Finite Group actions on Manifolds

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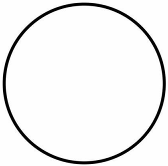
In Mathematics, automorphism group represnts symmetry.

Definition

A group is a set G, together with an operation(multiplication), and there exists an identity element and the operation satisfies asscoiative law.

- For example, the set of rotations and reflections is the symmetry information of circle.
- And it indeed forms a group, the operation is composition.



Defintion

- 1. A subgroup of G is a subset H which is still a group under the operation of G.
- 2. A group G is called abelian group, if the operation is commutative.

Jordan Theorem

$\forall n \in N$, there exists a constant C_n such that if G is a finite subgroup of $GL(n, R)$(invertible linear transformation), then G has an abelian subgroup A, $[G : A] \leq C_n$.

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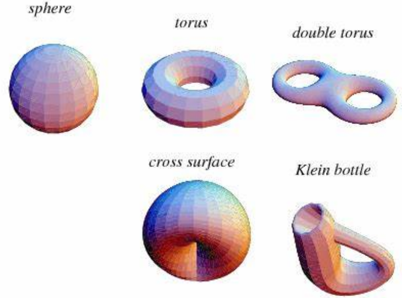
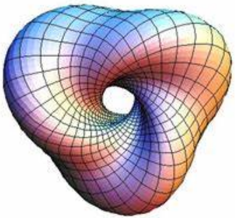
Definition

A n dimensional smooth manifold is locally like R^n, and with smooth condition.

- Homology of manifold are alebraic information of manifold, can be view as ID of the space.Denoted by $H_*(X)$.
- Euler characteristc is also algebraic information of a space. Denoted by χ.

Let's see some examples of manifold.





- Roughly speaking, the paper proofs if G is finite subgroup of diffeomorphism group of some manifold, then it satisfies the Jordan property.

Notation

Suppose that C is a set of finite groups, in the proof, we let C be the set of finite subgroups of $Diff(X)$.

- P(C) be the set of all $T \in C$, the order of T is a prime power.
- We denote by $T_A(C)$ the set of all $T \in C$ such that there exist primes p and q, an abelian Sylow p-subgroup P of T, and a normal abelian Sylow q-subgroup Q of T, such that $T = PQ$.

Definition

Let c and d be positive integers. We say that a set of groups C satisfies (the Jordan property) $J(c,d)$, if each $G \in C$ has an abelian subgroup A such that $[G:A] \leq c$ and A can be generated by d elements.

Lemma 3.2

Let d and M be positive integers. Let C be a set of finite groups which is closed under taking subgroups and such that $P(C) \cup T_A(C)$ satisfies $J(M,d)$. Then there exists a positive integer C_0 such that C satisfies $J(C_0,d)$.

Theorem 1.2

Let X be a smooth manifold belonging to one of following three collections, if finite group G acting effectively and smoothly on X, then G satisfies J(C,d), and C,d **depending only on** $dimX, H_*(X)$.

- 1. Acyclic manifolds, $H_0(X) = \mathbb{Z}, H_i(X) = 0, i > 0$.
- 2. Connected compact manifolds with $\chi \neq 0$
- 3. $H_*(X, \mathbb{Z}) \simeq H_*(S^n, \mathbb{Z})$

- Since G acting effectively and smoothly on X, G is a subgroup of $Diff(X)$.

For group action, we have following lemma.

Lemma 2.1

Let a finite group Γ act effectively and smoothly on a **connected** manifold X, and let $x \in X^{\Gamma}$.
Then we get an inclusion $\Gamma \to GL(T_x X) = GL(n, \mathbb{R})$.

- For p -group action case, we have following lemma.

Lemma 2.2

p be a prime, let X be a $\mathbb{Z}_p$ acyclic manifold, $\mathbb{Z}_p$ acts on X, then F (fix point space) is not empty, and F is $\mathbb{Z}_p$ acyclic manifold.

Lemma 2.3

Let Y be a **compact** smooth manifold, satisfying $\chi(Y) \neq 0$. Let p be a prime, and let G be a finite p-group acting smoothly on Y . Let r be the biggest nonnegative integer such that p^r divides $\chi(X)$. There exists some $y \in Y$ whose stabilizer G_y satisfies $[G : G_y] \leq p^r$.

Lemma 2.4

Let X be a manifold, let p be a prime, and let G be a finite p-group acting continuously on X. We have

$$\sum_j b_j(X^G, \mathbb{Z}_p) \leq \sum_j b_j(X, \mathbb{Z}_p)$$

- For $T_A(C)$ type group action, we have following.

Lemma 4.2

Suppose that X is a smooth connected manifold and that $G \in T_A(C)$. Assume that there is a G-invariant **connected** submanifold $Y \subset X$. Let G_Y be the group consisting of all diffeomorphisms of Y which are induced by restricting to Y the action of the elements of G. Let $r := dimX - dimY$. If G_Y is abelian, then there is an abelian subgroup $A \subset G$ satisfying $[G : A] \leq r!$.

Sketch of proof of case 1

- For p-group case, by lemma 2.2, X^G is $\mathbb{F}_p$ acyclic, therefore it is nonempty and connected, then by lemma 2.1, G is subgroup of $GL(n, R)$, then use Jordan theorem.
- For $T_A(C)$-group case, let $G = PQ, Y = X^Q$,for the same reason, we have Y is nonempty and connected.
- Since Q is normal in G, Y is G-invariant space, by lemma 4.2, G has Jordan property.

Sketch of proof of case 2,3

- p-group case, by lemma 2.3, there is a point x, $[G : G_x] \leq p^r$.
- By lemma 2.1 G_x is the subgroup of $GL(T_x) = GL(n, \mathbb{R})$,then use Jordan theorem, G_x has Jordan property, so G has.
- $T_A(C)$ case, first find $Q' \subset Q$, which is normal in G, and it has fix point.
- Then, by lemma 2.4, the component of $X^{Q'}$ is bounded, for a component Y, find $P' \subset P$ which the fix point space is not empty, and P' preserves x, then for $P'Q'$ use lemma 4.2.
- Case 3, $p-$group case is known, for $T_A(C)$ group case, the key thing is to use Borel formula.

Summary

- A finite group acting effectively and smoothly on three types of manifolds, then it has Jordan property.
- The main idea of proof is to reduced to p-group case and $T_A(C)$group case.
- For p-group case,we find a fix point then use 2.1.
- For $T_A(C)$, we find a submanifold, and using lemma 4.2.

Summary

- The theorem reflects the symmetry information of some geometric spaces.
- For example, The finite subgroups of $SO(3)$ (rotation group of 3 dimensional space) exactly corresponds to all regular polyhedrons and sphere!
- What about other manifolds(Spaces)?