No confusion will arise as composition is compatible with suspension; i.e. the diagram

$$\begin{tikzcd} \pi_* LAS_{q+1} \arrow[r, "\Sigma\alpha"] & \pi_{*+1} LAS_{n+1} \\ \pi_* LAS_q \arrow[u, "\Sigma"] \arrow[r, "\alpha"] & \pi_* LAS_n \arrow[u, "\text{Susp}"'] \end{tikzcd}$$

is commutative for every $\alpha \in \pi_q LAS_n$. Of course, all these are also hold for the groups $\pi_* L^u AS_n$ where L^u is the free unrestricted Lie algebra functor.

Definition 3.5. Let V be a simplicial vector space, let $y \in V_p, z \in V_q$. Then define $y\underline{\otimes}z \in (V \otimes V)_{p+q}$ to be

$$y\underline{\otimes}z = \sum_{(b,a)} \pm s_b y \otimes s_a z.$$

$s_a = s_{a_q} \cdots s_{a_1}$, $s_b = s_{b_p} \cdots s_{b_1}$. Where sum is taken over all permutations $\{a_1, \ldots, a_q, b_1, \ldots, b_p\}$ of $\{0, \ldots, p+q-1\}$ for which $a_1 < \cdots < a_q$ and $b_1 < \cdots < b_p$. The sign is the sign of permutation. (If $p = q = 0$, this is just the tensor product.)

Remark 3.6. This is also denoted as $y \nabla z$, called the shuffle product.

Definition 3.7. (The elements λ_n)

Let $i_n \in AS_n$ be the element represents the unique class in $\pi_n AS_n$, then we define

$$1 = [i_n] \in \pi_n L_1 AS_n,$$
$$\lambda_n = [i_n \underline{\otimes} i_n] \in \pi_{2n} L_2 AS_n.$$

Since we need to compute $\pi_* LAS$, we need the stable version of this , thus λ_n will also stand for any of its suspension.

Proposition 3.8. *(Additive structure of $\pi_* LAS$) The compositions*

$$\lambda_{i_1} \cdots \lambda_{i_k} 1$$

for which $k > 0$ and $i_{j+1} \leq 2i_j, j > 0$, form a basis for $\pi_* LAS$.

In order to prove it, we just need the following unstable version proposition.

Proposition 3.9. *(Additive structure of $\pi_* LAS_n$) The compositions*

$$\lambda_{i_1} \cdots \lambda_{i_k} 1$$

for which $k > 0, i_1 \geq n$ and $i_{j+1} \leq 2i_j, j > 0$, form a basis for $\pi_* LAS_n$.

Proposition 3.9 is a consequnce of following 2 lemmas combining with induction.

Lemma 3.10. *The inclusion map $L^u AS_n \to LAS_n$ and the function "composition on the right with λ_0 " induces isomorphisms*

$$\pi_* L_r AS_n \simeq \pi_* L_r^u AS_n, \quad r\, odd.$$
$$\pi_* L_r AS_n \simeq \pi_* L_r^u AS_n + \pi_* L_{\frac{r}{2}} AS_n, \quad r\, even.$$

Proof. (Sketch)

This is follows form,

(1) notice that we have decomposition

$$LM = L^u M \times LM$$

as set, through squaring map.

(2) The map on homotopy groups induced by squaring map is composition with λ_0. ☐

Lemma 3.11. *Let $n > 0$. Then the suspension homomorphism $\Sigma : \pi_{*-1} L^u AS_{n-1} \to \pi_* LAS_n$ and the composition homomorphism $\lambda_n : \pi_* L^u AS_{2n} \to \pi_* L^u AS_n$ induces isomorphisms*

$$\pi_* L_r^u AS_n \simeq \pi_* L_r^u AS_{n-1}, \quad r\, odd,$$

$$\pi_* L_r^u AS_n \simeq \pi_{*-1} L_r^u AS_{n-1} + \pi_* L_{\frac{r}{2}}^u AS_{2n}, \quad r\, even.$$

Remark 3.12. If we have already proved Lemma 3.11, we are be able to prove Proposition 3.9.

Remark 3.13. By notice the module structure of $\pi_* LAX$, we will get

$$\pi_* \Gamma_r / \Gamma_{r+1} FX = \pi_* L_r AX = 0, \text{if } r \neq 2^k,$$

for some k.

This k stand for the length of monomial $\lambda_{i_1} \cdots \lambda_{i_k}$.

Proof of Theorem 3.1 (1) and Theorem 3.2 (1). It remains to prove the statements about the differential in part (1) of Theorem 3.1 and 3.2. Form the Remark 3.13 let

$$\alpha \in \pi_s L_{2^i} AX = \pi_s \left(\Gamma_{2^i}/\Gamma_{2^{i+1}}\right) FX$$

$$\beta \in \pi_t L_{2^j} AS = \pi_t \left(\Gamma_{2^j}/\Gamma_{2^{j+1}}\right) FS$$

Let $b \in \Gamma_{2^j} FS$ be such that $d_0 b \in \Gamma_{2^{j+1}} FS, d_i b = *$ for $i \neq 0$ and $\operatorname{proj} b \in \beta$. Then proj $d_0 b \in d\beta$.

• This is because we view $\beta \in \pi_{t+s}(\Gamma_{2^j}/\Gamma_{2^{j+1}})FS_s)$, form the definition of homotopy group of simplicial spectrum. We then have following diagram.

$$\begin{tikzcd}
 & \Gamma_{2^{j+1}} FS_s & \Gamma_{2^j} FS_s & \Gamma_{2^j}/\Gamma_{2^{j+1}} FS_s & \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & N'_{t+s+1} \arrow[r] \arrow[d, "d_0"'] & N_{t+s+1} \arrow[r] \arrow[d, "d_0"'] & N''_{t+s+1} \arrow[r] \arrow[d, "d_0"'] & 0 \\
0 \arrow[r] & N'_{t+s} \arrow[r] \arrow[d, "d_0"'] & N_{t+s} \arrow[r] \arrow[d, "d_0"'] & N''_{t+s} \arrow[r] \arrow[d, "d_0"'] & 0 \\
0 \arrow[r] & N'_{t+s-1} \arrow[r] \arrow[d] & N_{t+s-1} \arrow[r] \arrow[d] & N''_{t+s-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Write b in the form $B(Fi)$ where B is a formula involving only degeneracy operators and the operations product and inverse and where $i \in S$ is the only non-degenerate simplex.

Let m be the largest integer for which B involves the operator s_m and let k be an integer such that $2k > t + m$.

Then there exists a simplex $a \in \Gamma_{2^i} FX$ such that $d_{2k} a \in \Gamma_{2^{i+1}} FX, d_i a = *$ for i $\neq 2k$ and proj $a \in \alpha$ and hence proj $d_{2k} a \in d\alpha$.

• This is because we view $\alpha \in \pi_{2k=s+(2k-s)}(\Gamma_{2^i}/\Gamma_{2^{i+1}} FX_{2k-s})$. And from the similar diagram as following:

$$
\begin{tikzcd}
 & \Gamma_{2^{i+1}}FX_{2k-s} & \Gamma_{2^i}FX_{2k-s} & \Gamma_{2^i}/\Gamma_{2^{i+1}}FX_{2k-s} & \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & N'_{2k+1} \arrow[r] \arrow[d, "d_{2k+1}"'] & N_{t+s+1} \arrow[r] \arrow[d, "d_{2k+1}"'] & N''_{t+s+1} \arrow[r] \arrow[d, "d_{2k+1}"'] & 0 \\
0 \arrow[r] & N'_{2k} \arrow[r] \arrow[d, "d_{2k}"'] & N_{2k} \arrow[r] \arrow[d, "d_{2k}"'] & N''_{2k} \arrow[r] \arrow[d, "d_{2k}"'] & 0 \\
0 \arrow[r] & N'_{2k-1} \arrow[r] \arrow[d] & N_{2k-1} \arrow[r] \arrow[d] & N''_{2k-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

- Recall the composition map:

$$AS_{t+s} \xrightarrow{\beta} L_{2^j}AS_s \xrightarrow{L(\alpha)} L_{2^{i+j}}AX$$

And it turns to be:

$$AS_{t+s+(2k-s)} \xrightarrow{\beta} L_{2^j}AS_{2k} \xrightarrow{L(\alpha)} L_{2^{i+j}}AX_{2k-s}$$

This implies that the simplex $B(a) \in \Gamma_{2^{i+j}}FX_{2k-s}$, such that proj $B(a) \in \beta\alpha$, proj $d_0B(a) \in \beta(d\alpha), \text{proj}\, d_{2k+t}B(a) \in (d\beta)\,\alpha$ and $d_iB(a) = *$ for $i \neq 0, 2k+t$.
For $1 \leq l \leq 2k$, we have

$$d_lB(Fi) = 0,$$

thus

$$d_lB(a) = 0.$$

For $2k < l < 2k+t$, since the choice of $2k$ we can use the simplicial relation of $d_l$and s_j move d_l next to a, i.e.

$$d_l(B(a)) = B(d_{l-t}a) = 0.$$

We have following diagram,

$$\begin{tikzcd}
& \Gamma_{2^{i+j+1}}FX_{2k-s} & \Gamma_{2^{i+j}}FX_{2k-s} & \Gamma_{2^{i+j}}/\Gamma_{2^{i+j+1}}FX_{2k-s} & \\
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & N'_{2k+t+1} \arrow[r] \arrow[d, "d_0"'] & N_{2k+t+1} \arrow[r, "\text{proj}"] \arrow[d, "d_0"'] & N''_{2k+t+1} \arrow[r] \arrow[d, "d_0"'] & 0 \\
0 \arrow[r] & N'_{2k+t} \arrow[r] \arrow[d, "d_0"'] & N_{2k+t} \arrow[r, "\text{proj}"] \arrow[d, "d_0"'] & N''_{2k+t} \arrow[r] \arrow[d, "d_0"'] & 0 \\
0 \arrow[r] & N'_{2k+t-1} \arrow[r] \arrow[d] & N_{2k+t-1} \arrow[r, "\text{proj}"] \arrow[d] & N''_{2k+t-1} \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

And,

$$\begin{tikzcd}
C_{2k+t} \arrow[r, "\partial"] \arrow[d, "\pi"] & C_{2k+t-1} \arrow[d, "\pi"] \\
N_{2k+t} \arrow[r, "d_0"] & N_{2k+t-1}.
\end{tikzcd}$$

Which means we can calculate $\partial(B(a)) = \sum_{i=0}^{2k+t}(-1)^i d_i B(a) = \sum_{i=0}^{2k+t} d_i B(a)$ and project it to the normalized group. But $d_0 B(a) + d_{2k+t}B(a)$ already in N_{2k+t-1}, which means $d_0 B(a) + d_{2k+t}B(a)$ is what we want.

i.e.,

$$d(\beta\alpha) = \text{proj}(\partial B(a)) = \text{proj}(d_0 B(a) + (-1)^{2k+t} d_{2k+t}B(a)) = (d\beta)\alpha + (-1)^t \beta(d\alpha).$$

$\square$

Remark 3.14. The d above is the differential map of E_1X, we write d for $d_1 : E_1X \to E_1X$ to distinguish with the face map $d_1 : X_k \to X_{k-1}$.

In order to prove the lemmas, we need some facts are the analogue of the Whitehead lemma.

Definition 3.15. A free simplicial Lie algebra Y is an s.s. Lie algebra for which there exists submodules $B_n \subseteq Y_n$ with the properties

(1) Y_n is the free unrestricted Lie algebra on B_n, for all n.

(2) if $b \in B_n$, and $0 \le i \le n$, then $s_i b \in B_{n+1}$.

Let V and W be two vector spaces, and

$$p : V \oplus W \to W,$$

be the projection map.

Then the kernel of map

$$L^u(p) : L^u(V \oplus W) \to L^u W,$$

is $L^u(U)$.[8, proposition 3.1]

where

$$U = V \otimes T(W) = V \oplus (V \otimes W) \oplus \cdots \oplus (V \otimes W^{\otimes n}) \oplus \cdots .$$

$L^u(U) \to L^u(V \oplus W)$ is defined by

$$v \otimes w_1 \otimes \cdots \otimes w_n \to [\ldots, [v, w_1], \ldots, w_n].$$

- If $V_1, V_2, \ldots V_r$ are vector spaces, we have

$$L^u(\oplus_{i=1}^r V_i) \simeq \oplus_{i=1}^r L^u(V_i).$$

- The lower central series of a simplicial Lie algebra R is defined by $\Gamma_1 R = R$, and $\Gamma_r R = [\Gamma_{r-1} R, R]$.
- The Abelianization of Lie algebra R is $AbR = R/[R,R] = R/\Gamma_2 R$.

Proposition 3.16. *(Whithead Lemma for simplicial Lie algebras) Let $f : Y \to Y'$ be a Lie map between connected free simplicial Lie algebras. If $\pi_* Abf$ is a isomorphism, then so is $\pi_* f$.*

Proposition 3.17. *If R is a connected free simplicial Lie algebra, then $\Gamma_r R$ is $log_2 r$ conneted, i.e.,$\pi_q(\Gamma_r R) = 0$ for $q < log_2 r$.*

Remark 3.18. The important here is that the connectivity $\Gamma_r R \to \infty$.

Proposition 3.19. *Let $f : R \to R'$ be a homomorphism of free simplicial Lie algebras. Then if $(Abf)_*$ is an isomorphism, so is*

$$(\Gamma_r f/\Gamma_{r+1} f)_*.$$

We now already to prove Lemma 3.11.

Proof of Lemma 3.11. We shall proof it for $n > 1$.

Let W be the simplicial Lie algebra freely generated by simplices x, y and z in dimensions $n-1, n$and $2n$ respectively, with faces $d_n y = x, d_i y = *$ for $i \neq n$, and $d_i z = *$, for all i.

$$W \simeq L^u(AS_{n-1} \oplus AS_n \oplus AS_{2n}).$$

And let $f : W \to L^u AS_n$ be the Lie map given by $y \to i_n$ and $z \to i_n \underline{\otimes} i_n$.

f correspondes to the projection:

$$X \oplus Y \oplus Z \to Y$$

$$(x, y, z) \to (0, y, y \underline{\otimes} y)$$

The complex $T = \ker f$ then is a free simplicial Lie algebra which in every dimension is freely generated by the simplices of the form

$$\begin{array}{rll} \mathrm{I} & [x, y, \ldots, y] & r \geq 1 \\ \mathrm{II} & [(y \underline{\otimes} y - z), y, \ldots, y] & r \geq 1 \end{array}$$

where the s_{α_i} are iterated degeneracy operators, and AbT is an simplicial $\mathbb{Z}/2$-module on the same generators. Let $U \subset AbT$ be the submodule generated by the simplices of the form I and let $V = (AbT)/U$. Then one readily verifies that $\pi_i U = \mathbb{Z}/2$ with generator $[x, y, \ldots, y]$

whenever $i = rn - 1$ for some $r \geq 1$ and $\pi_i U = 0$ otherwise and that $\pi_i V = \mathbb{Z}/2$ with generator $[(y \otimes y - z), y, \ldots, y]$ whenever $i = rn$ for some $r \geq 2$ and $\pi_i V = 0$ otherwise. However, in AbT the generators in dimension $> n-1$ kill each other, i.e.

$$d_{rn}[(y \underline{\otimes} y - z), y, \ldots, y] = [x, y, y, \ldots, y]$$
$$d_i[(y \underline{\otimes} y - z), y, \ldots, y] = 0 \quad \text{for } i < rn$$

These are follows from :

$$d_i(y \underline{\otimes} y - z) = d_i(y \underline{\otimes} y) = d_i(\sum s_\alpha y \otimes s_\beta y) = \sum d_i s_\alpha y \otimes d_i s_\beta y = \sum s_{\alpha'} d_{i'} y \otimes s_{\beta'} d_{i''} y.$$

- Those α, β can not move d_i next to y will kill each other. This can only happen when $i-1, i$ not on the same half shuffle. Since, suppose $i-1, i \in \alpha$, then by the simplicial relations of d_i, s_j, we can know d_i can move next to y. Suppose $i-1 \in \alpha, i \in \beta$, we take a new shuffle α', β' only exchange the position of $i-1$ and i. And it indeed is a shuffle.

Therefore we have

$$d_i(y \underline{\otimes} y - z) = d_i(y \underline{\otimes} y) = 0, \quad \text{if } i \neq 2n$$

$$\begin{aligned} d_{2n}(y \underline{\otimes} y) &= \sum d_{2k} s_\alpha y \otimes d_{2k} s_\beta y \\ &= \sum s_{\alpha'} d_n y \otimes s_{\beta'} y + \sum s_{\alpha''} y \otimes s_{\beta''} d_n y \\ &= \sum s_a x \otimes s_b y + \sum s_{a'} x \otimes s_{b'} y \\ &= x \otimes y + y \otimes x = [x, y]. \end{aligned} \tag{3.1}$$

- One may use

$$\partial(y \underline{\otimes} y) = \partial y \underline{\otimes} y + y \underline{\otimes} \partial y.$$

Therefore we have

$$d_i([x, y]) = 0.$$

Again by calculate

$$\begin{aligned} d_i([y \otimes y - z, y, \ldots, y]) &= 0, \quad \text{if } i \neq rn. \\ d_{rn}([y \otimes y - z, y, \ldots, y]) &= [x, y, \ldots, y]. \end{aligned} \tag{3.2}$$

we can have

$$d_i([x, y, \ldots, y]) = 0.$$

And thus $\pi_{n-1} AbT = \mathbb{Z}/2$ with generator x and $\pi_i AbT = 0$ for $i \neq n-1$. As $n > 1$ the Whitehead Lemma implies that the Lie maps $g : L^u AS_{n-1} \to T$ and $h : W \to L^u AS_{2n}$ given by $i_{n-1} \to x$ and $z \to i_{2n}$ induce isomorphisms of the homotopy groups. As the composition

$$L^u AS_{n-1} \xrightarrow{g} T \xrightarrow{\text{incl}} W \xrightarrow{h} L^u AS_{2n}$$

is trivial it follows that $T \xrightarrow{\text{incl}} W$ is trivial on the homotopy groups. The exactness of the homotopy sequence of the fibre map $f : W \to L^u AS_n$ now yields the desired result.

And we shall notice that

$$\begin{tikzcd} \pi_* L^u AS_n \arrow[r, "\partial"'] & \pi_{*-1} L^u AS_{n-1} \arrow[l, bend right, "\sigma"'] \end{tikzcd}$$

Therefore we have

$$\pi_* LAS_{2n} \simeq \pi_* W \xrightarrow{f} \pi_* L^u AS_n \xrightarrow{\partial} \pi_{*-1} L^u AS_{n-1}$$

is split.

Which means we have

$$\pi_* L^u_r AS_n \simeq \pi_{*-1} L^u_r AS_{n-1} + \pi_* L^u_{\frac{r}{2}} AS_{2n}, \quad r \text{ even.}$$

□

3.2. **Relations of lambda algebra.** In this section, we will prove Theorem 3.1 (3),(4).

As $\pi_* L_2 AS$ is generated by the λ_n and as the map

$$\pi_* L_1 AX = \mathrm{H}_* X \xrightarrow{d_1} \pi_* L_2 AX = \mathrm{H}_* X \otimes \pi_* L_2 AS$$

is natural. It follows from the duality between $\mathrm{H}_* X$ and $\mathrm{H}^* X$ that there are unique elements $T_n \in \mathscr{A}$(Steenrod algebra), with degree $T_n = n$ such that

$$a \xrightarrow{d_1} \sum_{n>0} (aT_n \otimes \lambda_{n-1})$$

for all $a \in \mathrm{H}_* X$ and we thus have to prove that $T_n = Sq^n$ for all $n > 0$.

This we will do using the following fact.

Remark 3.20. $Sq^n \in \mathscr{A}$ is the only non-zero element of degree n which vanishes on $\mathrm{H}_{2n-1} AS_{n-1}$.

Let X be the spectrum such that $X_0 = AS_{n-1}$ and X_q is the q-fold suspension of X_0 for $q \geq 0$. Then we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}_{2n-1} X_0 = \pi_{2n-2}(\Gamma_1/\Gamma_2)GX_0 & \xrightarrow{d_1} & \pi_{2n-3}(\Gamma_2/\Gamma_3)GX_0 = \Sigma(\mathrm{H}_i X_0 \otimes \pi_{2n-3} L_2 AS_{i-1}) + u \\
\simeq \downarrow f & & \downarrow g \\
\mathrm{H}_{2n-1} X = \pi_{2n-1}(\Gamma_1/\Gamma_2)FX & \xrightarrow{d_1} & \pi_{2n-2}(\Gamma_2/\Gamma_3)FX = \Sigma(\mathrm{H}_i X \otimes \pi_{2n-2-i} L_2 AS)
\end{array}$$

where G is as in [3, Section 7] and f and g are induced by the "inclusion" $X_0 \subset X$. The upper left equality is proved as in [3, Theorem 15.1] and implies that $f : \mathrm{H}_{2n-1} X_0 \to \mathrm{H}_{2n-1} X$ is the isomorphism induced by the "inclusion". The upper right equality is a consequence of 3.9; as cross effects are killed by suspension. one can choose the direct summand U in such a manner that it is killed by g. And as on $\Sigma(\mathrm{H}_i X_0 \otimes \pi_{2n-3} L_2 AS_{i-1})$ the map g is the map induced by the "inclusions" $X_0 \subset X$ and $S_{i-1} \subset S$ it follows 3.9 that an element $b \otimes \lambda_{n-1}$ is in the image of g only if $b = 0$. Thus $aT_n = 0$ for all $a \in \mathrm{H}_{2n-1} X$ and therefore also for all $a \in \mathrm{H}_{2n-1} X_0$ and it thus (Remark 3.20) remains to show that $T_n \neq 0$.

In order to do this we take a closer look at the spectral sequence for AS.

First we recall from [10].

Remark 3.21. $\mathrm{H}_* AS$ is a polynomial algebra on generators ξ_i of degree $2^i - 1 (i \geq 0)$ with one relation $\xi_0 = 1$. The Sq^n operate on the right on $\mathrm{H}_* AS$ according to the formulas

$$\begin{array}{ll}
\xi_i Sq = \xi_i + \xi_{i-1} & \text{where } Sq = \sum Sq^n \\
(\xi\xi')\, Sq = (\xi Sq)\,(\xi' Sq) & \text{for all } \xi, \xi' \in \mathrm{H}_* AS.
\end{array}$$

Now $T_1 = 0$ would imply that $1 \otimes \lambda_0 \in E_\infty AS$, which would contradict the convergence of the spectral sequence [6, Theorem 4.1].

- Since $d_1(1 \otimes \lambda_0) = 0$, and $T_1 = 0$, therefore it cann't lives in $im(d_1)$. Thus $1 \otimes \lambda_0 \in E_2^{0,1} AS$, and in $E_\infty AS$.

Proposition 3.22. [6, Theorem 4.1] *If X is simply connected and has finitely generated homotopy groups, then $E_t X$ is weakly convergent, and $E_\infty X$ is the graded group associated with filtration of $\pi_*(X;p)$.*

Remark 3.23. Then spectral sequence $E_t X$ mentioned above is just the unstable version we construed in this section.

Thus $T_1 = Sq^1$. Similarly $T_2 = Sq^2$ (because otherwise $1 \otimes \lambda_1$ or $1 \otimes \lambda_1 + \xi_1 \otimes \lambda_0$ is in $E_\infty AS$).

Therefore assume inductively that $T_n = Sq^n$ for $n < 2k$(k ≥ 1) and suppose T_{2k+1} were 0 . Then a simple calculation (using 3.20) yields that $d_1 d_1 \left(\xi_1 \xi_2^k\right)$ is a polynomial in the ξ_i with coefficients in $\pi_* L_4 AS$ of which the constant term is $\lambda_k \lambda_{2k-1}$.

- This is basically because

$$d^1 d^1(\xi_1 \xi_2^k) = \sum_{i>0, j>0} ((\xi_1 \xi_2^k) Sq^i) Sq^j \otimes \lambda_{j-1} \lambda_{i-1} + \sum_{i>0} \xi_1 \xi_2^k Sq^i \otimes d^1 \lambda_{i-1} \tag{3.3}$$

For the reason of degree, the second term above can't have constant term. And in the first term, i must be $2k$, $j = k+1$.

But as $d_1 d_1 = 0$ this would imply $\lambda_k \lambda_{2k-1} = 0$, in contradiction to 3.9. Thus $T_{2k+1} = Sq^{2k+1}$. Applying the same argument to $\xi_1^2 \xi_2^k$ we get also that $T_{2k+2} = Sq^{2k+2}$.

Proof of Theorem 3.1 (3) and (4). One can write $d_1 d_1 \left(\xi_1^n \xi_2^m\right) (m \geq 1)$ as a polynomial in the ξ_i with coefficients in $\pi_* L_4 AS$.

Let's compute the constant term of $d_1 d_1(\xi_1^n \xi_2^m)$ and $d_1 d_1(\xi_1^n)$.

First, we compute the constant term of $d_1 d_1(\xi_1^n)$.

Since we have

$$\begin{aligned}(\xi_1^n) Sq &= (\xi_1 Sq)^n \\ &= (\xi_1 + 1)^n \\ &= \sum_{i=0}^{n} \binom{n}{i} \xi_1^{n-i}\end{aligned} \tag{3.4}$$

Since $Sq^i : \mathrm{H}_*(AS) \to \mathrm{H}_{*-i}(AS)$, therefore we have $\xi_1^n Sq^i = \binom{n}{i} \xi_{n-i}$.

$$\begin{aligned}d_1 d_1(\xi_1^n) &= d_1(\sum_{i>0} \xi_1^n Sq^i \otimes \lambda_{i-1}) \\ &= d_1(\sum_{i>0} \xi_1^n Sq^i) \otimes \lambda_{i-1} + \sum_{i>0} \xi_1^n Sq^i \otimes d\lambda_{i-1} \\ &= \sum_{i,j>0} (\xi_1^n Sq^i) Sq^j \otimes \lambda_{j-1} \lambda_{i-1} + \sum_{i>0} \xi_1^n Sq^i \otimes d\lambda_{i-1}\end{aligned} \tag{3.5}$$

Since we are computing constant term, therefore $i + j = n$, constant term in the sencond term i must be n.

i.e. we have

$$\begin{aligned}\sum_{i,j>0}\binom{n}{i}\binom{n-i}{j}\lambda_{j-1}\lambda_{i-1}+d\lambda_{n-1}=0\\ \sum_{i,j}\binom{n}{i}\lambda_{i-1}\lambda_{j-1}=d\lambda_{n-1}\end{aligned}\tag{3.6}$$

$$\begin{aligned}(\xi_1^n\xi_2^m)Sq&=(\xi_2Sq)^m(\xi_1Sq)^n\\ &=(\xi_1+\xi_2)^m(\xi_1+1)^n\\ &=(\sum_{\alpha=0}^{m}\binom{m}{i}\xi_2^{m-\alpha}\xi_1^{\alpha})(\sum_{\beta=0}^{n}\binom{n}{\beta}\xi_1^{n-\beta})\\ &=\sum_{\alpha,\beta}\binom{m}{\alpha}\binom{n}{\beta}\xi_2^{m-\alpha}\xi_1^{\alpha+n-\beta}\end{aligned}\tag{3.7}$$

$$\begin{aligned}d_1d_1(\xi_1^n\xi_2^m)&=d_1(\sum_{i>0}\xi_1^n\xi_2^mSq^i\otimes\lambda_{i-1})\\ &=d_1(\sum_{i>0}\xi_1^n\xi_2^mSq^i)\otimes\lambda_{i-1}+\sum_{i>0}\xi_1^n\xi_2^mSq^i\otimes d\lambda_{i-1}\\ &=\sum_{i,j>0}(\xi_1^n\xi_2^mSq^i)Sq^j\otimes\lambda_{j-1}\lambda_{i-1}+\sum_{i>0}\xi_1^n\xi_2^mSq^i\otimes d\lambda_{i-1}\end{aligned}\tag{3.8}$$

For the reason of degree, the second term in the last row of (3.8) have no constant term. Then the constant term of (3.8) must be

$$\sum_{2\alpha+\beta=i,2\alpha'+\beta'=j,\alpha+\alpha'=m,\beta+\beta'=n+m}\binom{m}{\alpha}\binom{n}{\beta}\binom{m-\alpha}{\alpha'}\binom{\alpha+n-\beta}{\beta'}\xi_2^{m-\alpha-\alpha'}\xi_1^{\alpha'+\alpha+n-\beta-\beta'}\otimes\lambda_{j-1}\lambda_{i-1}$$

But for the reason of degree agin, α must be m.
Then the constant term is

$$\sum_{i=2m+\beta,j=m+n-\beta}\binom{n}{\beta}\lambda_{j-1}\lambda_{i-1}=\sum_{i'+j'=n,i'=n-\beta}\binom{n}{i'}\lambda_{i'+m-1}\lambda_{j'+2m-1}$$

Then in Λ we have

$$\sum_{i'+j'=n,i'=n-\beta}\binom{n}{i'}\lambda_{i'+m-1}\lambda_{j'+2m-1}=0.$$

And these are all relations, because these relations just tell us nonadmissble monomials how to represent by admissble basis.

□

Remark 3.24. As said before, there is a way to get the relations in 1.2.
Which is by calculate

$$d_1d_1(x)=0=\sum_{i,j>0}xSq^iSq^j\otimes\lambda_{j-1}\lambda_{i-1}+\sum_{i>0}xSq^i\otimes d\lambda_{i-1}.$$

using Adem relations:

$$Sq^a Sq^b = \sum_{c=0}^{[a/2]} \binom{b-1-c}{a-2c} Sq^{a+b-c} Sq^c, \quad \text{for } 0 < a < 2b.$$

4. Comparison with Adams spectral sequence

In this section, we will prove following result.

Theorem 4.1. *Let X be a simplicial spectrum, the spectral sequence $\{E_i X, d_i X\}$we constructed in section 3 is exactly the Adams spectral sequence of X, when $i \geq 2$.*

We only need to prove

$$\begin{array}{c} \mathrm{H}_* \, \Gamma_{2^{j+1}} FX = \pi_* A \Gamma_{2^{j+1}} FX \\ \big\downarrow \\ \mathrm{H}_* \, \Gamma_{2^j} FX = \pi_* A \Gamma_{2^j} FX \end{array}$$

is trivial map.

By [5, 15.4] the naturality result, we have following diagram.

$$\begin{tikzcd} \pi_* A \Gamma_{2^{j+1}} FX \arrow[r] \arrow[d, "\simeq"] & \pi_* A \Gamma_{2^j} FX \arrow[d, "\simeq"] \\ \pi_* \Gamma_{2^{j+1}} FAX \arrow[r] & \pi_* \Gamma_{2^j} FAX \end{tikzcd}$$

we only need to show the bottom map is trivial.

Proposition 4.2. *The spectral sequence collapses for spectra of the form AX, i.e. if X is a spectrum, then*

$$E_2 AX = E_\infty AX.$$

Proof. It suffices to prove this if $X = S$. By 3.1, 3.2 and Remark 3.20 $E_1 AS$ is freely generated by the elements

$$\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \lambda_{i_1} \cdots \lambda_{i_m} \tag{4.1}$$

for which $\alpha_i \geq 0$ for all i and $i_{j+1} \leq 2 i_j$ for all $j > 0$. Moreover, a straightforward calculation, using Remark 3.20 and caculation in section 3, yields

$$d_1 \left(\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \lambda_{i_1} \cdots \lambda_{i_m} \right) = \xi_1^{\alpha_2} \cdots \xi_{k-1}^{\alpha_k} \lambda_j \lambda_{i_1} \cdots \lambda_{i_m} + \sum \xi_1^{\beta_1} \cdots \xi_k^{\beta_k} \lambda_{j_0} \cdots \lambda_{j_m} \tag{4.2}$$

where $j = \alpha_k \cdot 2^{k-1} + \cdots + \alpha_1 - 1$ and where the sum is taken over certain generators with the property that

$$(0, \alpha_k, \ldots, \alpha_2) < (\beta_k, \ldots, \beta_1) \leq (\alpha_k, \ldots, \alpha_1)$$

in the lexicographical ordering.

For every integer $t > 0$ let F^t be the submodule generated by the generators of the form (4.1) for which $k + m \geq t$ where k is the largest integer for which $\alpha_k > 0$. By (4.2) $\gamma \in F^t$ implies $d_1 \gamma \in F^t$ and hence $Q^t = F^t / F^{t+1}$ is again a differential module. Now for fixed t and $s \leq t$ let $G^s \subset Q^t$ be generated by the generators of the form (4.1) for which either $m > s$ or $m = s$ and $i_1 \leq \alpha_k \cdot 2^k + \cdots + \alpha_1 \cdot 2 - 2$. Then $R^s = G^s / G^{s+1}$ is again a differential module and it follows readily from (4.2) that $\mathrm{H}_* R^s = 0$ for all s, which implies that $\mathrm{H}_* Q^t = 0$ for all $t > 0$. Again, using (4.2), a standard argument now yields that $E_2 AS = \mathbb{Z}/2$

- This is basically because we have

$$0 \to F^2 \to F^1 \to Q^1 \to 0$$

and

$$0 \to F^3 \to F^2 \to Q^2 \to 0$$
$$0 \to F^{i+1} \to F^i \to Q^i \to 0$$

Since $\mathrm{H}_* Q^i = 0, i > 0$.
We get

$$0 = \mathrm{H}_{*+1} Q^1 \longrightarrow \mathrm{H}_* F^2 \longrightarrow \mathrm{H}_* F^1 \longrightarrow \mathrm{H}_* Q^1 = 0$$

$$0 = \mathrm{H}_{*+1} Q^i \longrightarrow \mathrm{H}_* F^{i+1} \longrightarrow \mathrm{H}_* F^i \longrightarrow \mathrm{H}_* Q^i = 0$$

But for fixed $* > 0$ when $i \to \infty$ $\mathrm{H}_*(F^i) = 0$, therefore $E_2 AS = \mathrm{H}_0(E_1 AS) = \mathbb{Z}/2$.
Therefore $E_2 AS = E_\infty AS$.

$\square$

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